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Vector Fixed Point Approach to Control of Kolmogorov Differential Systems

Alexandru Hofman[1], **Radu Precup**[2, 3*]

[1]Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania
[2]Faculty of Mathematics and Computer Science and Institute of Advanced Studies in Science and Technology, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania
[3]Tiberiu Popoviciu Institute of Numerical Analysis, Romanian Academy, P.O. Box 68-1, 400110 Cluj-Napoca, Romania
E-mail: r.precup@math.ubbcluj.ro

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Abstract: The paper presents a vector approach to control problems for systems of equations. The method is described in the case of Kolmogorov systems which arise frequently in the dynamics of populations. Three types of problems are discussed: problems with control of both per capita growth rates, problems with control parameters acting on the growth rates, and problems which combine the first two types. The controllability is obtained via a vector approach based on the Perov fixed point theorem and matrices which are convergent to zero. Four concrete illustrative examples are added.

Keywords: Kolmogorov system, control problem, fixed point, matrix convergent to zero, differential equations and systems, Volterra-Fredholm integral equation, Lotka-Volterra system

MSC: 34H05, 37N25, 34A12, 34K35

1. Introduction

Differential equations and systems represent a dedicated class of models of many real processes, giving mathematical expression to specific laws. As a rule, they incorporate a number of parameters, some fixed and specific to the quantities involved and others susceptible to being influenced in order to reach a certain objective, the controllability condition. This change is made mathematically using some control parameters whose expression can, in many cases, be expressed in terms of state variables. These expressions, once inserted into the equations, transform them into functional-differential equations whose study can be reduced to that of the fixed points of some nonlinear operators. In this way, we speak of the fixed point method for control problems. It was frequently used in studies related to control theory in a particular way, specific to each investigated problem (see, [1-5] for example, and the monograph [6]). A general, unifying formulation of the method was given in the work [7]. We describe it as follows.

The problem is to find (w, λ), a solution to the following system

$$\begin{cases} w = N(w, \lambda) \\ w \in W,\ \lambda \in \Lambda,\ (w, \lambda) \in D \end{cases} \tag{1}$$

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associated to the fixed point equation $w = N(w, \lambda)$. Here w is the *state variable*, λ is the *control variable*, W is the *domain of the states*, Λ is the *domain of controls* and D is the *controllability domain*, usually given by means of some condition/property imposed to w, or to both w and λ. Notice that all involved sets are not necessarily structured sets and N is any mapping from $W \times \Lambda$ to W.

One says that the equation $w = N(w, \lambda)$ is controllable in $W \times \Lambda$ with respect to D, providing that problem (1) has a solution (w, λ). If the solution is unique, we say that the equation is *uniquely controllable*.

Let Σ be the set of all possible solutions, (w, λ) of the fixed point equation, and Σ_1 be the set of all w that are first components of some solutions of the fixed point equation, that is

$$\Sigma = \{(w, \lambda) \in W \times \Lambda : w = N(w, \lambda)\},$$
$$\Sigma_1 = \{w \in W : \text{ there is } \lambda \in \Lambda \text{ with } (w, \lambda) \in \Sigma\}.$$

Then, the set of all solutions to the control problem (1) is equal to $\Sigma \cap D$.

Define the set-valued map $F : \Sigma_1 \to \Lambda$ by

$$F(w) = \{\lambda \in \Lambda : \ (w, \lambda) \in \Sigma \cap D\}.$$

Thus, F gives the 'expression' of the control variable in terms of the state variable.

It is easily seen that if for some extension $\tilde{F} : W \to \Lambda$ of F from Σ_1 to W, the fixed point inclusion

$$w \in N\left(w, \tilde{F}(w)\right),$$

has a solution $w \in W$, that is

$$w = N(w, \lambda),$$

for some $\lambda \in \tilde{F}(w)$, then the couple (w, λ) solves the control problem (1).

In many cases, F and $\tilde{F}$ are single-valued maps and the extension $\tilde{F}$ can be done using the expression of F.

In applications, this principle should be accompanied by a fixed point principle to solve the resulting fixed point problem. The fixed point theorems of Banach, Schauder, and Leray-Schauder are currently used. Some illustrative examples were given in papers [7-10].

The purpose of this paper is to draw attention to the vector technique of the fixed point theory, based on the use of the concept of contraction in the sense of Perov and on matrices instead of constants in the Lipschitz and growth conditions. As first shown in [11] (see also [12, Chapter 10]), the vector approach, compared to the scalar one, proves to be more suitable for the study of systems of equations. It is consistent with the vector structure of a system viewed as a single equation decomposed on a product space.

We use the vector fixed point approach to discuss three control problems related to Kolmogorov systems, which, for example, model the dynamics of several species that mutually influence their per capita growth rates (see, e.g., [13-19]). The problems consist of finding appropriate changes to growth rates or per capita growth rates so that at a given time, certain desired levels are reached. Such issues are extremely important in controlling epidemics and ecological balances. In the models that describe the evolution of the production of components of a certain product, the control is carried out through production policies to reach the desired level of production. Control issues are also important in medicine, where control is achieved by dosing the drug in order to achieve the desired result.

