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Research Article

Modelling the VaR Using Integral Probability and Tail Dependence

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Abstract: In this paper, a method is proposed to estimate value at risk (VaR) in a multivariate context by employing the probability integral transformation (PIT) and tail dependence function to model extreme dependence structure. The proposed method involves two vectorial measures of VaR, which use either the distribution function or the survival function. Moreover, properties of these risk measures are analysed, and a connection between them and tail dependence functions is established.

***Keywords*:** copulas, stochastic finance, risk measures, extreme values theory, tail dependence, probability integral transformation (PIT)

MSC: 62H05, 60H99, 91G70, 62G32, 46F12

1. Introduction

The stochastic financial modelling provides useful indicators for measuring the risk associated with a financial asset. Dealing with a portfolio of assets, this modelling takes into consideration information on the dependence structure and knowledge of some marginal situations. Copulas theory is a helpful tool for modelling dependence and estimating quantile risk measures in a multivariate framework since it allows for an efficient representation and interpretation of the dependence structure of multivariate variables.

Indeed, the advantage of copulas in dependence analysis can be obtained from Sklar's theorem. This pioneering result shows $(X_1, \cdots, X_d)$ is a vector of random variables that are continuous with distribution function F. Therefore, there is a unique copula C_d such as

$$F(x_1, \cdots, x_d) = C_d\left(F_1(x_1), \cdots, F_d(x_d)\right), \tag{1}$$

where $(F_1, \cdots, F_d)$ are the univariate distribution functions of the random variables $X_1, \cdots, X_d$.

This result enables the marginal distribution to be joined to the multivariate joint distribution. The foregoing relationship, in the survival analysis, offers the survival copula in function of the survival law S, of F by

$$\bar{C}_d\left(S_1(x_1), \cdots, S_d(x_d)\right) = S(x_1, \cdots, x_d), \tag{2}$$

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where S is a d-dimensional survival function with margins $S_1, \cdots, S_d$.

The following describes the link between the copula C_d and the survival copula $\overline{C}_d$.

$$\overline{C}_d(u_1, \cdots, u_\kappa, \cdots, u_d) = \sum_{\kappa=0}^{d}\left[(-1)^\kappa \sum_{\mathbf{v}(\mathbf{u})\in \mathcal{Z}(d-\kappa, d, 1)} C_d(\mathbf{1}-\mathbf{v})\right], \tag{3}$$

where $\mathbf{u} = u_i, i = 1, \cdots, d; \mathbf{1}-\mathbf{v} = (1-v_1, \cdots, 1-v_d)$ and $\mathcal{Z}(m, d, \epsilon)$ denotes the set $\{\mathbf{v} \in [0, 1]^d \,|v_\kappa \in \{u_\kappa, \epsilon\}, \sum_{\kappa=1}^{d} \chi_{\{\epsilon\}}(v_\kappa) = \text{m}\}$.

A specific class of copulas is the extreme values one which satisfies the following relation for any vector $(u_1, \cdots, u_d) \in [0, 1]^d$ and for all $t > 0$;

$$C(u_1^t, \cdots, u_d^t) = C^t(u_1, \cdots, u_d). \tag{4}$$

Extreme value theory is the tool used to model and describe the occurrence and intensity of extreme events. According to Equation (1), a multidimensional extreme value distribution corresponds to a d-dimensional copula whose marginals are the suitable normalized limits of the law of maxima such that

$$F_i(x_i) = \lim_{n\to+\infty} P\left[\frac{M_n - b_n}{a_n} \le x_i\right], \; i = 1, \cdots, d, \tag{5}$$

with $(a_n) > 0$ and $(b_n) \in \mathbb{R}$ being the normalizing coefficients and M_n as the maximum of an independently and identically distributed sequence of random variables $\{X_1, \cdots, X_d\}$. The extremal copulas appear as possible limits of the maxima copulas in independent and identically distributed sample components and allow for an adequate modelling of multivariate extreme values (see [1] for more details). These copulas are of particular relevance in joint extreme risk modelling because the tails of the distribution are important for creating adequate models for the dependence structure between severe losses.

