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Research Article

Central Limit Theorem and Law of Large Numbers Analogues for the Total Progeny in the Q-Processes

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Abstract: In this paper we are dealing with population size growth systems called Q-processes. The class of trajectories of these systems is a subset of the family of all possible trajectories of the ordinary Galton-Watson branching system, provided that they do not decay in the remote future. We observe the total progeny of a single founder-individual, generated by the reproduction law of the Q-process up to time n. By analogy with branching systems models, this variable is of great interest in studying the deep properties of the Q-process. Our main results are analogues of Central Limit Theorem and Law of Large Numbers for S_n, denoting the total progeny of a single founder-individual, generated by the reproduction law of the Q-process up to time n. We find that the total progeny as a random variable approximates the standard normal distribution function under a second moment assumption for the initial Galton-Watson system offspring law. We estimate the speed rate of this approximation. We also prove an analogue of the law of large numbers with an estimate of the approximation rate to the degenerate distribution.

Keywords: branching system, Q-process, Markov chain, transition probabilities, invariant distribution, total progeny, central limit theorem, law of large numbers

MSC: 60J80, 60J85

1. Introduction and main results

Models of stochastic branching systems are an important mathematical simulation for describing an evolution of various population processes. Since the origin of the fundamental theory of branching systems, numerous branching schemes have been developed depending on the context of the problems being studied. Nowadays, there is great interest in these models. The origin of the theory of branching models is due to the prospect of estimating the survival probability of the population of monotypic individuals using branching schemes. A discrete-time branching model was introduced by the English statisticians Henry Watson and Francis Galton in the late 19th century. Studying eproductive rates of English lords, they developed a mathematical model for the population family growth is now called the Galton-Watson Branching (GWB) system; see [1–7]. Models of branching systems play an extremely important role both in theory and in applications of random processes. In the family of random trajectories of branching models, there is a class of positive ones such that they continue indefinitely long time. In the case of the GWB model, the class of such trajectories forms another

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stochastic model called the Q-process; see [2] and [8]. In the case of continuous-time Markov branching systems, a similar model, called *Markov Q-process*, was first introduced in [9]. Among the random trajectories of branching systems, there are those that continue a long time. Some properties of Q-process in discrete and continuous type are studied in [8–10] and [11].

GWB system is also useful to model species extinction, infectious diseases propagation, and many other phenomena. We can use this system to model the fiscal multiplier and the total consuming impact on the economy. The above model can be used to describe the behaviour of neutrons in nuclear fission reactions, such as that in an atomic bomb. In this case, the particles being considered are the neutrons. Their offspring are the neutrons released from the splitting of larger nuclei into smaller nuclei by collision with a neutron. This problem was initially discussed by Feller [12].

Consider the ordinary GWB system with branching rates $\{p_k, k \in \mathbb{N}_0\}$, where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{N} = \{1, 2, \ldots\}$. Let $Z(n)$ denotes a population size at the moment n in the system. The evolution of the system occurs according to the following branching mechanism. Each individual lives a unit length life time and then produces $k \in \mathbb{N}_0$ descendants with probability p_k. The system $\{Z(n), n \in \mathbb{N}_0\}$ is a reducible, homogeneous-discrete-time Markov chain with a state space consisting of two classes: $\mathcal{S}_0 = \{0\} \cup \mathcal{S}$, where $\{0\}$ is absorbing state, and $\mathcal{S} \subset \mathbb{N}$ is the class of possible essential communicating states.

Example 1 Figure 1 illustrates GWB system.

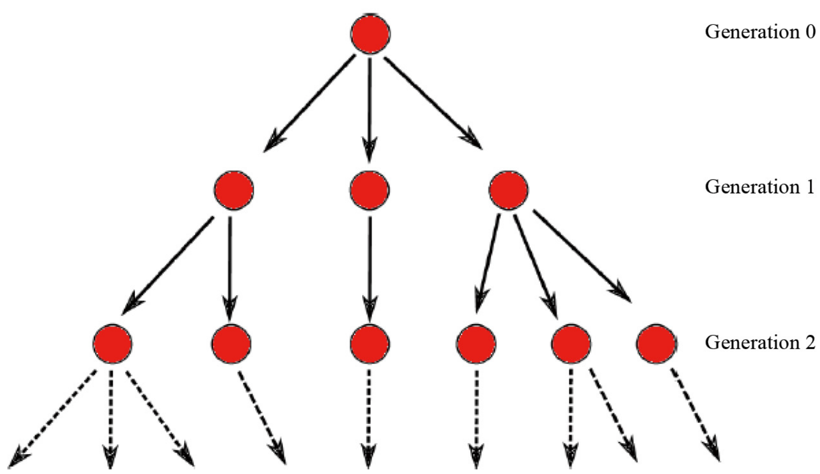

Figure 1. Branching scheme example

Let $Z(0) = 1$. If we set $Z(n)$ as the number of particles in n-th generation, then we have

$$Z(n+1) = \begin{cases} \sum_{i=1}^{Z(n)} \xi_{n+1}^{(i)}, & \text{if} \quad Z(n) > 0, \\ 0, & \text{if} \quad Z(n) = 0, \end{cases}$$

where $\xi_{n+1}^{(i)}$ is the number of generations produced by the i-th particle in the n-th generation. For example, in Figure 1, we have

$$Z(1) = \xi_1^{(1)} = 3 \quad \text{and} \quad Z(2) = \xi_2^{(1)} + \xi_2^{(2)} + \xi_2^{(3)} = 2 + 1 + 3 = 6.$$

In this paper we consider the system $\{Z(n)\}$ in the Schröder case, i.e., $p_0 > 0$ and $p_0 + p_1 > 0$. We assume also that

$$p_0 + p_1 < 1 \quad \text{and} \quad m := \sum_{k\in\mathcal{S}} k p_k < \infty.$$

In the Schröder case, the particle can either die or leave behind 1 offspring. In Böttcher case ($p_0 = p_1 = 0$), the particle does not die and at the end of its life leaves behind at least 2 generations.