For simplicity, we shall consider two-dimensional Kolmogorov systems, but the technique used and the results obtained can be adapted to the general case of n-dimensional systems. More exactly, we are concerned with the solvability of the control problems from below. In all cases, x_0, y_0 are the initial states at time $t = 0$, and x_T, y_T are the desired levels at a given time T. Also, x, y are the state variables, and λ, μ are the control parameters. Thus, the controllability conditions are

$$x(T) = x_T, \quad y(T) = y_T.$$

Problem 1 (with control of both per capita growth rates):

$$\begin{cases} x'(t) = x(t)\big(f(x(t), y(t)) - \lambda\big) \\ y'(t) = y(t)\big(g(x(t), y(t)) - \mu\big). \end{cases} \tag{2}$$

Problem 2 (with control on both growth rates):

$$\begin{cases} x'(t) = x(t) f\big(x(t), y(t)\big) - \lambda \\ y'(t) = y(t) g\big(x(t), y(t)\big) - \mu. \end{cases} \tag{3}$$

Problem 3 (with control of the per capita growth rate of one species and control of the growth rate of the other one):

$$\begin{cases} x'(t) = x(t)\big(f(x(t), y(t)) - \lambda\big) \\ y'(t) = y(t) g\big(x(t), y(t)\big) - \mu. \end{cases} \tag{4}$$

Notations and auxiliary results. Throughout this work, by $\|\cdot\|_\infty$ we shall denote de max norm on the space $C[0, T]$, i.e., $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$.

By a matrix that converges to zero we mean a square matrix M with nonnegative entries and the property that its power M^k converges to the zero matrix as $k \to \infty$. It is well-known that this property is equivalent to the fact that the spectral radius of M is strictly less than one, and also the fact that the matrix $I - M$ (I being the unit matrix of the same size) is nonsingular and its inverse also has nonnegative entries. We mention that a square matrix of size two $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with nonnegative entries is convergent to zero if and only if

$$\operatorname{tr} M < \min\{2, 1 + \det M\}, \tag{5}$$

that is

$$a + d < 2 \quad \text{and} \quad a + d < 1 + ad - bc. \tag{6}$$

We shall use this notion in two situations: in order to obtain the existence and uniqueness of the solution of a system, by means of Perov's fixed point theorem, and to guarantee the invariance condition to a given operator, when we will be led to solving vector inequations.

For the first situation, a matrix that converges to zero plays the role of the contraction constant from Banach's fixed point theorem. More exactly, we have the vector version of the contraction principle, namely Perov's fixed point theorem that we present here in a form sufficient for us.

Theorem 1.1. (Perov). Let $(X, \|\cdot\|)$ be a Banach space, D a closed subset of $X \times X$ and $N : D \to D$, $N = (N_1, N_2)$, $N_i : D \to X$ $(i = 1, 2)$ be an operator satisfying the following vector inequality

$$\begin{bmatrix} \|N_1(x) - N_1(y)\| \\ \|N_2(x) - N_2(y)\| \end{bmatrix} \le M \begin{bmatrix} \|x_1 - y_1\| \\ \|x_2 - y_2\| \end{bmatrix}$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in D$, where M is a convergent to zero matrix of size two. Then, N has a unique fixed point in D which is the limit of the sequence $\left(N^k(x)\right)_{k\ge1}$ of successive approximations starting from any $x \in D$.

For the second situation, trying to solve in x and y a vector inequation of the form

$$M\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}a\\b\end{bmatrix}\leq\begin{bmatrix}x\\y\end{bmatrix},$$

or equivalently, the inequation

$$(I-M)\begin{bmatrix}x\\y\end{bmatrix}\geq\begin{bmatrix}a\\b\end{bmatrix},$$

we shall multiply by $(I-M)^{-1}$ to obtain the solution

$$\begin{bmatrix}x\\y\end{bmatrix}\geq(I-M)^{-1}\begin{bmatrix}a\\b\end{bmatrix}.$$

This is possible with keeping the inequality sense, if $I-M$ is nonsingular and its inverse has nonnegative entries, that is, if M is convergent to zero.

2. Main results

In this section, the general method of solving control problems for fixed-point equations that was presented in Section 1 is followed for each of the three problems: (2), (3) and (4).

In the following we use the following numbers involving the initial and final values:

$$C_1:=\left|\ln x_0\right|+\left|\ln\frac{x_0}{x_T}\right|,\quad C_2:=\left|\ln y_0\right|+\left|\ln\frac{y_0}{y_T}\right|. \tag{7}$$

2.1 *First control problem*

Consider the control problem (2). The first result guarantees the unique controllability of the system, with a given bound of the states x, y.