Multivariate risk assessments rely heavily on the loss distribution function. The use of multivariate probability integral transformation (PIT) is an innovative approach in multivariate analysis applied to portfolio management to develop risk indicators which induce total order in $\mathbb{R}^d$ and contain information about the dependence structure of the financial instruments that make up a portfolio. A relevant instance of the multivariate probability integral transform is given by the Kendall distribution function, defined as follows:

$$K_d(t) = \mu_C\left(\{(u_1, \cdots, u_d) \in I^d |\, C_d(u_1, \cdots, u_d) \le t\}\right), \tag{6}$$

where C is the copula corresponding to the multivariate random vector $\mathbf{X}$ distribution (see [2] and [3] for more details).

This paper provides an estimation of the value at risk (VaR) in the approach of the PIT. The second section provides an overview of copulas, extremal models and risk measures. In the third section, using PIT approach to model risk dependence, formulas for computing multidimensional VaR are proposed and some properties have been analysed. Furthermore, a link between these risk measures and the tail dependence function has been established.

2. Preliminaries

In this section, we summarise theoretical framework that guides research and thinking about extreme values theory, copulas and risk measures on multivariate analysis used to risk management. These results are necessary for our approach.

2.1 *An overview of extremal structures*

The mathematical and statistical foundations of copulas have been explored in many studies including [4] and [1] by providing theoretical and practical introduction to the copulas analysis. The fitting of multivariate distributions to financial data was dealt by [5-7].

A d-dimensional copula is a function C, mapping $[0, 1]^d$ to $[0, 1]$ that meets the following conditions for all $d \in \mathbb{N}^*$;

(a) for all $(u_1, \cdots, u_d) \in [0, 1]^d$, $C(u_1, \cdots, u_{i-1}, 0, u_{i+1}, \cdots, u_d) = 0$ (grounded);

(b) for all $(u_1, \cdots, u_d) \in [0, 1]^d$, $C(1, \cdots, 1, u_i, 1, \cdots, 1) = u_i$;

(c) for all $(u_1^{(1)}, \cdots, u_d^{(1)}) \in [0, 1]^d$ such as $u_j^{(1)} \le u_j^{(2)}$, $\sum_{(i_1, \cdots, i_d) \in \{1, 2\}^d} (-1)^{\Sigma_{j=1}^d ij} C(u_1^{(i_1)}, \cdots, u_d^{(i_d)}) \ge 0$ (2-increasing).

There are several families of copulas including Archimedean or harmonic ones. Here, we are interested in extreme values copulas and Archimedean ones, which are the most commonly used to generate adequate models for the dependence structure between severe losses.

In particular, extremal copulas derive from extensions of multivariate extreme values distributions which is of great interest in risk modelling. Indeed, a risk measure works for extreme events and for smaller events. Thus, the determination of the VaR of portfolio's assets takes into account the extreme losses of the portfolio.

Archimedean copulas are another type of copula that has received a lot of interest in stochastic financial modelling. Copulas are called Archimedean when they may be expressed for all $(u_1, \cdots, u_d) \in [0, 1]^d$ in the form

$$C(u_1, \cdots, u_d) = \varphi^{-1}\left(\varphi(u_1) + \cdots + \varphi(u_d)\right), \tag{7}$$

where $\varphi(t)$ is a continuous, strictly decreasing and convex function from $[0, 1]$ to $[0, \infty]$ which contain all the information on d-dimensional dependence structure such that $\varphi(1) = 0$. The inverse function given by $\varphi^{-1}(u) = \inf\{x \in [0, 1] : \varphi(x) \le u\}$ satisfies

$$(-1)^k \frac{\partial^k}{\partial u^k} \varphi^{-1}(u) \ge 0, \text{ for } k = 1, \cdots, d. \tag{8}$$

Archimedean copulas are often used to simulate tails of distributions.

