Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Characteristic Equations of Chebyshev Polynomials of Third and Fourth Kinds and Their Generating Matrices

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Received: 12 April 2023; **Revised:** 16 May 2023; **Accepted:** 25 June 2023

Abstract: The main goal of the article is to obtain matrix representation for the third and fourth kinds of Chebyshev polynomials by using a tridiagonal matrix. We present a connection between the determinant of the tridiagonal matrix and the third and fourth kinds of Chebyshev polynomials. We also determine the characteristic equations for the third and fourth kinds of the Chebyshev polynomials up to degree three. We also prove some properties relating to matrix representation. We obtain a connection between the second, third kind and fourth kinds of Chebyshev polynomials and matrix power. It elaborates the theorem to validate through examples. The applications of the Chebyshev polynomials is also discussed. The practical application of the Chebyshev polynomials of the third kind in approximation theory is also detailed.

Keywords: chebyshev polynomials, matrix representation, matrix power, characteristic equations, determinant, composition identities

MSC: 41A50, 11B39

1. Introduction

The eminent Chebyshev polynomials for $m \geq 0$ of four kinds are defined as follows:
The first kind $T_m(p)$ is defined as:

$$T_0(p) = 1,\ T_1(p) = p,\ T_m(p) = 2pT_{m-1}(p) - T_{m-2}(p),\ \text{for } m \geq 2,\ 3,\ \ldots$$

The second kind $U_m(p)$ is defined as:

$$U_0(p) = 1,\ U_1(p) = 2p,\ U_m(p) = 2pU_{m-1}(p) - U_{m-2}(p),\ \text{for } m \geq 2,\ 3,\ \ldots$$

DOI: https://doi.org/10.37256/cm.5420242819

The third kind $V_m(p)$ is defined as:

$$V_0(p) = 1,\ V_1(p) = 2p - 1,\ V_m(p) = 2pV_{m-1}(p) - V_{m-2}(p),\ \text{ for } m \geq 2,\ 3,\ \ldots$$

The fourth kind $W_m(p)$ is defined as:

$$W_0(p) = 1,\ W_1(p) = 2p - 1,\ W_m(p) = 2pW_{m-1}(p) - W_{m-2}(p),\ \text{ for } m \geq 2,\ 3,\ \ldots$$

Hence, these polynomials are a succession of recurrence relations. Chebyshev polynomials comes under the class of classical orthogonal polynomials. In literature [1], there are many types of classical orthogonal polynomials i.e., Laguerre polynomials, Jacobi polynomials, and Hermite polynomials. Ugur Duran et al. studied Hermite polynomials, $(p,\ q)$-Bernstein polynomials with their modifications. Khan et al. [2] investigated the properties related to Laguerre polynomials. It obtained Laguerre-based Hermite-Bernoulli polynomials. These polynomials are closely related to Chebyshev polynomials. Many authors studied the third and fourth kinds of Chebyshev polynomials and also discovered many lemmas and identities related to these polynomials [3–7]. The basic definitions of the Chebyshev polynomials were studied from very fantastic books written by Doman and Manson [8, 9]. Da Fonseca [10, 11] provided explicit inverses for tridiagonal matrices using the second kind of Chebyshev polynomials. They used an invertible matrix to find results related to tridiagonal matrices. Yang and Zheng [12] used the Riordan array to give the determinant representation of Chebyshev polynomials, Fibonacci numbers, and Pell numbers. Bucer et al. [13] gave the Characteristic Equation (C.E.) of the Chebyshev matrix of the first kind, found associated polynomials of Chebyshev, and presented an explicit formula from them. Singh et al. [14] examined properties related to extension with two variables of the second kind of Chebyshev polynomials matrix second kind. And also obtained a generalized Chebyshev matrix of the second kind. Zhao et al. [15] discussed the coupling system for electron-phonon via the product of the Chebyshev pseudo-site matrix. Raslan et al. [16] discussed the Fredholm-Volterra integrodifferential equations with the first kind of shifted Chebyshev polynomials contingent on the operational matrix. The techniques are based on the first kind of shifted Chebyshev polynomials. Metwally et al. [17] described and study the second kind of Chebyshev matrix polynomials and discuss 3-term recurrence relations. Primo et al. [18] obtained many identities for the first and second kinds of Chebyshev polynomials with a non-singular complex matrix. They employ the matrix's power, trace, and determinant, yielding the following result:

Let X be a non-singular 2 × 2 matrix. The integral power of X, for $m \geq 2$

$$X^m = a_2^{\frac{(m-1)}{2}} U_{m-1}\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) X - a_2^{\frac{m}{2}} U_{m-2}\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) I,$$

where I is the identity matrix and $U_m(p)$, is the second kind of Chebyshev polynomials [18].