Theorem 2.1. Let $\rho>0$ be such that $\ln\rho>C_1$, $\ln\rho>C_2$, and let f, $g:[0,\rho]^2\to\mathbb{R}$ be bounded by a constant C. Assume that f and g satisfy the Lipschitz conditions

$$\left|f(x,y)-f(\overline{x},\overline{y})\right|\leq a_{11}\left|x-\overline{x}\right|+a_{12}\left|y-\overline{y}\right|, \tag{8}$$

$$\left|g(x,y)-g(\overline{x},\overline{y})\right|\leq a_{21}\left|x-\overline{x}\right|+a_{22}\left|y-\overline{y}\right| \tag{9}$$

for all $x,y,\overline{x},\overline{y}\in[0,\rho]$. Then, for each

$$0<T\leq\min\left\{\frac{\ln\rho-C_1}{C},\frac{\ln\rho-C_2}{C}\right\} \tag{10}$$

for which the matrix

$$M:=\rho T[a_{ij}]_{1\leq i,j\leq 2} \tag{11}$$

converges to zero, the control problem (2) has a unique solution $(x^*,y^*,\lambda^*,\mu^*)$ with x^*,y^* positive and $\left\|x^*\right\|_\infty\leq\rho$, $\left\|y^*\right\|_\infty\leq\rho$.

Proof. Looking for positive x and y, we may take them under the form $x = e^u$ and $y = e^v$. In the new variables u, v, the initial conditions are $u(0) = u_0$ where $v(0) = v_0$, where $u_0 = \ln x_0$ and $v_0 = \ln y_0$. Also, the controllability conditions become $u(T) = u_T$, $v(T) = v_T$, where $u_T = \ln x_T$ and $v_T = \ln y_T$. Substitution and integration then yield the Volterra type integral system

$$\begin{cases} u(t) = u_0 + \int_0^t f(e^{u(s)}, e^{v(s)})ds - \lambda t, \\ v(t) = v_0 + \int_0^t g(e^{u(s)}, e^{v(s)})ds - \mu t. \end{cases} \tag{12}$$

Using the controllability conditions $u(T) = u_T$ and $v(T) = v_T$ gives the expression of the control parameters in terms of the variables u and v, namely

$$\begin{aligned} \lambda &= \frac{1}{T}\left(u_0 - u_T + \int_0^T f(e^{u(s)}, e^{v(s)})ds\right), \\ \mu &= \frac{1}{T}\left(v_0 - v_T + \int_0^T g(e^{u(s)}, e^{v(s)})ds\right). \end{aligned} \tag{13}$$

Replacing in (12) we obtain a Volterra-Fredholm type integral system which can be seen as a fixed point equation for the operator $N = (A, B)$ giving by

$$\begin{aligned} A(u,v)(t) &= u_0 - \frac{t}{T}(u_0 - u_T) - \frac{t}{T}\int_0^T f(e^u, e^v)ds + \int_0^t f(e^u, e^v)ds, \\ &= u_0 - \frac{t}{T}(u_0 - u_T) + \left(1 - \frac{t}{T}\right)\int_0^t f(e^u, e^v)ds - \frac{t}{T}\int_t^T f(e^u, e^v)ds. \\ B(u,v)(t) &= v_0 - \frac{t}{T}(v_0 - v_T) - \frac{t}{T}\int_0^T g(e^u, e^v)ds + \int_0^t g(e^u, e^v)ds, \\ &= v_0 - \frac{t}{T}(v_0 - v_T) + \left(1 - \frac{t}{T}\right)\int_0^t g(e^u, e^v)ds - \frac{t}{T}\int_t^T g(e^u, e^v)ds. \end{aligned}$$

We shall apply Perov's theorem (see [12]) in the set

$$D_R := \left\{(u,v) \in C([0,T]; \mathbb{R}^2) : \|u\|_\infty \le R,\ \|v\|_\infty \le R\right\},$$

where $R = \ln \rho$. Let $(u,v), (\overline{u}, \overline{v}) \in D_R$. Using the Lipschitz condition on f, we obtain the following estimate

$$\begin{aligned} &|A(u,v)(t) - A(\overline{u},\overline{v})(t)| \\ &= \left|\left(1 - \frac{t}{T}\right)\int_0^t \left(f(e^u, e^v) - f(e^{\overline{u}}, e^{\overline{v}})\right)ds - \frac{t}{T}\int_t^T \left(f(e^u, e^v) - f(e^{\overline{u}}, e^{\overline{v}})\right)ds\right| \\ &\le \int_0^T |f(e^u, e^v) - f(e^{\overline{u}}, e^{\overline{v}})|\, ds \\ &\le \int_0^T (a_{11}\,|e^u - e^{\overline{u}}| + a_{12}\,|e^v - e^{\overline{v}}|)\, ds. \end{aligned}$$

Now using Lagrange's mean value theorem we obtain

$$\begin{aligned} &|A(u,v)(t) - A(\overline{u},\overline{v})(t)| \\ &\le \int_0^T \rho(a_{11}|u(s) - \overline{u}(s)| + a_{12}\,|v(s) - \overline{v}(s)|)\, ds \\ &\le \rho T(a_{11}\|u - \overline{u}\|_\infty + a_{12}\,\|v - \overline{v}\|_\infty). \end{aligned}$$