Letting

$$P_{ij}(n) := \mathsf{P}\left\{Z(n+k) = j \,\middle|\, Z(k) = i\right\} \quad \text{for any} \quad k \in \mathbb{N}_0$$

be the n-step transition probabilities and using the Kolmogorov-Chapman equation, we observe that the probability generating function (GF)

$$\sum_{j\in\mathcal{S}_0} P_{ij}(n) s^j = \left[f_n(s)\right]^i, \tag{1}$$

where

$$f_n(s) := \sum_{k\in\mathcal{S}_0} \mathsf{p}_k(n) s^k,$$

herein $\mathsf{p}_k(n) := P_{1k}(n)$, and the GF $f_n(s)$ is n-fold iteration of the offspring GF

$$f(s) := \sum_{k\in\mathcal{S}_0} p_k s^k.$$

Evidently that $f_n(0) = \mathsf{p}_0(n)$ is a vanishing probability of the system initiated by single founder. Note that this probability approaches monotonously to a finite limit q as $n \to \infty$, which called an extinction probability of the system, i.e., $\lim_{n\to\infty} \mathsf{p}_0(n) = q$. The extinction probability

- $q = 1$ if $m \le 1$;
- $q < 1$ if $m > 1$.

Based on this, according to the values of the parameter m, the system is called

- sub-critical if $m < 1$;
- critical if $m = 1$;
- super-critical if $m > 1$; see [2] [Ch.I].

In what follows, the symbols $\mathsf{P}_i\{*\}$ and $\mathsf{E}_i[*]$ will denote the probability distribution and mean provided that the system initiated by $\{Z(0) = i\}$ founder and, we will write $\mathsf{P}\{*\}$ and $\mathsf{E}[*]$ instead of $\mathsf{P}_1\{*\}$ and $\mathsf{E}_1[*]$ respectively. We are dealing with the GWB system conditioned on the event $\{n < \mathcal{H} < \infty\}$, where $\mathcal{H} := \min\{n \in \mathbb{N} \colon Z(n) = 0\}$ is the system extinction time. Define the following conditioned probability measure:

$$\mathsf{P}_i^{\mathcal{H}(n+k)}\{*\} := \mathsf{P}_i\left\{* \,\middle|\, n + k < \mathcal{H} < \infty\right\} \quad \text{for any} \quad k \in \mathbb{N}.$$

In [2] [p. 58] proved, that

$$\mathcal{Q}_{ij}(n) := \lim_{k\to\infty} \mathsf{P}_i^{\mathcal{H}(n+k)}\left\{Z(n)=j\right\} = \frac{jq^{j-i}}{i\beta^n} P_{ij}(n), \tag{2}$$

where $\beta := f'(q)$ and $\sum_{j\in\mathbb{N}} \mathcal{Q}_{ij}(n) = 1$ for each $i\in\mathbb{N}$. Thus, the probability measure $\mathcal{Q}_{ij}(n)$ can determine a new population growth system with the state space $\mathcal{E}\subset\mathbb{N}$ which we denote by $\{W(n), n\in\mathbb{N}_0\}$. This is a discrete-homogeneous-time irreducible Markov chain defined in [2] [p. 58] and called *the Q-process*. Undoubtedly $W(0)\overset{d}{=}Z(0)$ and transition probabilities

$$\mathcal{Q}_{ij}(n) := \mathsf{P}_i\left\{W(n)=j\right\} = \mathsf{P}_i\left\{Z(n)=j \,\middle|\, \mathcal{H}=\infty\right\},$$

so that the Q-process can be interpreted as a "long-living" GWB system.

Recent results on limiting structures of Q-process can be found in [13]. Some limit results on continuous-state Q-processes are available in [10].

Let's consider a GF

$$w_n^{(i)}(s) := \sum_{j\in\mathcal{E}} \mathcal{Q}_{ij}(n)s^j.$$

Combining relations (1) and (2) we get

$$w_n^{(i)}(s) = w_n(s)\left[\frac{f_n(qs)}{q}\right]^{i-1}, \tag{3}$$

where $w_n(s) := w_n^{(1)}(s) = \mathsf{E}s^{W(n)}$ has a form of

$$w_n(s) = s\frac{f_n'(qs)}{\beta^n} \quad \text{for any} \quad n\in\mathbb{N}. \tag{4}$$

Using iterations over $f(s)$ in (3), we obtain the following functional equation:

$$w_{n+1}^{(i)}(s) = \frac{w(s)}{f_q(s)} w_n^{(i)}\left(f_q(s)\right), \tag{5}$$

where $w(s) := w_1(s)$ and $f_q(s) = f(qs)/q$. Thus, Q-process is completely defined by setting the GF

$$w(s) = s\frac{f'(qs)}{\beta}. \tag{6}$$

The asymptotic behavior of trajectories of the Q-process is completely regulated by its structural parameter $\beta > 0$. In fact, by [2] [p. 59, Theorem 2], that

- $\mathcal{E}$ is positive recurrent if $\beta < 1$;
- $\mathcal{E}$ is transient if $\beta = 1$.

By default, the positive recurrent case $\beta < 1$ of Q-process is in a definition character of the non-critical case $m \neq 1$ of the initial GWB system. Note that $\beta \leq 1$ and nothing but.

In this paper we deal with the positive recurrent case assuming that the first moment $\alpha\colon = w'(1-)$ be finite. Then differentiating with respect to s at the point $s = 1$ we obtain $\alpha = 1 + \gamma_q \cdot (1 - \beta)$, where

$$\gamma_q\colon = \frac{qf''(q)}{\beta(1-\beta)}.$$

At the same time it follows from (3) and (4) that $\mathsf{E}_i W(n) = (i-1)\beta^n + \mathsf{E}W(n)$, where

$$\mathsf{E}W(n) = 1 + \gamma_q \cdot \left(1 - \beta^n\right).$$

It is obvious, that when initial GWB system is sub-critical, the condition $\alpha < \infty$ is equivalent to that $f''(1-) < \infty$. Further we everywhere will be accompanied by this condition by default.

Our purpose is to investigate asymptotic properties of a random variable

$$S_n = W(0) + W(1) + \cdots + W(n-1),$$

denoting the total progeny of a single founder-individual, generated by the reproduction law of the Q-process up to time n, see e.g., [6, 7, 11, 14].