Let X be a non-singular 2 × 2 matrix . For any integer $m \geq 0$,

$$\text{Trace } X^m = 2a_2^{\frac{m}{2}} T_m\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right),$$

where, $T_m(p)$ is the first kind of Chebyshev polynomials [18].

Andreescu and Mushkarov [19] discussed the quadratic form and determinant representation of the Chebyshev polynomials matrix for the first type of Chebyshev polynomials, obtained the following identity connected to the first kind of Chebyshev polynomials matrix.

The following result holds for all integers $m \geq 3$,

$$X_m(2p) = 2T_m(p) + 2(-1)^m.$$

Matrix X have the following Eigenvalues;

$$\lambda_k = cos\left(\frac{(2k-1)\pi}{m}\right), \ \ 1 \leq k \leq m.$$

Minimal and maximal Eigenvalues of X are:

$$\lambda_{min} = \begin{cases} -2cos\left(\frac{\pi}{m}\right), & \text{if } m \text{ is even.} \\ -2, \ \ \text{if } m \text{ is odd.} \end{cases}$$

$$\lambda_{min} = 2cos\left(\frac{\pi}{m}\right).$$

Arya and Verma [20] gave identities for Fibonacci polynomials and generalized Fibonacci numbers. They also obtained an exceptional representation in the form of a matrix by using obtained identities. Metwally et al. [21] investigated associated matrix polynomials linked with the second kind of Chebyshev matrix polynomials. They found many results related to the associated Chebyshev matrix polynomial. Sanford [22] presented the solution to a problem from the American mathematical journal's problem section. The Chebyshev polynomials of the second kind with matrix representations was central to the problem. Qi et al. [23] linked tridiagonal determinants with Fibonacci polynomials, Fibonacci numbers, and Chebyshev polynomials. They also presented two formulas to calculate tridiagonal determinants. Kocik [24] found a matrix representation of the Chebyshev polynomials, Fibonacci series and Lucas polynomials by using the symmetric tensor and power of a particular matrix. Milica et al. [25] found a two-determinant generalized formula situated on the second kind of Chebyshev polynomials. For this purpose, they utilized a tridiagonal matrix and a Heisenberg matrix. Da Fonseca [10] discussed the relationship between the second kind of Chebyshev polynomials and Fibonacci numbers. To obtain the result, he used a tridiagonal matrix determinant and presented some new identities. Oteles et al. [26] worked on a family of tridiagonal matrices relating to the first kind of Chebyshev polynomials and obtained Eigen vectors and Eigen values. Altın and Cekim [27] presented recurrence relations for Chebyshev matrix polynomials, especially for the second kind. They also found generating matrix functions and several identities for this second kind of Chebyshev polynomials. Ahmed [28] derived an algorithm for the Lane-Emden equation with the help of shifted Chebyshev polynomials of the first kind. This method makes a well approximate solution. Regmi et al. [29] discussed the application of Chebyshev polynomials for nonlinear equations under weak conditions. Erdmanna and Schroll [30] derived the results on the Chebyshev polynomials of the second kind by using symmetric matrices. Aiyub et al. [31] presented a binomial matrix connected to Poisson Fibonacci. They specified some results of rough statistical convergence. And also deduce that approximation theory consolidated the rough statistical convergence. Pucanovic and Pesovic [32] used the properties of circulant matrices and Chebyshev polynomials. They connect circulant matrices and Chebyshev polynomials. Many authors [12, 33–38] also worked on the first and second kinds and gave many theorems with their applications. Fouad [39] studied a shifted Chebyshev polynomials of the third kind for operational matrices. Chishti [40] studied a shifted Chebyshev polynomials of the fourth kind for operational matrices.