Furthermore, the notion of tail dependence which has been widely explored in financial modelling for market and credit risks, is a highly relevant tool which has a significant impact on the VaR computation. According to Diakarya [8], the lower-tail dependence function and the upper-tail dependence function consider κ variables and the conditional probability corresponding to the residual d-κ variables, according to the following formula:

$$\lambda_{U,\kappa} = \lim_{u \to 1^-} P\left(X_1 > F_1^{-1}(u), \cdots, X_\kappa > F_\kappa^{-1}(u) \middle| X_{\kappa+1} > F_{\kappa+1}^{-1}(u), \cdots, X_d > F_d^{-1}(u)\right), \tag{9}$$

and

$$\lambda_{L,\kappa} = \lim_{u \to 0^+} P\left(X_1 \le F_1^{-1}(u), \cdots, X_\kappa \le F_\kappa^{-1}(u) \middle| X_{\kappa+1} \le F_{\kappa+1}^{-1}(u), \cdots, X_d \le F_d^{-1}(u)\right), \tag{10}$$

where F_i^{-1}, $i = 1, \cdots, d$ represent the quantile functions of the respective marginal distributions.

2.2 *An overview on the VaR*

Risk management from the perspective of mathematics, is a process for establishing a loss distribution. The VaR is a financial risk measurement and a control tool intrinsically linked to the loss distribution function. It is determined by two parameters: the probability level and the time horizon. For a univariate risk X, with distribution function F_t, the VaR of level α, is the minimal amount of the loss which accumulates a probability α in the left tail.

$$\mathrm{VaR}_\alpha(X) = \inf\left\{x \in \mathbb{R} : F_t(x) \geq \alpha\right\} = F_t^{-1}(\alpha). \tag{11}$$

Equivalently, the univariate VaR can be defined using the survival function S_t associated with the risk X as follows:

$$\mathrm{VaR}_\alpha(X) = \inf\left\{x \in \mathbb{R} : S_t(x) \leq 1-\alpha\right\}. \tag{12}$$

Similar to the univariate case, in multivariate analysis for a d-dimensional risk portfolio, VaR can be defined using the distribution function and the survival function, denoted by the lower-tail VaR (VaR_α^L) and the upper-tail VaR (VaR_α^U) respectively. As shown by Loyara et al. [9] the concept of copula as a tool for generating distribution functions from a set of marginals is adequate for expressing multidimensional VaR. So, using Sklar's theorem, the two variants of the multidimensional VaR at probability level α, for a random vector $\mathbf{X} = (X_1, \cdots, X_d)$ are defined respectively as follows for $i = 1, \cdots, d$:

$$\mathrm{VaR}_\alpha^L(\mathbf{X}) = \mathbb{E}\left[F_i^{-1}(U_i) \mid \mathbf{U} \in \mathcal{L}_\alpha^C\right], \tag{13}$$

where F_i^{-1} denote the inverses of the marginal distribution functions of $\mathbf{X}$, and C is a parametric copula satisfying regularity conditions associated with $\mathbf{X}$, $\mathbf{U} = (U_1, \cdots, U_d)$ with $U_i = F_i(X_i)$ uniformly distributed random variables having a joint distribution C such as $\mathcal{L}_\alpha^C = \left\{\mathbf{U} \in I^d : C(\mathbf{U}) = \alpha\right\}$.

$$\mathrm{VaR}_\alpha^U(\mathbf{X}) = \mathbb{E}\left[S_i^{-1}(V_i) \mid \mathbf{V} \in \overline{\mathcal{L}}_{1-\alpha}^C\right], \tag{14}$$

where S_i^{-1} denote the inverses of the marginal survival functions of $\mathbf{X}$, and $\overline{C}$ the survival copula associated with $\mathbf{X}, \mathbf{V} = (V_1, \cdots, V_d)$ with $V_i = S_i(X_i)$ uniformly distributed random variables having a joint distribution $\overline{C}$ such as $\overline{\mathcal{L}}_{1-\alpha}^C = \left\{\mathbf{V} \in I^d : \overline{C}(\mathbf{V}) = 1-\alpha\right\}$.

As the Kendall distribution function is the distribution that describes the level sets of a copula, we discuss in the following sections the modelling of multidimensional VaR conditional on copula level sets using this distribution.

3. Main results

In this study, we provide a multidimensional VaR model based on the Kendall distribution that captures the dependence structure between the losses of a portfolio of one-period financial risks. We propose some results on the multivariate PIT and tail dependence function applied to portfolio risk management. In which follows, let consider a random continuous loss vector $\mathbf{X} = (X_1, \cdots, X_d)$ distributed according to an extreme values law F with respect to the Lebesgue measure μ on $\mathbb{R}^d$ Moreover, we make assumption that the random vector $\mathbf{X}$, is non-negative. In addition, a links between the PIT and the tail dependence function on the multidimensional VaR is established.