Throughout the paper we will use famous Landau symbols o, $\mathcal{O}$ and $\mathcal{O}^*$ to describe kinds of bounds on asymptotic varying rates of positive functions $f(x)$ and $g(x)$. So, $f = o(g)$ means that $\lim_x f(x)/g(x) = 0$, and we write $f = \mathcal{O}(g)$ if $\limsup_x f(x)/g(x) < \infty$ and also we write $f = \mathcal{O}^*(g)$ if the ratio $f(x)/g(x)$ has a positive explicit limit. i.e., $\lim_x f(x)/g(x) = C < \infty$. Moreover, $f(x) \sim g(x)$ means that $\lim_x f(x)/g(x) = 1$.

Our main results are analogues of Central Limit Theorem and Law of Large Numbers for S_n. Let $\mathcal{N}\left(0, \sigma^2\right)$ be a normal distributed random variable with the zero mean and the finite variance σ^2 and $\Phi_{0,\,\sigma^2}(x)$ is its distribution function.

Theorem 1 Let $\beta < 1$ and $\alpha < \infty$. Then there exists a positive real-valued sequence $\mathcal{K}_n$ such that $\mathcal{K}_n = \mathcal{O}^*\left(\sqrt{n}\right)$ and

$$\frac{S_n - \mathsf{E}S_n}{\mathcal{K}_n} \xrightarrow{\mathsf{D}} \mathcal{N}\left(0, \sigma^2\right) \quad \text{as} \quad n \to \infty,$$

where the symbol "$\xrightarrow{\mathsf{D}}$" means the convergence in distribution and σ^2 is a (finite) positive number that depends on beta and alpha moments.

Remark 1 (Explanation of Theorem 1) The asymptotic behaviour of S_n has been widely studied in probability theory, in which the Law of Large Numbers and Central Limit Theorems play an important role. The Central Limit allows us to describe the class consisting of the limits of the distribution function, where the sum of the sequence of independent random variables is infinitely small compared to the sum of the contributions of each participant. For example, consider a process where the generating function is fractional-linear:

$$f(s) = \frac{a+bs}{1-\delta s}, \quad 0 < \delta < 1, \quad a+b+\delta = 1, \quad k \in \mathbb{N}_0.$$

In that case, the sequence of random variables $(S_n - \mathsf{E}S_n)/\mathcal{K}_n$ approaches a normal (or Gaussian) random variable with parameter $(0, \sigma^2)$ in distribution.

In the theory of probability, the problem of finding the convergence rate is considered very relevant and complex, and this rate allows us to conclude about the solution of the problem. Below is such a convergence rate.

Theorem 2 Let $\beta < 1$ and $\alpha < \infty$. Then there exists slowly varying function at infinity $\mathcal{L}(*)$ such that

$$\left| \mathsf{P}\left\{ \frac{S_n - \mathsf{E}S_n}{\mathcal{K}_n} < x \right\} - \Phi_{0,\, \sigma^2}(x) \right| \leq \frac{\mathcal{L}(n)}{n^{1/4}}$$

uniformly in x.

Let I_a be a degenerate distribution concentrated at the point a, i.e.

$$I_a(B) = \begin{cases} 1 & \text{if} \quad x \in B, \\ 0 & \text{if} \quad x \notin B. \end{cases}$$

Theorem 3 Let $\beta < 1$ and $\alpha < \infty$. Then

$$\frac{S_n}{n} \xrightarrow{\mathsf{D}} 1 + \gamma_q \quad \text{as} \quad n \to \infty.$$

Moreover there exists slowly varying function at infinity $\mathcal{L}_\gamma(*)$ such that

$$\left| \mathsf{P}\left\{ \frac{S_n}{n} < x \right\} - I_{1+\gamma_q}(x) \right| \leq \frac{\mathcal{L}_\gamma(n)}{\sqrt{n}}$$

uniformly in x, where

$$I_{1+\gamma_q}(x) = \begin{cases} 0 & \text{if} \quad x \leq 1 + \gamma_q, \\ 1 & \text{if} \quad x > 1 + \gamma_q. \end{cases}$$

Remark 2 (Explanation of Theorem 3) The arithmetic mean of the first n terms of a sequence of independent random variables loses its randomness at sufficiently large n and approaches the arithmetic mean of the mathematical expectation. This assertion is called the law of large numbers. In particular, consider a sequence of random variables with a $1-p$ parameter geometric distribution:

$$p_k = (1-p)p^k, \quad 0 < p < 1, \quad k \in \mathbb{N}_0 \quad \text{and} \quad f(s) = (1-p)/(1-ps).$$

Then, by the Theorem 3, we have

$$\frac{S_n}{n} \xrightarrow{\mathsf{D}} 1+\frac{2pq}{(1-pq)(1-\beta)} \quad \text{as} \quad n \to \infty$$

with a rate of $\mathcal{O}^*\left(\sqrt{n}\right)$.

The rest of this paper is organized as follows. Section 2 provides auxiliary statements that will be essentially used in the proof of our theorems. Section 3 is devoted to the proof of main results.

2. Preliminaries

Further we need the joint GF of the variables $W(n)$ and S_n

$$J_n(s;\, x) = \sum_{j\in\mathcal{E}} \sum_{l\in\mathbb{N}} \mathsf{P}\left\{W(n) = j,\, S_n = l\right\} s^j x^l$$

on a two-dimensional domain

$$\mathbb{K} = \left\{(s;\, x) \in \mathbb{R}^2\colon\; s\in[0,\,1],\;\; x\in[0,\,1],\;\; \sqrt{(s-1)^2+(x-1)^2} > 0\right\}.$$

Now due to the Markov nature of the Q-process we see that the two-dimensional one-step joint-transition probabilities

$$\mathsf{P}\left\{W(n+1) = j,\, S_{n+1} = l \,\middle|\, W(n) = i,\, S_n = k\right\} = \mathsf{P}_i\left\{W(1) = j,\, S_1 = l\right\} \delta_{l,\,i+k},$$

where δ_{ij} is the Kronecker's delta function:

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Therefore, we have

$$\begin{aligned} \mathsf{E}_i\left[s^{W(n+1)} x^{S_{n+1}} \,\middle|\, S_n = k\right] &= \sum_{j\in\mathcal{E}} \sum_{l\in\mathbb{N}} \mathsf{P}_i\left\{W(1) = j,\, S_1 = l\right\} \delta_{l,\,i+k} s^j x^l \\ &= \sum_{j\in\mathcal{E}} \mathsf{P}_i\left\{W(1) = j\right\} s^j x^{i+k} = w^{(i)}(s) x^{i+k}. \end{aligned}$$

Next, we obtain

$$J_{n+1}(s;\,x) = \mathsf{E}\left[\mathsf{E}\left[s^{W(n+1)}x^{S_{n+1}}\,\middle|\,W(n),\,S_n\right]\right] = \mathsf{E}\left[w^{W(n)}(s)x^{W(n)+S_n}\right]$$

$$= \mathsf{E}\left[\big(w(s)f_q(s)\big)^{W(n)-1}x^{W(n)+S_n}\right] = \frac{w(s)}{f_q(s)}\mathsf{E}\left[\big(xf_q(s)\big)^{W(n)}x^{S_n}\right].$$

In the last line we used formula (3). Thus for $(s;\,x) \in \mathbb{K}$ and any $n \in \mathbb{N}$

$$J_{n+1}(s;\,x) = \frac{w(s)}{f_q(s)}J_n\big(xf_q(s);\,x\big). \tag{7}$$

Using relation (7), we can now obtain an explicit expression for the GF $J_n(s;\,x)$. Indeed, applying it consistently, taking into account (6) and, after standard transformations, we have

$$J_n(s;\,x) = \frac{s}{\beta^n}\frac{\partial H_n(s;\,x)}{\partial s}, \tag{8}$$

where the function $H_n(s;\,x)$ is defined for any $(s;\,x) \in \mathbb{K}$ by the following recursive relations:

$$\begin{cases} H_0(s;\,x) = s; \\ H_{n+1}(s;\,x) = xf_q\big(H_n(s;\,x)\big). \end{cases} \tag{9}$$

Since $\partial J_n(s;\,x)/\partial x\Big|_{(s;\,x)=(1;\,1)} = \mathsf{E}S_n$, from (8) and (9), we find that

$$\mathsf{E}S_n = (1+\gamma_q)n - \gamma_q\frac{1-\beta^n}{1-\beta}. \tag{10}$$

Remark 3 Needles to say that the GF $f_q(s) = f(qs)/q$ generates a sub-critical GWB system. Denoting the population in this system as $Z_q(n)$, we define the sum $V_n = \sum_{k=0}^{n-1} Z_q(k)$ which is a total progeny of individuals that participated in the evolution of the system $\big\{Z_q(n),\, n \in \mathbb{N}_0\big\}$, up to the n-th generation. It is known that the GF of the joint distribution $\big(Z_q(n),\, V_n\big)$ satisfies the recursive equation (9); see [14] [p. 126]. Thus, the function $H_n(s;\,x)$ is a two-dimensional GF for all $n \in \mathbb{N}$ and $(s;\,x) \in \mathbb{K}$ and obeys to all properties of the GF $\mathsf{E}\left[s^{Z_q(n)}x^{V_n}\right]$.

By virtue of what said in Remark 3, in studying $H_k(s;x)$ we use the properties of the GF $\mathsf{E}\left[s^{Z_q(n)}x^{V_n}\right]$. Since the system $\big\{Z_q(n)\big\}$ is sub-critical, it goes extinct with probability 1. Therefore, there exists a proper random variable $V = \lim_{n\to\infty} V_n$, which means the total number of individuals participated in the whole evolution of the system. So

$$h(x) := \mathsf{E}x^V = \lim_{n\to\infty}\mathsf{E}x^{V_n} = \lim_{n\to\infty}H_n(1;\,x)$$

and, according to (9) it satisfies the functional equation

$$h(x) = x f_q\big(h(x)\big). \tag{11}$$

Further, we note that

$$\mathsf{P}\big\{Z_q(n) = 0,\, V_n = k\big\} = \mathsf{P}\big\{Z_q(n) = 0,\, V = k\big\}.$$

Then, we find

$$\mathsf{P}\big\{V = k\big\} - \sum_{i \in \mathbb{N}} \mathsf{P}\big\{Z_q(n) = i,\, V_n = k\big\} s^i \le \mathsf{P}\big\{V = k,\, Z_q(n) > 0\big\}.$$

Therefore, denoting

$$R_n(s;\, x) \colon = h(x) - H_n(s;\, x)$$

for $(s;\, x) \in \mathbb{K}$, we have

$$R_n(s;\, x) \le \sum_{k \in \mathbb{N}} \mathsf{P}\left\{V = k,\, Z_q(n) > 0\right\} x^k = R_n(0;\, x).$$

It can be seen $R_n(0;\, x) \le R_n(0;\, 1) = \mathsf{P}\left\{Z_q(n) > 0\right\}$. Then

$$\big|R_n(s;\, x)\big| \le \mathsf{P}\left\{Z_q(n) > 0\right\} \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{12}$$

On the other hand, due to the fact that $|h(x)| \le 1$ and $|H_n(s;\, x)| \le 1$ we have

$$R_n(s;\, x) = x\left[f_q\big(h(x)\big) - f_q\big(H_{n-1}(s;\, x)\big)\right]$$

$$= x\mathsf{E}\big[h(x) - H_{n-1}(s;\, x)\big]^{Z_q(n)} \le \beta R_{n-1}(s;\, x)$$

for all $(s;\, x) \in \mathbb{K}$. This implies that

$$\big|R_n(s;\, x)\big| \le \beta^{n-k}\big|R_k(s;\, x)\big| \tag{13}$$

for any $n \in \mathbb{N}$ and $k = 0,\, 1,\, \ldots,\, n$.