A similar estimate is obtained for B. Taking the maximum for $t \in [0,T]$, we have

$$\| A(u,v) - A(\overline{u},\overline{v}) \|_\infty \le \rho T(a_{11} \| u - \overline{u} \|_\infty + a_{12} \| v - \overline{v} \|_\infty),$$
$$\| B(u,v) - B(\overline{u},\overline{v}) \|_\infty \le \rho T(a_{21} \| u - \overline{u} \|_\infty + a_{22} \| v - \overline{v} \|_\infty).$$

These two inequalities can be put in the vector form

$$\begin{bmatrix} \| A(u,v) - A(\overline{u},\overline{v}) \|_\infty \\ \| B(u,v) - B(\overline{u},\overline{v}) \|_\infty \end{bmatrix} \le M \begin{bmatrix} \| u - \overline{u} \|_\infty \\ \| v - \overline{v} \|_\infty \end{bmatrix},$$

where the matrix M is assumed to converge to zero. Hence the operator $N = (A, B)$ is a Perov contraction. It remains to prove the invariance of the set D_R, that is,

$$\| u \|_\infty \le R, \| v \|_\infty \le R \text{ imply } \| A(u,v) \|_\infty \le R, \ \| B(u,v) \|_\infty \le R.$$

One has

$$|A(u,v)(t)| \le |u_0| + |u_0 - u_T| + \int_0^T |f(e^u, e^v)| \, ds \le C_1 + TC \le R,$$

since $T \le \dfrac{\ln \rho - C_1}{C}$. Similarly,

$$|B(u,v)(t)| \le |v_0| + |v_0 - v_T| + \int_0^T |g(e^u, e^v)| \, ds \le C_2 + TC \le R,$$

since $T \le \dfrac{\ln \rho - C_2}{C}$. Therefore, the operator $N = (A, B)$ invariants the set D_R and thus Perov's fixed point theorem applies and guarantees a unique fixed point $(u^*, v^*) \in D_R$. Finally, $x^* = e^{u^*}$, $y^* = e^{v^*}$ and λ^*, μ^* calculated according to (13) give the solution of the control problem (2).

For the next result instead of the Lipschitz conditions on f and g, we assume a logarithmic growth. The bounds of the states x, y is not imposed from the beginning, but they are obtained by calculation.

Theorem 2.2. Let $f, g : \mathbb{R}_+^2 \to \mathbb{R}$ be continuous and satisfy logarithmic growth conditions

$$\begin{aligned} |f(x,y)| &\le a_{11} |\ln x| + a_{12} |\ln y| + b_1, \\ |g(x,y)| &\le a_{21} |\ln x| + a_{22} |\ln y| + b_2, \end{aligned} \tag{14}$$

for all $x, y \in (0, \infty)$ and some constants $a_{ij}, b_i \in \mathbb{R}_+$ $(i, j = 1, 2)$. Then for each $T > 0$ for which the matrix

$$M = T[a_{ij}]$$

converges to zero, the control problem (2) has at least one solution $(x^*, y^*, \lambda^*, \mu^*)$ with $x^* > 0$ and $y^* > 0$.

Proof. We shall apply Schauder's fixed point theorem (see, e.g., [20]) to the operator (A, B) in a bounded set D of the form

$$D = B_{R_1} \times B_{R_2},$$

where $B_{R_i} = \left\{ w \in C([0,T]; \mathbb{R}_+) : \|w\|_\infty \le R_i \right\}$ $(i = 1, 2)$. We need to prove that one can find two positive numbers R_1 and R_2 such that the following invariance condition is satisfied:

$$\|u\|_\infty \le R_1, \|v\|_\infty \le R_2 \text{ imply } \|A(u,v)\|_\infty \le R_1, \ \|B(u,v)\|_\infty \le R_2.$$

Using (14) we have

$$\begin{aligned}\left|A(u,v)(t)\right| &\le C_1 + \int_0^T |f(e^u, e^v)|ds \\ &\le C_1 + \int_0^T (a_{11}|u| + a_{12}|v| + b_1)ds \\ &\le C_1 + T(a_{11}R_1 + a_{12}R_2 + b_1).\end{aligned}$$

A similar estimate holds for B. Hence,

$$\begin{aligned}\|A(u,v)\|_\infty &\le T(a_{11}R_1 + a_{12}R_2) + C_1 + Tb_1, \\ \|B(u,v)\|_\infty &\le T(a_{21}R_1 + a_{22}R_2) + C_2 + Tb_2,\end{aligned}$$

that is, in the vector form

$$\begin{bmatrix}\|A(u,v)\|_\infty \\ \|B(u,v)\|_\infty\end{bmatrix} \le M\begin{bmatrix}R_1 \\ R_2\end{bmatrix} + \begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix},$$

where $\alpha_1 := C_1 + Tb_1$ and $\alpha_2 := C_2 + Tb_2$. Thus, for the desired invariance property, we would like to have

$$M\begin{bmatrix}R_1 \\ R_2\end{bmatrix} + \begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix} \le \begin{bmatrix}R_1 \\ R_2\end{bmatrix},$$

equivalently

$$\begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix} \le (I - M)\begin{bmatrix}R_1 \\ R_2\end{bmatrix}.$$