3.1 *Modelling the VaR with Kendall distribution*

Consider a random variable X with distribution function F_t, the quantile function of X at level $u \in (0, 1)$, which is usually unique, is given by,

$$Q_t(u) = F_t^{-1}(u) = \inf\left\{z : F_t(z) \geq u\right\}. \tag{15}$$

Similar reasoning applies in the multivariate case. Under the previous conditions, given $\alpha \in (0, 1)$, the multivariate quantile function of level α is given by

$$\mathcal{L}_\alpha^F = \left\{(x_1, \cdots, x_d) \in \mathbb{R}^d : F(x_1, \cdots, x_d) = \alpha\right\}. \tag{16}$$

$\mathcal{L}_\alpha^F$ is an iso-line for bivariate distributions because it is the iso-hyper-surface (with dimension d-1) when F equals the constant value α. Due to the PIT, we can connect to F a parametric copula C such that, for all $(u_1, \cdots, u_d) \in [0, 1]^d$, $\mathcal{L}_\alpha^F$ is the corresponding of $\mathcal{L}_\alpha^F$ in $[0, 1]^d$.

The quantile function is informative as it expresses the level of dependence between the assets in a portfolio for a fixed probability level. The Kendall distribution function can be used to fix this probability level in a suitable way. Indeed, using a reasoning similar to that of [10], we show that,

$$\alpha = K_C(q_\alpha) = P\left\{C\left(F_1(X_1), \cdots, F_d(X_d)\right) \le q_\alpha\right\}, \tag{17}$$

where $q_\alpha \in \mathbf{I}$ is the Kendall quantile of order α.

In what follows, by coupling the PIT and quantiles, this result is obtained.

Proposition 1 Let $\mathbf{X} = (X_1, \cdots, X_d)$ be a continuous random loss vector with distribution function F and survival distribution $\overline{F}$ satisfying both the regularity conditions, provided that the densities of copulas associated with $\left(X_i, F(\mathbf{X})\right)$ and $\left(X_i, \overline{F}(\mathbf{X})\right), i = 1, \cdots, d$ exist, then for $\alpha \in (0, 1)$,

(i) the marginal lower tail VaR at probability level α of $\mathbf{X}$ is given by,

$$\mathrm{VaR}_\alpha^L(X_i) = \int_\alpha^1 \mathrm{VaR}_{u_i}(X_i) c(u_i, v) du_i, \tag{18}$$

where $(u_i, v) = (F_i(x_i), K_C(\alpha))$ and $c(u, v) = \dfrac{\partial^2 C(u, v)}{\partial u \partial v}$ is the density of the copula $C(u, v)$;

(ii) the marginal upper tail VaR at probability level α of $\mathbf{X}$ is given by:

$$\mathrm{VaR}_\alpha^U(X_i) = \int_0^{1-\alpha} \mathrm{VaR}_{1-u_i}(X_i) c(u_i, \overline{v}) du_i, \tag{19}$$

where $(u_i, \overline{v}) = \left(F_i(x_i), K_{\hat{C}}(\alpha)\right)$ and $K_{\hat{C}}(\alpha)$ denotes the survival distribution function of $F(\mathbf{X})$.

Proof.

(i) Following Equation (11) and using the limit procedure in [11], we have

$$\mathrm{VaR}_\alpha^L(X_i) = \mathbb{E}\left[X_i \middle| F(\mathbf{X}) = \alpha\right] = \frac{1}{K'_C(\alpha)} \int_{F_i^{-1}(\alpha)}^\infty x_i f_{(X_i, F(\mathbf{X}))}(x_i, \alpha) dx_i, \tag{20}$$

for $i = 1, \cdots, d$ where $K'_C(\alpha) = \dfrac{dK_C(\alpha)}{d\alpha}$ is the density of the Kendall distribution. So, the joint density of the pair $(X_i, F(\mathbf{X}))$, $i = 1, \cdots, d$ can be written as,

$$f_{(X_i, F(\mathbf{X}))}(x_i, \alpha) = \frac{\partial^2 F(x_i, \alpha)}{\partial x_i, \partial \alpha} = \frac{\partial^2 C\left(F_i(x_i), K_C(\alpha)\right)}{\partial x_i, \partial \alpha} = c\left(F_i(x_i), K_C(\alpha)\right) f_i(x_i), K'_C(\alpha). \tag{21}$$