In what follows, where the function $R_n(s;\, x)$ will be used, we deal with the domain $\mathbb{K}$, where this function does not vanish. By virtue of (12), taking into account (9), (11), we obtain the asymptotic formula

$$R_{n+1}(s;\,x) = xf_q'\big(h(x)\big)R_n(s;\,x) - x\frac{f_q''\big(h(x)\big) + \eta_n(s;\,x)}{2}R_n^2(s;\,x), \tag{14}$$

where $|\eta_n(s;\,x)| \to 0$ as $n \to \infty$ uniformly in $(s;\,x) \in \mathbb{K}$. Since $R_n(s;\,x) \to 0$, it follows from (14) that

$$R_n(s;\,x) = \frac{R_{n+1}(s;\,x)}{xf_q'\big(h(x)\big)}(1 + o(1)) \quad \text{as} \quad n \to \infty.$$

Using last equality, we transform (14) to the form

$$R_{n+1}(s;\,x) = xf_q'\big(h(x)\big)R_n(s;\,x) - \left[\frac{f_q''\big(h(x)\big)}{2f_q'\big(h(x)\big)} + \varepsilon_n(s;\,x)\right]R_n(s;\,x)R_{n+1}(s;\,x)$$

and, therefore

$$\frac{u(x)}{R_{n+1}(s;\,x)} = \frac{1}{R_n(s;\,x)} + \upsilon(x) + \varepsilon_n(s;\,x), \tag{15}$$

where

$$u(x) = xf_q'\big(h(x)\big) \quad \text{and} \quad \upsilon(x) = x\frac{f_q''\big(h(x)\big)}{2u(x)}$$

and $\sup_{(s;\,x)\in\mathbb{K}}\big|\varepsilon_n(s;\,x)\big| \le \varepsilon_n \to 0$ as $n \to \infty$. By successively applying (15), we find the following representation for $R_n(s;\,x)$:

$$\frac{u^n(x)}{R_n(s;\,x)} = \frac{1}{R_0(s;\,x)} + \frac{\upsilon(x)\big[1 - u^n(x)\big]}{1 - u(x)} + \sum_{k=1}^{n}\varepsilon_k(s;\,x)u^k(x). \tag{16}$$

In what follows, our discussions will essentially be based on formula (16). Now, for convenience, we write

$$J_n(s;\,x) = s\prod_{k=0}^{n-1}\frac{xf_q'\big(H_k(s;\,x)\big)}{\beta}$$

which is a direct consequence of formulas (8) and (9). In our notation, it is almost obvious that $T_n(x) := \mathsf{E}x^{S_n} = J_n(1;\,x)$. Then it follows that

$$T_n(x) = \prod_{k=0}^{n-1}u_k(x), \tag{17}$$

where

$$u_n(x) = \frac{x f_q'\big(h_n(x)\big)}{\beta},$$

at that $h_n(x) = \mathsf{E}x^{V_n}$ which satisfies a recurrence equation $h_{n+1}(x) = x f_q\big(h_n(x)\big)$. Accordingly, the function $\Delta_n(x){:} = h(x) - h_n(x)$ satisfies the inequality

$$\big|\Delta_n(x)\big| \le \beta^{n-k}\big|\Delta_k(x)\big| \tag{18}$$

which is a consequence of (13). Successive application of the inequality (18) gives

$$\big|\Delta_n(x)\big| = \mathcal{O}\big(\beta^n\big) \to 0 \quad \text{as} \quad n \to \infty \tag{19}$$

uniformly in $x \in \mathbb{K}$. Similarly to the case $R_n(s; x)$, taking into account (19) we find the following representation:

$$\frac{u^n(x)}{\Delta_n(x)} = \frac{1}{h(x)-1} + \frac{v(x)\big[1-u^n(x)\big]}{1-u(x)} + \sum_{k=1}^{n} \varepsilon_k(x) u^k(x), \tag{20}$$

where $\sup_{x\in\mathbb{K}}\big|\varepsilon_n(x)\big| \le \varepsilon_n \to 0$ as $n \to \infty$.

In our further discussion we will also need expansions functions $h(x)$ and $u(x)$ in the left neighborhood of the point $x = 1$.

Lemma 1 Let $\beta < 1$ and $\alpha < \infty$. Then for GF $h(x) = \mathsf{E}x^V$ the following local expansion holds:

$$1 - h(x) \sim \frac{1}{1-\beta}(1-x) - \frac{2\beta(1-\beta)+b_q}{2(1-\beta)^3}(1-x)^2 \quad \text{as} \quad x \uparrow 1, \tag{21}$$

where $b_q{:} = f_q''(1-)$.

Proof. We write the Peano's form Taylor expansion for $h(x) = \mathsf{E}x^V$:

$$h(x) = 1 + h'(1-)(x-1) + \frac{h''(1-)}{2}(x-1)^2 + o(x-1)^2 \quad \text{as} \quad x \uparrow 1. \tag{22}$$

Formula (11) and standard calculations produce that

$$h'(1-) = \frac{1}{1-\beta} \quad \text{and} \quad h''(1-) = \frac{2\beta(1-\beta)+b_q}{(1-\beta)^3}.$$

Substituting these expressions in the expansion (22), entails (21).

The lemma is proved. □

Similar arguments can be used to verify the validity of the following lemma.

Lemma 2 Let $\beta < 1$ and $\alpha < \infty$. Then

$$u(x) = \beta x\left[1 - \gamma_q(1-x)\right] + \rho(x), \tag{23}$$

where

$$\frac{\rho(x)}{(1-x)^2} \to const \quad \text{as} \quad x \uparrow 1.$$

Proof. Write the Taylor expansion with Lagrange error bound for $f_q'(y)$:

$$f_q'(y) = \beta + f_q''(1)(y-1) + r(y),$$

where $r(y) \leq A \cdot (y-1)^2$ as $y \uparrow 1$ and $A = const$. Since $u(x) = x f_q'\big(h(x)\big)$, taking herein $y = h(x)$ and using (21) leads to (23).

The lemma is proved. □

The following two results directly follow from Lemma 1 and Lemma 2 respectively.