If the matrix M converges to zero, then $(I-M)^{-1} \in \mathcal{M}_{2\times 2}(\mathbb{R}_+)$ and thus, we can multiply and preserve inequality sign. It turns out that

$$(I-M)^{-1}\begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix} \le \begin{bmatrix}R_1 \\ R_2\end{bmatrix}.$$

This inequality allows the choice of the radii $R_1, R_2 > 0$ to guarantee the invariance property. Thus, Schauder's fixed point theorem can be applied in $B_{R_1} \times B_{R_2}$.

2.2 *Second control problem*

We consider now the control problem (3), when the control parameters act on the growth rates.

Theorem 2.3. Assume that the functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ satisfy the following conditions:

$$\begin{aligned}\left|xf(x,y) - \bar{x}f(\bar{x},\bar{y})\right| &\le a_{11}\,|x-\bar{x}| + a_{12}\,|y-\bar{y}|, \\ \left|yg(x,y) - \bar{y}g(\bar{x},\bar{y})\right| &\le a_{21}|x-\bar{x}| + a_{22}\,|y-\bar{y}|,\end{aligned}$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}$, and the matrix

$$M = T\left[a_{ij}\right]_{1\le i,j\le 2}$$

converges to zero. Then the control problem (3) has a unique solution.

Proof. Integration leads to the integral system

$$\begin{cases} x(t) = x_0 + \int_0^t x(s) f(x(s), y(s)) ds - \lambda t, \\ y(t) = y_0 + \int_0^t y(s) g(x(s), y(s)) ds - \mu t. \end{cases} \tag{15}$$

Using the controllability conditions, we find the expressions of λ and μ, namely

$$\lambda = \frac{1}{T}\left(x_0 - x_T + \int_0^T x(s) f(x(s), y(s)) ds\right).$$
$$\mu = \frac{1}{T}\left(y_0 - y_T + \int_0^T y(s) g(x(s), y(s)) ds\right).$$

Replacing in (15) we obtain a Volterra-Fredholm type integral system which can be seen as a fixed point equation in $C([0, T]; \mathbb{R}^2)$, for the operator $N = (A, B) : C([0,T]; \mathbb{R}^2) \to C([0,T]; \mathbb{R}^2)$, defined by

$$\begin{aligned} A(x, y)(t) &= x_0 - \frac{t}{T}(x_0 - x_T) - \frac{t}{T}\int_0^T xf(x, y) ds + \int_0^t xf(x, y) ds \\ &= x_0 - \frac{t}{T}(x_0 - x_T) + \left(1 - \frac{t}{T}\right)\int_0^t xf(x, y) ds - \frac{t}{T}\int_t^T xf(x, y) ds. \\ B(x, y)(t) &= y_0 - \frac{t}{T}(y_0 - y_T) - \frac{t}{T}\int_0^T yg(x, y) ds + \int_0^t yg(x, y) ds \\ &= y_0 - \frac{t}{T}(y_0 - y_T) + \left(1 - \frac{t}{T}\right)\int_0^t yg(x, y) ds - \frac{t}{T}\int_t^T yg(x, y) ds. \end{aligned}$$

We apply Perov's fixed point theorem in the whole space $C([0, T]; \mathbb{R}^2)$. Similarly to the proof of the previous theorems, we have the following estimate

$$\left|A(x, y)(t) - A(\overline{x}, \overline{y})(t)\right| \le \int_0^T |xf(x, y) - \overline{x}f(\overline{x}, \overline{y})| \, ds.$$

Using the Lipschitz conditions from the hypothesis gives

$$\begin{aligned} \left|A(x, y)(t) - A(\overline{x}, \overline{y})(t)\right| &\le \int_0^T (a_{11}|x - \overline{x}| + a_{12}|y - \overline{y}|) ds \\ &\le Ta_{11}\|x - \overline{x}\|_\infty + Ta_{12}\|y - \overline{y}\|_\infty. \end{aligned}$$

In this way we obtain the estimates

$$\|A(x, y) - A(\overline{x}, \overline{y})\|_\infty \le Ta_{11}\|x - \overline{x}\|_\infty + Ta_{12}\|y - \overline{y}\|_\infty,$$
$$\|B(x, y) - B(\overline{x}, \overline{y})\|_\infty \le Ta_{21}\|x - \overline{x}\|_\infty + Ta_{22}\|y - \overline{y}\|_\infty.$$

We write them in the vector from

$$\begin{bmatrix} \|A(x, y) - A(\overline{x}, \overline{y})\|_\infty \\ \|B(x, y) - B(\overline{x}, \overline{y})\|_\infty \end{bmatrix} \le M \begin{bmatrix} \|x - \overline{x}\|_\infty \\ \|y - \overline{y}\|_\infty \end{bmatrix},$$

where the matrix M is convergent to zero. Thus, the operator N is a Perov contraction on $C([0, T]; \mathbb{R}^2)$. Its unique fixed point gives the solution of the control problem.