Thus, $\mathbb{E}[X_i | F(\mathbf{X}) = \alpha]$ can be written as follows:

$$\mathbb{E}\left[X_i \middle| F(\mathbf{X}) = \alpha\right] = \frac{1}{K'_C(\alpha)} \int_{F_i^{-1}(\alpha)}^\infty x_i c\left(F_i(x_i), K_C(\alpha)\right) f_i(x_i) K'_C(\alpha) dx_i. \tag{22}$$

By performing the change of variable $F_i(x_i) = u_i$, and $K_C(\alpha) = v, (u_i, v) \in I^2$, we have

$$\mathbb{E}\left[X_i \,|F(\mathbf{X})\right] = \int_\alpha^1 F_i^{-1}(u_i)c(u_i, v)du_i. \tag{23}$$

The result is obtained by using the one-dimensional VaR definition.

(ii) In the same way, following Equation (12) one has:

$$\text{VaR}_\alpha^U(X_i) = \mathbb{E}[X_i \mid \bar{F}(X) = 1-\alpha] = \frac{1}{K'_{\hat{C}}(1-\alpha)} \int_0^{F_i^{-1}(1-\alpha)} x_i f_{(X_i, \bar{F}(\mathbf{X}))}(x_i, 1-\alpha)dx_i, \tag{24}$$

and in terms of copula function as:

$$\text{VaR}_\alpha^U(X_i) = \int_0^{F_i^{-1}(1-\alpha)} x_i c\left(F_i(x_i), K_{\bar{C}}(1-\alpha)\right) f_{X_i}(x_i)dx_i. \tag{25}$$

Let's take $u_i = F_i(x_i)$ and $\bar{v} = K_{\bar{C}}(1-\alpha)$, it comes that:

$$\text{VaR}_\alpha^U(X_i) = \int_0^{1-\alpha} F_i^{-1}(u_i)c(u_i, \bar{v})du_i. \tag{26}$$

Finally, using the definition of univariate VaR, we have the result.

From the previous result (of Proposition 1), we deduce the conditional distribution of X_i for probability level α:

$$F_{(X_i|F(\mathbf{X}))}(x) = \frac{1}{K'_C(\alpha)} \int_{F_{X_i}^{-1}(\alpha)}^x f_{(X_i, F(\mathbf{X}))}(t, \alpha)dt, \tag{27}$$

with the conditional density

$$f_{(X_i|F(X))}(x) = c(u, v)f_i(x). \tag{28}$$

As a result of Sklar's theorem, we obtain the following result.

Corollary 2 Let $\mathbf{X} = (X_1, \cdots, X_d)$ be a d-dimensional random loss vector whose VaR is given by Equation (18), then the conditional random variable $X_i|F(\mathbf{X})$ is distributed as follows,

$$F_{(X_i|F(\mathbf{X}))}(x_i) = \frac{\partial C(u_i, v)}{\partial v} - \frac{\partial C(\alpha, v)}{\partial v}\Big|_{(u_i, v)=(F_i(x_i), K_C(\alpha))}. \tag{29}$$

Proof. Following Equation (27), we have:

$$F_{(X_i|F(\mathbf{X}))}(x) = \int_{F_i^{-1}(\alpha)}^x c\left(F_i(t), K_C(\alpha)\right) f_{X_i}(t)dt. \tag{30}$$

Using the change of variables $w = F_i(t)$, we obtain

$$F_{(X_i|F(\mathbf{X}))}(x) = \int_\alpha^u c(w, v)dw. \tag{31}$$

The computation of this integral gives the result.