Lemma 3 Let $\beta < 1$, $\alpha < \infty$. Then

$$h\left(e^{\theta}\right) - 1 \sim \frac{\theta}{1-\beta} + \frac{2+\beta\gamma_q}{2(1-\beta)^2}\theta^2 \quad \text{as} \quad \theta \to 0. \tag{24}$$

Lemma 4 Let $\beta < 1$, $\alpha < \infty$. Then

$$\frac{u\left(e^{\theta}\right)}{\beta} - 1 = (1+\gamma_q)\theta + \rho(\theta), \tag{25}$$

where $\rho(\theta) = \mathcal{O}^*\left(\theta^2\right)$ as $\theta \to 0$.

Next Lemma follows from combination of (20), (24) and (25).

Lemma 5 Let $\beta < 1$, $\alpha < \infty$. Then

$$\frac{\Delta_n\left(e^{\theta}\right)}{u^n\left(e^{\theta}\right)} = \frac{1}{1-\beta}\theta + \mathcal{O}^*\left(\theta^2\right) \quad \text{as} \quad \theta \to 0 \tag{26}$$

for any fixed $n \in \mathbb{N}$.

Now we prove the following lemma.

Lemma 6 Let $\beta < 1$, $\alpha < \infty$. Then

$$\ln \prod_{k=0}^{n-1} u_k\left(e^{\theta}\right) \sim -\left(1 - \frac{u\left(e^{\theta}\right)}{\beta}\right) n - \gamma_q\theta \cdot \sum_{k=0}^{n-1} u^k\left(e^{\theta}\right) \quad \text{as} \quad \theta \to 0 \tag{27}$$

for any fixed $n \in \mathbb{N}$.

Proof. Using the inequality $\ln(1-y) \geq -y - y^2/(1-y)$, which is valid for $0 \leq y < 1$, we have

$$\ln \prod_{k=0}^{n-1} u_k\left(e^{\theta}\right) = \sum_{k=0}^{n-1} \ln\left\{1 - \left[1 - u_k\left(e^{\theta}\right)\right]\right\}$$

$$= \sum_{k=0}^{n-1} \left[u_k\left(e^{\theta}\right) - 1\right] + \rho_n^{(1)}(\theta) =: I_n(\theta) + \rho_n^{(1)}(\theta), \tag{28}$$

where

$$I_n(\theta) = -\sum_{k=0}^{n-1} \left[1 - u_k\left(e^{\theta}\right)\right], \tag{29}$$

and

$$-\sum_{k=0}^{n-1} \frac{\left[1 - u_k\left(e^{\theta}\right)\right]^2}{u_k\left(e^{\theta}\right)} \leq \rho_n^{(1)}(\theta) \leq 0.$$

It is easy to see that the sequence of functions $\{h_k(x)\}$ does not decrease in $k \in \mathbb{N}$. Then, by the property of the GF, and the function $u_k\left(e^{\theta}\right)$ is non-decreasing in k, for any fixed $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$. Therefore,

$$\frac{1 - u_0\left(e^{\theta}\right)}{u_0\left(e^{\theta}\right)} I_n(\theta) \leq \rho_n^{(1)}(\theta) \leq 0. \tag{30}$$

Due to the monotonously increasing property of probability GF and its derivatives, we will also verify that under the conditions of our theorem $1 - u_0\left(e^{\theta}\right) \to 0$ as $\theta \to 0$. Then, according to (30), $\rho_n^{(1)}(\theta) \to 0$ if only $I_n(\theta)$ has a finite limit as $\theta \to 0$.

Write the following Taylor expansion with Lagrange error bound:

$$f_q'(t) = f_q'(t_0) - f_q''(t_0)(t_0 - t) + (t_0 - t) g(t_0; t),$$

where $g(t_0; t) = (t_0 - t) f_q'''(\tau)/2$ and $t_0 < \tau < t$. Hence, letting $t_0 = h(x)$ and $t = h_k(x)$ we have the following relation:

$$u_k(x) = \frac{u(x)}{\beta} - \frac{x f_q''(h(x))}{\beta} \Delta_k(x) + \Delta_k(x) g_k(x),$$

where $g_k(x) = x \Delta_k(x) f_q'''(\tau)/2\beta$ and $h_k(x) < \tau < h(x)$. Therefore,

$$u_k(e^\theta) = \frac{u(e^\theta)}{\beta} - \frac{e^\theta f_q''\left(h(e^\theta)\right)}{\beta}\Delta_k(e^\theta) + \Delta_k(e^\theta) g_k(e^\theta).$$

Then (29) becomes

$$I_n(\theta) = -\left[1 - \frac{u(e^\theta)}{\beta}\right] n - \frac{e^\theta f_q''\left(h(e^\theta)\right)}{\beta} \sum_{k=0}^{n-1} \Delta_k(e^\theta) + \rho_n^{(2)}(\theta), \tag{31}$$

where

$$0 \le \rho_n^{(2)}(\theta) \le \Delta_0(e^\theta) \sum_{k=0}^{n-1} g_k(e^\theta).$$

We used the fact that $|\Delta_n(x)| \le \beta^n |\Delta_0(x)|$ in the last step, which directly follows from inequality (18). It follows from (24) that $\Delta_0(e^\theta) = \mathcal{O}(\theta)$ as $\theta \to 0$. And also estimation (19) implies that $g_k(e^\theta) = \mathcal{O}(\beta^k)$ as $k \to \infty$ and hence the functional series $\sum_{k=0}^{\infty} g_k(e^\theta)$ converges for all $\theta \in \mathbb{R}$. Therefore,

$$\Delta_0(e^\theta) \sum_{k=0}^{n-1} g_k(e^\theta) = \mathcal{O}(\theta) \to 0 \quad \text{as} \quad \theta \to 0.$$

Then the remainder term in (31)

$$\rho_n^{(2)}(\theta) \to 0 \quad \text{as} \quad \theta \to 0. \tag{32}$$

Assertion (26) implies that

$$\sum_{k=0}^{n-1} \Delta_k(e^\theta) = \frac{\theta}{1-\beta} \sum_{k=0}^{n-1} u^k(e^\theta) \left(1 + \mathcal{O}^*(\theta)\right) \quad \text{as} \quad \theta \to 0. \tag{33}$$

Since $e^\theta f_q''\left(h(e^\theta)\right) \to f_q''(1)$ as $\theta \to 0$, combining relations (28), (31)–(33) and, after standard calculations, we will come to (27).