Our work is motivated by the earlier work of Primo et al. [18]. The authors derived the theorems and many identities related to the Chebyshev polynomials of the first and second kind via 2×2 matrix. Another motivation for our passion in establishing the presented results is the research of Bucur et al. [13]. The authors obtained the character equations for Chebyshev polynomials of the first and second kind. We have prior studied Arya and Verma [20], Yang and Zheng [12], and Altın and Cekim [27] research to generate matrix representation and characteristic equations of Chebyshev polynomials of the third and fourth kind.

The outline of this paper is as follows:

The paper is mainly divided in to four segments. In the I^{st} section, we look back at the basic definitions of Chebyshev polynomials. In the next section, we present the determinant representation for the third and fourth kinds and discover the characteristic equation up to degree three for the third and fourth kind of the Chebyshev polynomials. In the next section, we obtained theorems which describe the relationship between matrix power and Chebyshev polynomials of second, third, and fourth kinds. The last section is about the applications related to the Chebyshev polynomials. The practical application of the Chebyshev polynomials in approximation theory is also discussed.

2. Matrix representation of chebyshev polynomials of third kind

Now we present determinant representation and characteristic equations for the Chebyshev polynomials of the third kind,

$$V_0(p) = 1,\ V_1(p) = 2p - 1,\ V_m(p) = 2pV_{m-1}(p) - V_{m-2}(p),\ \text{for } m\ \geq 2,\ 3,\ \ldots$$

Tri-diagonal matrix, $[b_{q,\ r}]$ represents a matrix sequence for the Chebyshev polynomials of the third kind:

$$[b_{q,\ r}] = \begin{cases} b_{q,\ r} = ap - 1, & \text{if } q = r = 1 \\ b_{q,\ r} = ap, & \text{if } q = r \geq 2 \\ b_{q,\ r} = 1, & \text{if } q = r + 1,\ q = r - 1 \\ b_{q,\ r} = 0, & \text{otherwise} \end{cases}$$

In general, determinant representation:

$$X(m) = \begin{vmatrix} ap-1 & 1 & 0 & \ldots & \ldots & 0 \\ 1 & ap & 1 & \ldots & \ldots & 0 \\ 0 & 1 & ap & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & ap & 1 \\ 0 & 0 & \ldots & \ldots & 1 & ap \end{vmatrix}$$

When $a = 2$;

$$X(m)=\begin{vmatrix} 2p-1 & 1 & 0 & \dots & \dots & 0 \\ 1 & 2p & 1 & \dots & \dots & 0 \\ 0 & 1 & 2p & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 2p & 1 \\ 0 & 0 & \dots & \dots & 1 & 2p \end{vmatrix}$$

$X(m)$ = Determinants are of type, i.e., the determinant of Chebyshev matrices $V_m(p)$,

$$|X(1)|=t_{1,\,1}=2p-1=V_1(p).$$

$$|X(2)|=t_{1,\,1}t_{2,\,2}-t_{2,\,1}t_{1,\,2}=Det\begin{pmatrix} 2p-1 & 1 \\ 1 & 2p \end{pmatrix}$$

$$=4p^2-2p-1=V_2(p).$$

$$|X(3)|=t_{3,\,3}|D(2)|-t_{3,\,2}t_{2,\,3}|D(1)|$$

$$=8p^3-4p^2-4p+1$$

$$=Det\begin{pmatrix} 2p-1 & 1 & 0 \\ 1 & 2p & 1 \\ 0 & 1 & 2p \end{pmatrix}=V_3(p).$$

$$|X(4)|=t_{4,\,4}|D(3)|-t_{4,\,3}t_{3,\,4}|D(2)|$$

$$=16p^4-8p^3-12p^2+4p+1$$

$$=Det\begin{pmatrix} 2p-1 & 1 & 0 & 0 \\ 0 & 1 & 2p & 1 \\ 0 & 0 & 1 & 2p \end{pmatrix}=V_4(p).$$

$$|X(5)| = t_{5,\ 5}|D(4)| - t_{5,\ 4}t_{4,\ 5}|D(3)|$$

$$= 32p^5 - 16p^4 - 32p^3 + 12p^2 + 6p - 1$$

$$= Det \begin{pmatrix} 2p-1 & 1 & 0 & 0 & 0 \\ 1 & 2p & 1 & 0 & 0 \\ 0 & 1 & 2p & 1 & 0 \\ 0 & 0 & 1 & 2p & 1 \\ 0 & 0 & 0 & 1 & 2p \end{pmatrix} = V_5(p).$$