In terms of the marginal distribution, the distribution function of X_i has the form

$$F_i(x) = \int_{-\infty}^{x} \int_{0}^{1} f(t, \alpha) d\alpha dt. \tag{32}$$

To implement the computation of the VaR in such context, it is necessary to have an explicit formula for the copula density considered. For two-dimensional extreme copulas using the formula of Faà di Bruno in [12], obtained a general formula for the density is obtained as follows,

$$c(u_1, \cdots, u_d) = \frac{C(u_1, \cdots, u_d)}{\Pi(u_1, \cdots, u_d)} \sum_{m=1}^{d} (-1)^{d-m} \sum_{\pi:|\pi|=m} \prod_{B\in\pi} D_B l(\tilde{u}_1, \cdots, \tilde{u}_d), \tag{33}$$

where $\tilde{u}_i = (-\log u_i),\ i = 1, \cdots, d,\ u_i \in (0,\ 1), \Pi(u_1, \cdots, u_d)$ is the independence copula, π scans the set Π of all the partitions in the set $\mathcal{J} = \{1, \cdots, d\}$, $B \in \pi$ means that B scans the list of all elements of the partition π and $D_B = \frac{\partial^{|B|}}{\prod_{j\in B} \partial \tilde{u}_j}$. $l : [0, \infty)^d \to [0, \infty)$ denotes the stable tail dependence function given by,

$$l(\mathbf{u}) = \int_{\Delta_{d-1}} \max_{1\le k\le d}(z_k u_k) dG(z_1, \cdots, z_d),\ \mathbf{u} \in [0, \infty)^d, \tag{34}$$

for a Borel measure G on Δ_{d-1} known as the spectral measure meeting the conditions

$$\int_{\Delta_{d-1}} z_k dG(z_1, \cdots, z_d) = 1,\ k \in \{1, \cdots, d\}. \tag{35}$$

We get for any bivariate extreme value copula,

$$c(u_1, u_2) = \frac{C(u_1, u_2)}{u_1 u_2} \left[\left(\frac{\partial l(\tilde{u}_1, \tilde{u}_2)}{\partial \tilde{u}_1} \frac{\partial l(\tilde{u}_1, \tilde{u}_2)}{\partial \tilde{u}_2} \right) - \frac{\partial^2 l(\tilde{u}_1, \tilde{u}_2)}{\partial \tilde{u}_1 \partial \tilde{u}_2} \right]. \tag{36}$$

The previous relation allows us to obtain marginal VaR expression for any random vector **X**, with an extremal dependence structure.

Corollary 3 If the derivative of the tail dependence function l exists and is well defined for the extremal copula $C(u_i, v)$ for $i = 1, \cdots, d$ the marginal lower tail VaR is given by the following expression,

$$\mathrm{VaR}_{\alpha}^{L}(X_i) = \int_{\alpha}^{1} VaR_{u_i}(X_i) \frac{C(u_i, v)}{u_i v} \left[\left(\frac{\partial l(\tilde{u}_i, \tilde{v})}{\partial \tilde{u}_i} \frac{\partial l(\tilde{u}_i, \tilde{v})}{\partial \tilde{v}} \right) - \frac{\partial^2 l(\tilde{u}_i, \tilde{v})}{\partial \tilde{u}_i \partial \tilde{v}} \right] du_i. \tag{37}$$

The bivariate extremal copula density formula allows for an analysis of how the VaR measure is affected by a change in the level of risk.

Proposition 4 Given $\mathbf{X} = (X_1, \cdots, X_d)$ as a random loss vector distributed according to an extremal copula, then any realization x_i of X_i is a decreasing function of the probability level α.

Proof. To prove this result we prove that $P(X_i | F(\mathbf{X}) = \alpha)$ is a decreasing function of α for any realization x_i. Which is equivalent to showing that for all $v \in [\alpha, 1]$, the derivatives $\frac{\partial C(u, v)}{\partial v}$ and $\frac{\partial C(\alpha, v)}{\partial v}$ are respectively decreasing in u and in α, almost everywhere and $C(u, v) > C(\alpha, v)$. So it comes that

$$\frac{\partial C(u,v)}{\partial v}=\frac{C(u,v)}{v}\{A(q)-qA'(q)\},\ \text{with}\ \ q=\frac{\log u}{\log u+\log v}, \tag{38}$$

where A is a convex function of [0, 1] in [0, 1], verifying $\max(q, 1 - q) \leq A(q) \leq 1$ and derivable almost everywhere such that its derivative $A'(q)$ verifies $-1 \leq A'(q) \leq 1$.