2.3 *Third control problem*

For problem (4), we apply again Perov's fixed point theorem by combining the techniques used for the first two problems. Thus, we require the Lipschitz continuity of $f(x, y)$ and $yg(x, y)$.
We look for solutions (x, y) with $x, y \in C[0,T], x > 0$ on $[0,T]$ and $\|x\|_\infty \le \rho$.
Theorem 2.4. Let $f, g : [0,\rho]\times\mathbb{R}\to\mathbb{R}$ be such that

$$\begin{aligned} |f(x,y)-f(\overline{x},\overline{y})| &\le a_{11}|x-\overline{x}|+a_{12}|y-\overline{y}|, \\ |yg(x,y)-\overline{y}g(\overline{x},\overline{y})| &\le a_{21}|x-\overline{x}|+a_{22}|y-\overline{y}|, \end{aligned}$$

for all $x, \overline{x}\in[0,\rho]$ and $y, \overline{y}\in\mathbb{R}$. Assume that

$$|f(x,y)|\le C$$

for $(x,y)\in[0,\rho]\times\mathbb{R}$,

$$C_1 + TC \le \ln\rho \tag{16}$$

and that the matrix

$$M = T\begin{bmatrix} \rho a_{11} & a_{12} \\ \rho a_{21} & a_{22} \end{bmatrix} \tag{17}$$

is convergent to zero. Then, the control problem has a unique solution $(x^*, y^*, \lambda^*, \mu^*)$ such that $x^* > 0$ and $\| x^* \|_\infty \le \rho$.

Proof. Let $x = e^u$ and denote $u_0 := u(0) = \ln x(0)$. Making substitutions and integrating we obtain

$$\begin{cases} u(t) = u_0 + \int_0^t f(e^{u(s)}, y(s))ds - \lambda t, \\ y(t) = y_0 + \int_0^t y(s)g(e^{u(s)}, y(s))ds - \mu t. \end{cases} \tag{18}$$

Using the controllability conditions $u(T) = u_T$ and $y(T) = y_T$, we find the expressions of the control parameters in terms of the state variables,

$$\begin{aligned} \lambda &= \frac{1}{T}\left(u_0 - u_T + \int_0^T f(e^{u(s)}, y(s))ds \right), \\ \mu &= \frac{1}{T}\left(y_0 - y_T + \int_0^T y(s)g(e^{u(s)}, y(s))ds \right). \end{aligned}$$

Replacing in (18) we arrive to the Volterra-Fredholm type integral system

$$\begin{cases} u(t) &= u_0 - \frac{t}{T}(u_0 - u_T) - \frac{t}{T}\int_0^T f(e^u, y)ds + \int_0^t f(e^u, y)ds \\ y(t) &= y_0 - \frac{t}{T}(y_0 - y_T) - \frac{t}{T}\int_0^T yg(e^u, y)ds + \int_0^t yg(e^u, y)ds, \end{cases}$$

which can be seen as a fixed point equation for the operator $N = (A, B)$, where

$$A(u,y)(t) = u_0 - \frac{t}{T}(u_0 - u_T) - \frac{t}{T}\int_0^T f(e^u, y)ds + \int_0^t f(e^u, y)ds$$
$$= u_0 - \frac{t}{T}(u_0 - u_T) + \left(1 - \frac{t}{T}\right)\int_0^t f(e^u, y)ds - \frac{t}{T}\int_t^T f(e^u, y)ds.$$
$$B(u,y)(t) = y_0 - \frac{t}{T}(y_0 - y_T) - \frac{t}{T}\int_0^T yg(e^u, y)ds + \int_0^t yg(e^u, y)ds$$
$$= y_0 - \frac{t}{T}(y_0 - y_T) + \left(1 - \frac{t}{T}\right)\int_0^t yg(e^u, y)ds - \frac{t}{T}\int_t^T yg(e^u, y)ds.$$

We shall apply Perov's theorem to the operator N in the set $D = B_R \times C[0,T]$, where $B_R = \{u \in C[0,T] : \|u\|_\infty \le R\}$. To this aim, using the Lipschitz conditions on $f(x, y)$ and $yg(x, y)$, we obtain estimates of A and B. One has

$$\left|A(u,y)(t) - A(\overline{u},\overline{y})(t)\right|$$
$$= \left|\left(1 - \frac{t}{T}\right)\int_0^t \left(f(e^u, y) - f(e^{\overline{u}}, \overline{y})\right)ds - \frac{t}{T}\int_0^t \left(f(e^u, y) - f(e^{\overline{u}}, \overline{y})\right)ds\right|$$
$$\le \int_0^T |f(e^u, y) - f(e^{\overline{u}}, \overline{y})|\, ds.$$