In general,

$$|X(m)| = t_{m,\ m}|D(m-1)| - t_{m,\ m-1}t_{m-1,\ m}|D(m-2)|$$

$$= Det \begin{pmatrix} 2p-1 & 1 & 0 & \dots & \dots & 0 \\ 1 & 2p & 1 & \dots & \dots & 0 \\ 0 & 1 & 2p & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 2p & 1 \\ 0 & 0 & \dots & \dots & 1 & 2p \end{pmatrix} = V_m(p).$$

2.1 *Characteristic equations of a chebyshev polynomials of the third kind*

Here we obtain the characteristic equations of the Chebyshev polynomials of the third kind up to degree three.
1. $\lambda - V_1 = 0.$
2. $\lambda^2 - (4p-1)\lambda + V_2 = 0.$
3. $\lambda^3 - (6p-1)\lambda^2 + (12p^2 - 4p - 2)\lambda - V_3 = 0.$

3. Matrix representation of chebyshev polynomials of fourth kind

Now we present determinant representations and characteristic equations for the fourth kind of Chebyshev polynomials. The Chebyshev polynomials of the fourth kind are defined by a recurrence relation,

$$W_0(p) = 1,\ W_1(p) = 2p+1,\ W_m(p) = 2pW_{m-1}(p) - W_{m-2}(p),\ \text{ for } m\ \geq 2,\ 3,\ \dots$$

Tri-diagonal matrix, $[d_{q,\ r}]$ represents a matrix sequence for the Chebyshev polynomials of the fourth kind:

$$
[d_{q,\,r}] = \begin{cases} d_{q,\,r} = ap+1, & \text{if } q = r = 1 \\ d_{q,\,r} = ap, & \text{if } q = r \geq 2 \\ d_{q,\,r} = 1, & \text{if } q = r+1,\ q = r-1 \\ d_{q,\,r} = 0, & \text{otherwise} \end{cases}
$$

In general, determinant representation:

$$
Y(m) = \begin{vmatrix} ap+1 & 1 & 0 & ... & ... & 0 \\ 1 & ap & 1 & ... & ... & 0 \\ 0 & 1 & ap & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ap & 1 \\ 0 & 0 & ... & ... & 1 & ap \end{vmatrix}
$$

When $a = 2$;

$$
Y(m) = \begin{vmatrix} 2p+1 & 1 & 0 & ... & ... & 0 \\ 1 & 2p & 1 & ... & ... & 0 \\ 0 & 1 & 2p & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & 2p & 1 \\ 0 & 0 & ... & ... & 1 & 2p \end{vmatrix}
$$

$Y(m)$ = Determinants are of type, i.e., the determinant of Chebyshev matrices $W_m(p)$,

$$
|Y(1)| = p_{1,\,1} = 2p+1 = W_1(p).
$$

$$
|Y(2)| = p_{1,\,1}p_{2,\,2} - p_{2,\,1}p_{1,\,2} = Det \begin{pmatrix} 2p+1 & 1 \\ 1 & 2p \end{pmatrix}
$$

$$
= 4p^2 + 2p - 1 = W_2(p).
$$

$$|Y(3)| = p_{3,\,3}|D(2)| - p_{3,\,2}p_{2,\,3}|D(1)|$$

$$= 8p^3 + 4p^2 - 4p - 1$$

$$= Det\begin{pmatrix} 2p+1 & 1 & 0 \\ 1 & 2p & 1 \\ 0 & 1 & 2p \end{pmatrix} = W_3(p).$$

$$|Y(4)| = p_{4,\,4}|D(3)| - p_{4,\,3}t_{3,\,4}|D(2)|$$

$$= 16p^4 + 8p^3 - 12p^2 - 4p + 1$$

$$= Det\begin{pmatrix} 2p+1 & 1 & 0 & 0 \\ 0 & 1 & 2p & 1 \\ 0 & 0 & 1 & 2p \end{pmatrix} = W_4(p).$$

$$|Y(5)| = p_{5,\,5}|D(4)| - p_{5,\,4}p_{4,\,5}|D(3)|$$

$$= 32p^5 + 16p^4 - 32p^3 - 12p^2 + 6p + 1$$

$$= Det\begin{pmatrix} 2p+1 & 1 & 0 & 0 & 0 \\ 1 & 2p & 1 & 0 & 0 \\ 0 & 1 & 2p & 1 & 0 \\ 0 & 0 & 1 & 2p & 1 \\ 0 & 0 & 0 & 1 & 2p \end{pmatrix} = W_5(p).$$