Moreover, $\frac{C(u,v)}{v}$ and $\left(A(q)-qA'(q)\right)$ are positive, we show in an analogous way to [13], that each of them is decreasing.

$$\frac{\partial}{\partial v}\left(\frac{C(u,v)}{v}\right)=\frac{C(u,v)}{v^2}\left(A(q)-qA'(q)\right)-\frac{C(u,v)}{v^2}. \tag{39}$$

From the properties of A and A' that for all $q \in [0, 1]$, we have:

$$A(q)-qA'(q)\geq \max(q,1-q)-q\geq 0\ \ \text{and}\ \ A'(q)\geq\frac{1-A(q)}{-q}. \tag{40}$$

Hence, for all $0 \leq u, v \leq 1$:

$$0\leq A(q)-qA'(q)\leq 1. \tag{41}$$

Therefore, one obtains

$$\frac{C(u,v)}{v^2}\geq\frac{C(u,v)}{v^2}\left\{A(q)-qA'(q)\right\}. \tag{42}$$

So, the function $\frac{C(u,v)}{v}$ is decreasing in v almost everywhere.

Furthermore, the function $A(q) - qA'(q)$ can be written for all for all $v \in [0, 1]$ fixed as $g(q(u))$, where $g(q) = A(q) - qA'(q)$ and $q(u)=\frac{\log u}{\log u+\log v}$ is a decreasing (0, 1] function in [0, 1]. Moreover, we have

$$g'(q)=-qq'A''(q). \tag{43}$$

Since $A(q)$ is a convex and differentiable function, then $A'(t)$ is increasing. Therefore $A''(t) \geq 0$ Moreover, $q(u)$ is a decreasing function, hence $q'(u) \leq 0$. Consequently, $g'(q) \geq 0$ and $g(q)$ is increasing, showing that $\frac{\partial C(u,v)}{\partial v}$ is decreasing.

The other part, $\frac{\partial C(\alpha,v)}{\partial v}$ is proved in a similar way. We also have $\alpha\leq u$, so $C(\alpha,v)\leq C(u,v)$, hence the result.

Figure 1 illustrates this result with the Gumbel copula of parameter $\theta = 2.94$ for a portfolio of risk whose marginals follow the Gumbel distribution of location 1 and scale 2.

Conditional distributions have been used by several authors to model financial risk. Using tail dependence coefficients extensions and extreme structures based on copulas in [14] which related to the VaR to a measure of tail dependence called the expected tail dependence coefficients. This new concept bridges the gap between multivariate VaR and tail concentration functions.

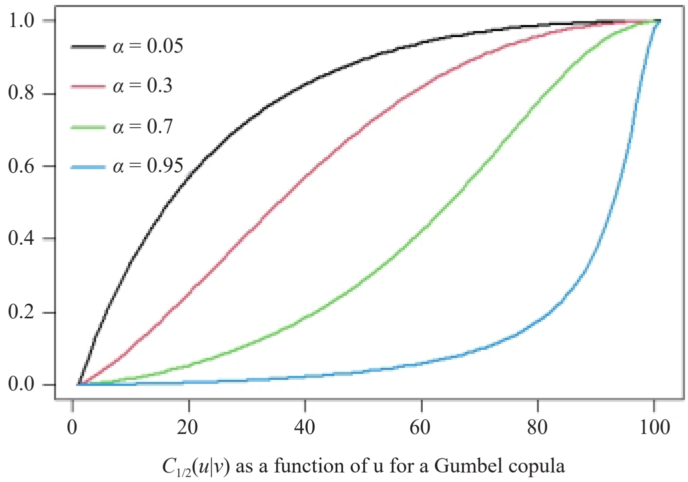

Figure 1. Conditional distribution of X given α for Gumbel copula

3.2 *Kendall distribution and tail dependence coefficients on VaR*

The notion of tail dependence is an adequate approach to estimate the occurrence of simultaneous risks of tails of distributions and it allows to model the concomitant occurrence of extreme values.

This notion was used by Mallam et al. [14] to define the expected tail function. The expected lower tail function ζ_κ^L and the expected upper tail function ζ_κ^U for a d-dimensional random vector with marginals F_i and a distribution function F are given by,

$$\zeta_\kappa^L(\alpha) = \mathbb{E}\left(X_{(\kappa)} \middle| X_{\kappa+1} \le F_{\kappa+1}^{-1}(\alpha), \cdots, X_d \le F_d^{-1}(\alpha)\right), \tag{44}$$

respectively,

$$\zeta_\kappa^U(\alpha) = \mathbb{E}\left(X_{(\kappa)} \middle| X_{\kappa+1} \le F_{\kappa+1}^{-1}(\alpha), \cdots, X_d \le F_d^{-1}(\alpha)\right). \tag{45}$$

The following results establish a link between multidimensional VaR and the coefficient of expected tails dependence.