Then, using the Lipschitz condition on f, we obtain

$$|A(u,y)(t) - A(\overline{u},\overline{y})(t)|$$
$$\le \int_0^T (a_{11}|e^u - e^{\overline{u}}| + a_{12}|y - \overline{y}|)ds$$
$$\le \int_0^T \left(\rho a_{11}|u - \overline{u}| + a_{12}|y - \overline{y}|\right)ds$$
$$\le \rho T a_{11}\|u - \overline{u}\|_\infty + T a_{12}\|y - \overline{y}\|_\infty.$$

Similarly,

$$|B(u,y)(t) - B(\overline{u},\overline{y})(t)|$$
$$= \left|\left(1 - \frac{t}{T}\right)\int_0^t \left(yg(e^u, y) - \overline{y}g(e^{\overline{u}}, \overline{y})\right)ds - \frac{t}{T}\int_0^t \left(yg(e^u, y) - \overline{y}g(e^{\overline{u}}, \overline{y})\right)ds\right|$$
$$\le \int_0^T |yg(e^u, y) - \overline{y}g(e^{\overline{u}}, \overline{y})|\, ds.$$

Furthermore,

$$|B(u,y)(t) - B(\overline{u},\overline{y})(t)|$$
$$\le \int_0^T (a_{21}|e^u - e^{\overline{u}}| + a_{22}|y - \overline{y}|)ds$$
$$\le \int_0^T \left(\rho a_{21}|u - \overline{u}| + a_{22}|y - \overline{y}|\right)ds$$
$$\le \rho T a_{21}\|u - \overline{u}\|_\infty + T a_{22}\|y - \overline{y}\|_\infty.$$

Thus, we have

$$\|A(u,y) - A(\overline{u},\overline{y})\|_\infty \le \rho T a_{11}\|u - \overline{u}\|_\infty + T a_{12}\|y - \overline{y}\|_\infty,$$
$$\|B(u,y) - B(\overline{u},\overline{y})\|_\infty \le \rho T a_{21}\|u - \overline{u}\|_\infty + T a_{22}\|y - \overline{y}\|_\infty.$$

Putting the above inequalities in the vector form

$$\begin{bmatrix} \| A(u,y)-A(\overline{u},\overline{y}) \|_\infty \\ \| B(u,y)-B(\overline{u},\overline{y}) \|_\infty \end{bmatrix} \le T \begin{bmatrix} \rho a_{11} & a_{12} \\ \rho a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} \| u-\overline{u} \|_\infty \\ \| y-\overline{y} \|_\infty \end{bmatrix},$$

we guarantee the Perov contraction condition under the assumption that matrix M is convergent to zero. Moreover, the invariance condition on B_R holds as follows:

$$\begin{aligned} &|A(u,y)(t)| \\ &\le |u_0| + |u_0 - u_T| + \left| \left(1-\frac{t}{T}\right)\int_0^t f(e^u,y)ds - \frac{t}{T}\int_t^T f(e^u,y)ds \right| \\ &\le C_1 + \int_0^T |f(e^u,y)|\, ds \le C_1 + TC \le \ln\rho = R. \end{aligned}$$

Perov's theorem can be applied on $B_R \times C[0, T]$, and which guarantees the existence of a unique fixed point (u^*, y^*) of the operator N. It yields the solution of the control problem $(x^*, y^*, \lambda^*, \mu^*)$ as desired.

Remark 2.1. The specific structure of the component equations in a Kolmogorov system makes an exponential change of a variable useful in order to explicitly obtain the expression of the control in terms of the state variables from the corresponding equivalent integral equations when the control acts on the per capita rate. Thus, for the first problem, both controls act on the per capita rates; for the second problem, no one of the controls acts on the per capita rate; and for the third problem, only one of the controls does. Correspondingly, both state variables have been changed for the treatment of Problem 1; no changes have been made for Problem 2; only one variable has been changed in the case of Problem 3.

Remark 2.2. The proofs of the previous theorems show the advantage of the vector method over the usual one, namely that it allows us, instead of a set of conditions imposed on the constants involved in Lipschitz or growth inequalities, to formulate a single condition imposed cumulatively by the matrix whose elements are these constants.

Example 1. This example illustrates Theorem 2.1. Consider the following self-limiting system

$$\begin{cases} x' = x\left(\dfrac{10^{-4}}{1+x^2+y^2} - \lambda\right) \\ y' = y\left(\dfrac{2\cdot 10^{-4}}{1+4x^2+y^2} - \mu\right), \end{cases}$$

where $T = 5$, $\rho = 100$, $x_0 = e$, $y_0 = e^2$ and the final controllability conditions are $x_5 = e^2$ and $y_5 = e$. We have that $C = 2\cdot 10^{-4}$,

$$\begin{aligned} &\left|\frac{\partial f}{\partial x}\right| = \left| -\frac{2\cdot 10^{-4} x}{(1+x^2+y^2)^2} \right| \le 10^{-4}, &&\left|\frac{\partial f}{\partial y}\right| = \left| -\frac{2\cdot 10^{-4} y}{(1+x^2+y^2)^2} \right| \le 10^{-4}, \\ &\left|\frac{\partial g}{\partial x}\right| = \left| -\frac{8\cdot 2\cdot 10^{-4} x}{(1+4x^2+y^2)^2} \right| \le 4\cdot 10^{-4}, &&\left|\frac{\partial g}{\partial y}\right| = \left| -\frac{2\cdot 2\cdot 10^{-4} y}{(1+4x^2+y^2)^2} \right| \le 2\cdot 10^{-4}. \end{aligned}$$