In general,

$$|Y(m)| = p_{m,\,m}|D(m-1)| - p_{m,\,m-1}p_{m-1,\,m}|D(m-2)|$$

$$= Det\begin{pmatrix} 2p+1 & 1 & 0 & ... & ... & 0 \\ 1 & 2p & 1 & ... & ... & 0 \\ 0 & 1 & 2p & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & 2p & 1 \\ 0 & 0 & ... & ... & 1 & 2p \end{pmatrix} = W_m(p).$$

3.1 *Characteristic equations of a chebyshev polynomials of the fourth kind*

Now we determine the characteristic equations up to degree three for the Chebyshev polynomials of the fourth kind.
1. $\lambda - W_1 = 0$.
2. $\lambda^2 - (4p+1)\lambda + W_2 = 0$.
3. $\lambda^3 - (6p+1)\lambda^2 + (12p^2+4p-2)\lambda - W_3 = 0$.

4. Identities related to matrix power and chebyshev polynomials

Here, we prove theorems involving matrix power with Chebyshev polynomials of the second, third, and fourth kind. Let X be a non-singular 2×2 matrix and let,

$$a_1 = \text{Trace } X, \;\; a_2 = \text{ Determinant } X \neq 0,$$

$$u = \left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) = \sqrt{\frac{1+p}{2}}.$$

And the characteristic equation is,

$$\lambda^2 - a_1\lambda + a_2 = 0.$$

Here λ denotes the Eigenvalues of X.

4.1 *Relation between matrix power and chebyshev polynomials of the third kind*

Here we consider the result that connects the trace of matrix powers with the first kind, and the second kind proved in [7]. We also used the identity that gave a relationship between the first kind, and third kind of Chebyshev polynomials.

Theorem 1 For any integer, $m \geq 0$, it follows:

$$\text{Trace } X^{2m+1} = 2ua_2^{\frac{2m+1}{2}} V_m(p).$$

$V_m(p)$ denotes the m^{th} degree for the Chebyshev polynomials of the third kind,

$$V_0(p) = 1,$$

$$V_1(p) = 2p - 1,$$

$$V_2(p) = 4p^2 - 2p - 1,$$

$$V_3(p) = 8p^3 - 4p^2 - 4p + 1, \ldots$$

Proof. Since trace $X = a_1$, trace $X^2 = a_1^2 - 2a_2$.
Using the formulas given by Newton-Girard, we can find the following:

$$\text{Trace } X^m = a_1(\text{trace } X^{m-1}) - a_2(\text{trace } A^{m-2}).$$

Now we used the result obtained in [18]

$$\text{Trace } X^m = 2a_2^{\frac{m}{2}} T_m\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right).$$

Put $m = 2m+1$ in the above equation,

$$\text{Trace } X^{2m+1} = 2a_2^{\frac{2m+1}{2}} T_{2m+1}\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right).$$

Using the identity in the above equation

$$V_m(p) = u^{-1} T_{2m+1}(u)$$

where $u = \left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) = \sqrt{\frac{1+p}{2}}$

$$\text{Trace } X^{2m+1} = 2a_2^{\frac{2m+1}{2}} V_m(p).$$

Put $m = 0$

$$\text{trace } X^1 = 2a_2^{\frac{1}{2}} V_0(p).$$

Put $m = 1$

$$\text{Trace } X^3 = 2a_2^{\frac{3}{2}} V_1(p).$$

Put $m = 2$

$$\text{Trace } X^5 = 2a_2^{\frac{5}{2}} V_2(p).$$

Example 2 To verify the result stated in theorem 1:

$$\text{Trace } X^{2m+1} = 2ua_2^{\frac{2m+1}{2}} V_m(p).$$

Proof. The theorem presents the connection between trace of matrix and Chebyshev polynomials of the third kind. $V_m(p)$, represents the m^{th} degree Chebyshev polynomials of the third kind. We have proved our theorem with a random non-singular 2×2 matrix for $m = 0, 1, 2, \ldots$