The lemma is proved. □

3. Proof of theorems

Proof of Theorem 1 Define a sequence of variables

$$\zeta_n := \frac{S_n - \mathsf{E}S_n}{\mathcal{K}_n}$$

for some positive real-valued sequence $\mathcal{K}_n$ such that $\mathcal{K}_n \to \infty$ as $n \to \infty$ and then an appropriate characteristic function

$$\varphi_{\zeta_n}(\theta) := \mathsf{E}\left[\exp\{i\theta\zeta_n\}\right] = \mathsf{E}\left[\theta_n^{S_n} \cdot \exp\left\{\frac{-i\theta \mathsf{E}S_n}{\mathcal{K}_n}\right\}\right],$$

where $\theta_n := \exp\left\{i\theta/\mathcal{K}_n\right\}$ and $\theta \in \mathbb{R}$. Using (10) we write

$$\ln \varphi_{\zeta_n}(\theta) \sim -(1+\gamma_q)\frac{i\theta}{\mathcal{K}_n}n + \ln T_n(\theta_n) \quad \text{as} \quad n \to \infty, \tag{34}$$

where $T_n(x) = \mathsf{E}x^{S_n}$. Simultaneously according to (17) and Lemma 6,

$$\ln T_n(\theta_n) \sim -\left(1 - \frac{u(\theta_n)}{\beta}\right)n - \frac{i\theta\gamma_q}{\mathcal{K}_n} \cdot \sum_{k=0}^{n-1} u^k(\theta_n) \tag{35}$$

as $n \to \infty$. In turn, (25) implies

$$-\left(1 - \frac{u(\theta_n)}{\beta}\right)n = (1+\gamma_q)\frac{i\theta}{\mathcal{K}_n}n + n\rho\left(\frac{i\theta}{\mathcal{K}_n}\right), \tag{36}$$

where $0 < \lim_{\theta\to 0}\rho(\theta)/\theta^2 =: C_\rho < \infty$. Now we readily choose

$$\mathcal{K}_n = \mathcal{O}^*\left(\sqrt{n}\right) \quad \text{as} \quad n \to \infty \tag{37}$$

which is equivalent to $\mathcal{K}_n/\sqrt{n} \to C_\mathcal{K} > 0$. Hence we see that

$$n\rho\left(\frac{i\theta}{\mathcal{K}_n}\right) \to -K\theta^2 \quad \text{as} \quad n \to \infty, \tag{38}$$

where $K := C_\rho / C_\mathcal{K}^2 > 0$. At the same time, since $u(x) = xf_q'\left(h(x)\right)$, in our assumptions we observe that $u(x) \le \beta$ uniformly in $x \in [0, 1]$. Therefore, one can choose $\varepsilon > 0$ so desirably small that

$$\left|u^k(\theta_n) - \beta^k\right| \le \varepsilon$$

for large enough n. This entails that $\lim_{n\to\infty}\sum_{k=0}^{n-1} u^k(\theta_n)$ converges uniformly in $\theta \in \mathbb{R}$. Eventually, after combination of asymptotic estimations (35)–(38), and denoting $\sigma^2 := 2C_\rho$, the relation (34) becomes

$$\ln \varphi_{\zeta_n}(\theta) = -\frac{\sigma^2\theta^2}{2} + \mathcal{K}_n(\theta), \tag{39}$$

where $\mathcal{K}_n(\theta) = \mathcal{O}^*\left(i\theta/\mathcal{K}_n\right)$ as $n \to \infty$. Finally, we conclude that

$$\varphi_{\zeta_n}(\theta) \longrightarrow \exp\left\{-\frac{\sigma^2\theta^2}{2}\right\} \quad \text{as} \quad n \to \infty$$

for any fixed $\theta \in \mathbb{R}$. The assertion follows now from the continuity theorem for characteristic functions.

The proof is completed. □

Proof of Theorem 2 The relation (39) and formal use of inequalities

$$\left|e^{iy}\right| \le 1 \quad \text{and} \quad \left|e^{iy} - 1 - y\right| \le \frac{|y|^2}{2}$$

imply

$$\begin{aligned}\left|\varphi_{\zeta_n}(\theta) - e^{-\sigma^2\theta^2/2}\right| &\le \left|e^{-\sigma^2\theta^2/2}\right| \left|e^{\mathcal{K}_n(\theta)} - 1\right| \\ &\le \left|e^{\mathcal{K}_n(\theta)} - 1 - \mathcal{K}_n(\theta)\right| + \left|\mathcal{K}_n(\theta)\right| \\ &\le \frac{\left[\mathcal{K}_n(\theta)\right]^2}{2} + \left|\mathcal{K}_n(\theta)\right| \qquad (40)\end{aligned}$$

for all n. By definition we write

$$\mathcal{K}_n(\theta) = C(n)\frac{i\theta}{\mathcal{K}_n},$$

where $\lim_{n\to\infty} C(n) = C < \infty$. Then, denoting

$$F_n(x) := \mathsf{P}\left\{\zeta_n < x\right\},$$

and using the estimation (40), we obtain the Berry-Esseen approximation bound [12] [p. 538] as follows:

$$\begin{aligned}\left|F_n(x) - \Phi_{0,\,\sigma^2}(x)\right| &\le \frac{1}{\pi}\int_{-T}^{T}\left|\frac{\varphi_{\zeta_n}(\theta) - e^{-\sigma^2\theta^2/2}}{\theta}\right| d\theta + \frac{24M}{\pi T} \\ &\le \frac{2}{\pi}\frac{C(n)}{\mathcal{K}_n}T + \frac{24M}{\pi T} \qquad (41)\end{aligned}$$

for all x and $T > 0$, where M is such that $\Phi'_{0,\,\sigma^2}(x) \le M$. It can be decidedly taken that $M = 1/\sigma\sqrt{2\pi}$.