Thus, the Lipschitz conditions (8), (9) become

$$\begin{aligned} &\left| f(x,y) - f(\overline{x},\overline{y}) \right| \le 10^{-4} \,|x-\overline{x}| + 10^{-4}\,|y-\overline{y}|, \\ &\left| g(x,y) - g(\overline{x},\overline{y}) \right| \le 4\cdot 10^{-4}\,|x-\overline{x}| + 2\cdot 10^{-4}\,|y-\overline{y}|. \end{aligned}$$

Also, in this case, using (7), we have $C_1 = 2$ and $C_2 = 3$. For $T = 5$, condition (10) is satisfied. In addition, matrix M given by (11) is

$$M = \begin{bmatrix} 5\cdot 10^{-2} & 5\cdot 10^{-2} \\ 2\cdot 10^{-1} & 10^{-1} \end{bmatrix}.$$

Recalling the necessary and sufficient condition (5) for a matrix of size two of being convergent to zero, it is easy to check that this condition holds for our matrix M. Indeed, we have

$$\operatorname{tr} M = 5\cdot 10^{-2} + 10^{-1} < 2,$$
$$\operatorname{tr} M < 1 + \det M = 1 + 5\cdot 10^{-2}\cdot 10^{-1} - 5\cdot 10^{-2}\cdot 2\cdot 10^{-1},$$

that is $\operatorname{tr} M < \min\{2, 1 + \det M\}$. Applying Theorem 2.1, it turns out that the control problem has a unique solution with $\| x^{*}\|_{\infty} \le 100$ and $\| y^{*}\|_{\infty} \le 100$.

Example 2. Theorem 2.2 in particular applies with the following choice of functions f and g

$$f(x,y) = \frac{1}{10}\frac{x}{x+y+1}\ln x + 1,$$
$$g(x,y) = \frac{1}{10}\frac{y}{x+y+1}\ln y + 1 \quad (x, y > 0),$$

extended by continuity to $x = 0$ and $y = 0$, respectively, that is $f(0, y) = g(x, 0) = 1$ ($x, y \in \mathbb{R}_+$). It is easy to check that the assumptions of Theorem 2.2 are satisfied for $T = 5$ and that the convergent to zero matrix M is

$$M = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

Thus, the corresponding Kolmogorov system is controllable for any initial and final values of x and y.

Example 3. The following functions make the assumptions of Theorem 2.3 to be fulfilled:

$$f(x,y) = \frac{1}{10}(1+\sin y)\frac{\sin x}{x},$$
$$g(x,y) = \frac{1}{10}(1+\sin x)\frac{\sin y}{y}.$$

Here, it is understood that $f(0,y) = \frac{1}{10}(1+\sin y)$ and $g(x,0) = \frac{1}{10}(1+\sin x)$. The assumptions of Theorem 2.3 are satisfied for $T = 3$ and that the convergent to zero matrix M is in this case

$$M = \begin{bmatrix} 0.6 & 0.3 \\ 0.3 & 0.6 \end{bmatrix}.$$

Example 4. Consider the functions

$$f(x,y) = \frac{1}{100\left(1+x^2+y^2\right)},$$
$$g(x,y) = \frac{1}{100}(1+\sin x)\frac{\sin y}{y},$$

for which $a_{11} = a_{12} = a_{21} = \frac{1}{100}$, $a_{22} = \frac{2}{100}$ and $C = \frac{1}{100}$, independently of ρ. Taking $x_0 = 1$ and $x_T = e$, we have $C_1 = 1$. Next, taking $T = 10$ and $\rho = e^2$. we have that condition (16) holds. In addition, matrix (17) is

$$M = \frac{1}{10}\begin{bmatrix} e^2 & 1 \\ e^2 & 2 \end{bmatrix}$$

and by checking (6), it is convergent to zero. Thus, Theorem 2.4 applies.

Remark 2.3. (Approximation and numerical methods) As can be seen from the above, solving control problems often leads to Volterra-Fredholm integral systems, whose numerical solving is a real challenge. Starting from this finding, in the paper [8], an algorithm was developed for the approximation of solutions to control problems for fixed point equations. The algorithm was successfully applied in the recent work [21] to a control problem related to a three-dimensional system that models stem cell transplantation. It can also be used for the numerical solution of control problems for Kolmogorov systems, as shown in the paper [10], and can be easily combined with the vector method that was the subject of this work.

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Conflict of interest

The authors declare no competing financial interest.

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