Let $X = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$ be a 2 × 2 non-singular matrix.

$$\text{Trace} = a_1 = 9$$

$$Det = a_2 = 2$$

$$u = \left(\frac{a_1}{2a_2^{\frac{1}{2}}} \right) = \sqrt{\frac{1+p}{2}},$$

Hence,

$$p = \frac{77}{4}.$$

Put $m = 0$,

$$\text{Trace } X^1 = 2a_2^{\frac{1}{2}} V_0(p).$$

$$\text{Left Hand Side (L.H.S.)} = \text{Trace } X^1 = a_1 = 9.$$

$$\text{Right Hand Side (R.H.S.)} = 2a_2^{\frac{1}{2}} V_0(p) = 2 \times \frac{9}{2\sqrt{2}} \times 2^{\frac{1}{2}} \times 1 = 9.$$

Hence result is true for $m = 0$

Put $m = 1$

$$\text{Trace } X^3 = 2a_2^{\frac{3}{2}} V_1(p).$$

$$X^3 = \begin{pmatrix} 61 & 158 \\ 237 & 614 \end{pmatrix}$$

$$\text{L.H.S.} = \text{Trace } X^3 = 675.$$

$$\text{R.H.S.} = 2a_2^{\frac{3}{2}} V_1(p) = 2 \times \frac{9}{2\sqrt{2}} \times 2^{\frac{3}{2}} \times \frac{75}{2} = 675.$$

Hence result is true for $m = 1$
Put $m = 2$,

$$\text{Trace } X^5 = 2a_2^{\frac{5}{2}} V_2(p).$$

$$X^5 = \begin{pmatrix} 4,693 & 12,158 \\ 18,237 & 47,246 \end{pmatrix}.$$

$$\text{L.H.S.} = \text{Trace } X^5 = 51,939.$$

$$\text{R.H.S.} = 2a_2^{\frac{5}{2}} V_2(p) = 2 \times \frac{9}{2\sqrt{2}} \times 2^{\frac{5}{2}} \times \frac{5,771}{4} = 51,939.$$

Hence result is true for $m = 2$.
This example verified the above result for $m = 0,\ 1,\ 2,\ \ldots$
Similarly we can prove our result for $m = 3,\ 4,\ 5\ \ldots$
Hence, this result is hold for any non-singular matrix X i.e.,

$$\text{Trace } X^{2m+1} = 2ua_2^{\frac{2m+1}{2}} V_m(p).$$

4.2 *Relation between second, fourth kind of chebyshev polynomials and matrix power*

Here we start with the result that connects the trace of matrix powers with the second kind of Chebyshev polynomials, proved in [18] and we also used the identity that gave a relationship between the second, and fourth kind of Chebyshev polynomials i.e.,

$$X^m = a_2^{\frac{(m-1)}{2}} U_{m-1} \left(\frac{a_1}{2a_2^{\frac{1}{2}}} \right) X - a_2^{\frac{m}{2}} U_{m-2} \left(\frac{a_1}{2a_2^{\frac{1}{2}}} \right) I.$$

Now use the identity that connects the Chebyshev polynomials of the second and fourth kind with each other i.e.,

$$U_{2m}(u) = W_m(p),$$

where $u = \sqrt{\dfrac{1+p}{2}}$.

Theorem 3 Let $m \geq 2$, the integral power of X is given by:

$$X^{2m+1} = a_2^m W_m(p)X - a_2^{\frac{2m+1}{2}} U_{2m-1}\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) I$$

where I denote the identity matrix, $U_m(p)$, and $W_m(p)$ are the m^{th} degree Chebyshev polynomials of the second and fourth kind respectively.

$$U_0(p) = 1,\ U_1(p) = 2p,\ U_2(p) = 4p^2 - 1,\ U_3(p) = 8p^2 - 4p,\ ...$$

$$W_0(p) = 1,\ W_1(p) = 2p+1,\ W_2(p) = 4p^2 + 2p - 1,$$

$$W_3(p) = 8p^3 + 4p^2 - 4p - 1,\ ...$$

Proof. By using the above-mentioned results we obtain,

$$X^{2m+1} = a_2^m U_{2m}\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) X - a_2^{\frac{2m+1}{2}} U_{2m-1}\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) I,$$

$$X^{2m+1} = a_2^m W_m(p)X - a_2^{\frac{2m+1}{2}} U_{2m-1}\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) I$$