We let now $T \to \infty$ and in the same time it is necessary to be $T = o\left(\sqrt{n}\right)$ since $\mathcal{K}_n = \mathcal{O}^*\left(\sqrt{n}\right)$ as $n \to \infty$. We can choose T in general, in the form of $T = n^{\delta}\mathcal{L}_T(n)$, where $0 < \delta < 1/2$ and $\mathcal{L}_T(n)$ slowly varies at infinity in the sense of Karamata. Then we reform (41) as follows:

$$\left|F_n(x) - \Phi_{0,\,\sigma^2}(x)\right| \le \frac{\mathcal{L}_C(n)}{n^{1/2-\delta}} + \frac{\mathcal{L}_M(n)}{n^{\delta}}, \tag{42}$$

where

$$\mathcal{L}_C(n) := \frac{2C(n)}{\pi}\mathcal{L}_T(n) \quad \text{and} \quad \mathcal{L}_M(n) := \frac{24M}{\pi}\frac{1}{\mathcal{L}_T(n)}.$$

To come up to optimum degree of an estimation of approximation in (42), we would choose value of δ such that $(1/2-\delta)\delta$ has reached the maximum value for $\delta \in (0,\,1/2)$. It happens only in a unique case when $\delta = 1/2-\delta$ or $\delta = 1/4$. Thus (42) becomes

$$\left|F_n(x) - \Phi_{0,\,\sigma^2}(x)\right| \le \frac{\mathcal{L}(n)}{n^{1/4}},$$

where $\mathcal{L}(n) = \mathcal{L}_C(n) + \mathcal{L}_M(n)$ slowly varies at infinity.

The proof is completed. □

Proof of Theorem 3 First we will show that

$$\frac{S_n}{n} \xrightarrow{\mathsf{D}} 1 + \gamma_q \quad \text{as} \quad n \to \infty. \tag{43}$$

Writing

$$\eta_n := \frac{S_n}{n} = \frac{\mathsf{E}S_n}{n} + \frac{\mathcal{K}_n}{n}\zeta_n,$$

and considering (10), we have

$$\begin{aligned} \varphi_{\eta_n}(\theta) \quad &:= \quad \mathsf{E}\left[\exp\left\{i\theta\eta_n\right\}\right] \\ &= \quad e^{i\theta\left(1+\gamma_q\right)}\left[\varphi_{\zeta_n}(\theta)\right]^{\mathcal{K}_n/n}\left(1 - \frac{i\theta\gamma_q}{1-\beta}\frac{1}{n}\left(1-\beta^n\right)\right), \end{aligned} \tag{44}$$

where $\varphi_{\zeta_n}(\theta) = \mathsf{E}\left[\exp\left\{i\theta\zeta_n\right\}\right]$. Relation (39) implies

$$\varphi_{\zeta_n}(\theta) = e^{-\sigma^2\theta^2/2}\left(1 + \mathcal{O}^*\left(i\theta/\mathcal{K}_n\right)\right) \quad \text{as} \quad n \to \infty$$

and hence $\left[\varphi_{\zeta_n}(\theta)\right]^{\mathcal{K}_n/n} \to 0$ as $n \to \infty$. Thus (44) entails

$$\varphi_{\eta_n}(\theta) \to e^{i\theta\left(1+\gamma_q\right)} \quad \text{as} \quad n \to \infty.$$

According to the continuity theorem, this is sufficient for being of (43).

From (44) we obtain

$$\begin{aligned}
\left|\varphi_{\eta_n}(\theta) - e^{i\theta\left(1+\gamma_q\right)}\right| &\leq \left|\left[\varphi_{\zeta_n}(\theta)\right]^{\mathcal{K}_n/n}\left(1 - \frac{i\theta\gamma_q}{1-\beta}\frac{1}{n}\left(1-\beta^n\right)\right) - 1\right| \\
&\leq \left|\frac{i\theta\gamma_q}{1-\beta}\frac{1}{n}\left(1-\beta^n\right)\right|.
\end{aligned}$$

We accounted in the last step that $|\varphi_*(\theta)| \leq 1$ for any characteristic function. Now we can write the Berry–Esseen bound as follows:

$$\begin{aligned}
\left|\mathsf{P}\left\{\eta_n < x\right\} - I_{1+\gamma_q}(x)\right| &\leq \frac{1}{\pi}\int_{-T}^{T}\left|\frac{\varphi_{\eta_n}(\theta) - e^{i\theta\left(1+\gamma_q\right)}}{\theta}\right| d\theta + \frac{24M_\eta}{\pi T} \\
&\leq \frac{\gamma_q}{\pi}\frac{\left(1-\beta^n\right)}{1-\beta}\frac{2T}{n} + \frac{24}{\pi T} \leq \frac{1}{\pi}\frac{2\gamma_q}{1-\beta}\frac{T}{n} + \frac{24}{\pi T},
\end{aligned}$$

where we put $M_\eta = 1$ which is suitable for the degenerate distribution function.

In this case we choose $T = n^\delta \mathcal{L}_T(n)$, where $0 < \delta < 1$ and $\mathcal{L}_T(n)$ slowly varies at infinity. Therefore

$$\left|\mathsf{P}\left\{\eta_n < x\right\} - I_{1+\gamma_q}(x)\right| \leq \frac{\mathcal{L}_\beta(n)}{n^{1-\delta}} + \frac{\mathcal{L}_1(n)}{n^\delta}, \tag{45}$$

where

$$\mathcal{L}_\beta(n) := \frac{1}{\pi}\frac{2\gamma_q}{1-\beta}\mathcal{L}_T(n) \quad \text{and} \quad \mathcal{L}_1(n) := \frac{24}{\pi}\frac{1}{\mathcal{L}_T(n)}.$$

We find $\delta = 1/2$ and (45) becomes

$$\left|\mathsf{P}\left\{\eta_n < x\right\} - I_{1+\gamma_q}(x)\right| \leq \frac{\mathcal{L}_\gamma(n)}{n^{1/2}},$$

where $\mathcal{L}_\gamma(n) = \mathcal{L}_\beta(n) + \mathcal{L}_1(n)$ slowly varies at infinity.

The proof is completed. □

Conflict of interest

The author declares no competing financial interest.

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