For $m = 1$

$$X^3 = a_2 W_1(p)X - a_2^{\frac{3}{2}} U_1\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) I.$$

For $m = 2$

$$X^5 = a_2 W_2(p)X - a_2^{\frac{5}{2}} U_3\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) I.$$

For $m = 3$

$$X^7 = a_2 W_3(p)X - a_2^{\frac{7}{2}} U_5\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) I.$$

Example 4 To verify the result stated in Theorem 3:

$$X^{2m+1} = a_2^m W_m(p)X - a_2^{\frac{2m+1}{2}} U_{2m-1}\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) I$$

Proof. The theorem presents the connection between matrix power and Chebyshev polynomials of the second kind and fourth kind. $U_m(p),\ W_m(p)$, represents the m^{th} degree Chebyshev polynomials of the second kind and fourth kind respectively. We have proved our theorem with a random non-singular 2×2 matrix for $m = 0,\ 1,\ 2,\ ...$

Let $X = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$ be a 2 × 2 non-singular matrix.

$$\text{Trace} = a_1 = 9$$

$$Det = a_2 = 2$$

$$u = \left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) = \sqrt{\frac{1+p}{2}},$$

Hence,

$$p = \frac{77}{4}$$

Put $m = 1$, we get

$$X^3 = a_2^1 W_1(p)X - a_2^{\frac{3}{2}} U_1\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) I.$$

$$\text{L.H.S.} = X^3 = \begin{pmatrix} 61 & 158 \\ 237 & 614 \end{pmatrix}$$

$$\text{R.H.S.} = a_2^1 W_1(p)X - a_2^{\frac{3}{2}} U_1\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) I.$$

$$= 2 \times \frac{79}{2} \times \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} - 2^{\frac{3}{2}} \times U_1\left(\frac{9}{2\sqrt{2}}\right) \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 79 & 158 \\ 237 & 632 \end{pmatrix} - 18 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 61 & 158 \\ 237 & 614 \end{pmatrix}$$

Hence result is true for $m = 1$.
Put $m = 2$, we get

$$X^5 = a_2^2 W_2(p)X - a_2^{\frac{5}{2}} U_3\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) I.$$

$$\text{L.H.S. } X^5 = \begin{pmatrix} 4,693 & 12,158 \\ 18,237 & 47,246 \end{pmatrix}.$$

$$\text{R.H.S.} = a_2^2 W_2(p)X - a_2^{\frac{5}{2}} U_3\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) I$$

$$= 4 \times \frac{6,079}{4} \times \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} - 2^{\frac{5}{2}} \times U_3\left(\frac{9}{2\sqrt{2}}\right) \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6,079 & 12,158 \\ 18,237 & 48,632 \end{pmatrix} - 1,386 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4,693 & 12,158 \\ 18,237 & 47,246 \end{pmatrix}$$

Hence result is true for $m = 2$.
This example verified the above result for $m = 1,\ 2$.
Similarly we can prove our result for $m = 3,\ 4,\ 5...$
Hence, this result is hold for any non-singular matrix X i.e.,

$$X^{2m+1} = a_2^m W_m(p)X - a_2^{\frac{2m+1}{2}} U_{2m-1}\left(\frac{a_1}{2a_2^{\frac{1}{2}}}\right) I.$$

This example verified the above result.

5. Applications

The Chebyshev polynomials are widely used to enhance the advanced technique for counting and to study the integer function. These polynomials play a vital role to solve other polynomials that are used to obtain new trigonometric identities. Chebyshev polynomials are in high demand in computer graphics to generate shapes, surfaces, and curves (Figure 1). The approximate solution of the second-order differential equations can be obtained with the help of these polynomials and the large data can be interpolated in the numerical and approximation theory. The approximate numerical solution can be obtained for differential and integral equations. Chebyshev polynomials play a crucial role in computer science to obtain signal processing, mainly in the design of filters known as Chebyshev filters. Chebyshev polynomials are high demand in computer graphics to generate shapes, surfaces and curves. We can approximate every polynomials in terms of Chebyshev polynomials. Here we approximate a polynomials in terms of the Chebyshev polynomials of the third kind that enlightens the practical application of the Chebyshev polynomials in approximation theory. The primary benefit of the given methods is the high accuracy of the approximation solution.

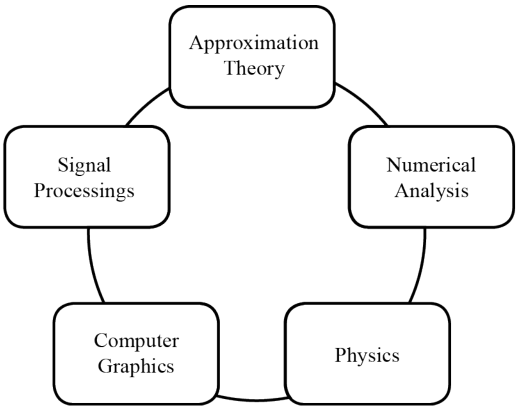

Figure 1. Application flow chart of Chebyshev polynomials

5.1 *Practical application*

Express $p^4 + 2p^3 - 3p - 2$ in terms of Chebyshev polynomials of third kind.

Sol: The first four Chebyshev polynomials of third kind are:

$$V_0(p) = 1,$$

$$V_1(p) = 2p - 1,$$

$$V_2(p) = 4p^2 - 2p - 1,$$

$$V_3(p) = 8p^3 - 4p^2 - 4p + 1,$$

$$V_4(p) = 16p^4 - 8p^3 - 12p^2 + 4p + 1,$$

From above equations, we get

$$p = \frac{1}{2}[V_0(p) + V_1(p)]$$

$$p^2 = \frac{1}{4}[2V_0(p) + V_1(p) + V_2(p)]$$

$$p^3 = \frac{1}{8}[3V_0(p) + 3V_1(p) + V_2(p) + V_3(p)]$$

$$p^4 = \frac{1}{16}[6V_0(p) - 4V_1(p) + 4V_2(p) + V_3(p) + V_4(p)]$$

Put above values in

$$p^4 + 2p^3 - 3p - 2$$

We get,

$$\frac{1}{16}[6V_0(p) - 4V_1(p) + 4V_2(p) + V_3(p) + V_4(p)]$$

$$+\frac{2}{8}[3V_0(p) + 3V_1(p) + V_2(p) + V_3(p)] - \frac{3}{2}[V_0(p) + V_1(p] - 2V_0(p)$$

Hence,

$$\frac{1}{16}V_4(p) + \frac{5}{16}V_3(p) + \frac{1}{2}V_2(p) - V_1(p) - \frac{19}{8}V_0(p).$$

6. Significance of the work

Here we recapitulate the significance of the present article in the following points:

• Introducing matrix representation of Chebyshev polynomials of the third kind and fourth kind.

• Obtaining characteristic equations of a Chebyshev polynomials of the third and fourth kinds.

• Identities related to matrix power and Chebyshev polynomials.

• Obtaining connections between the Chebyshev polynomials of the second and fourth kind with matrix power.

• Approximate a polynomial by using the Chebyshev polynomials of the third kind.

It has to be noted here that the above-obtained results and discussions are helpful. A few of their presumed uses are given below:

• The matrix representation helps solve linear and nonlinear differential equations.

• The matrix representation is beneficial to acquiring numerical solutions of linear and nonlinear differential equations.

• The characteristic equations of Chebyshev polynomials are helpful to obtaining Eigen values and Eigen vectors.

• The connections between the Chebyshev polynomials of the second, third, and fourth kinds with matrix power are very fruitful to obtaining the identities related to them.

• The Chebyshev polynomials are fruitful in approximation theory.

7. Conclusion

It is observed that characteristic equations for the third and fourth kinds of Chebyshev polynomials can be extended up to the m^{th} degree. Further, by using matrix power, we can describe more identities that connect Chebyshev polynomials. To obtain our result, we used the non-singular 2×2 matrix and utilized the trace and determinant of the matrix. The numerical examples authenticates the theoretical results.

Conflict of interest

The authors declare no conflicts of interest.

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