Proposition 5 Let $\mathbf{X} = (X_1, \cdots, X_d)$ a d-dimensional random vector with Archimedean copula C_d. Suppose the conditional density and survival conditional density of $\mathbf{X} = (X_1, \cdots, X_d)$ exist, then for $\alpha \in (0, 1)$,

(i) the marginal expected lower tail function is given by,

$$\zeta_{\kappa,i}^L(\alpha) = \frac{\partial}{\partial\alpha} VaR_\alpha^L(X_i); \tag{46}$$

(ii) the marginal expected upper tail function is given by,

$$\zeta_{\kappa,i}^U(\alpha) = \frac{\partial}{\partial(1-\alpha)} VaR_\alpha^U(X_i). \tag{47}$$

Proof.

(i) The lower tail expected function $\zeta_{\kappa,i}^L(\alpha)$, for all $i = 1, \cdots, \kappa$ is given by:

$$\zeta_{\kappa,i}^{L}(\alpha) = \mathbb{E}\left(X_i \,\middle|\, X_{\kappa+1} \le F_{\kappa+1}^{-1}(\alpha), \cdots, X_d \le F_d^{-1}(\alpha)\right); \tag{48}$$

which gives:

$$\zeta_{\kappa,i}^{L}(\alpha) = \int_{F_i^{-1}(\alpha)}^{\infty} x_i f_{X_i|X_{(d-\kappa)}}(x_i, x_{\kappa+1}, \cdots, x_d)dx_i, \tag{49}$$

where $f_{X_i|X_{(d-\kappa)}}$ is the density of the conditional variable $(X_i | X_{\kappa+1}, \cdots, X_d)$, which exists by assumption. The associated conditional distribution function $F_{X_i|X_{(d-\kappa)}}$ of $f_{X_i|X_{(d-\kappa)}}$ is given by,

$$F_{X_i|X_{(d-\kappa)}}(x_i, x_{\kappa+1}, \cdots, x_d) = \mathbb{P}(X_i \le x_i \,|\, X_{\kappa+1} \le x_{\kappa+1}, \cdots, X_d \le x_d). \tag{50}$$

Furthermore, we have

$$F_{X_i|X_{(d-\kappa)}}(x_i, x_{\kappa+1}, \cdots, x_d) = \frac{\mathbb{P}(X_i \le x_i, X_{\kappa+1} \le x_{\kappa+1}, \cdots, X_d \le x_d)}{\mathbb{P}(X_{\kappa+1} \le x_{\kappa+1}, \cdots, X_d \le x_d)}, \tag{51}$$

which in copula terms gives,

$$F_{X_i|X_{(d-\kappa)}}(x_i, x_{\kappa+1}, \cdots, x_d) = \frac{C_d(u_i, u_{\kappa+1}, \cdots, u_d)}{C_{d-\kappa}(u_{\kappa+1}, \cdots, u_d)}, \tag{52}$$

where $(u_i, u_{\kappa+1}, \cdots, u_d) = \left(F_i(x_i), F_{\kappa+1}(x_{\kappa+1}), \cdots, F_d(x_d)\right)$.

Let's consider the sequence $(U_i, U_{\kappa+1}, \cdots, U_d) = (F_i(X_i), F_{\kappa+1}(X_{\kappa+1}), \cdots, F_d(X_d))$ of uniformly distributed standard random variables, then $W_{d-\kappa} = C_{d-\kappa}(U_{\kappa+1}, \cdots, U_d)$ is a random variable distributes as

$$K_{d-\kappa}(w_{d-\kappa}) = \mathbb{P}(W_{d-\kappa} \le w_{d-\kappa}),\ w_{d-\kappa} \in [0,\ 1], \tag{53}$$

which is the Kendall distribution of $X_{(d-\kappa)}$. Consequently,

$$V_{d-\kappa} = K_{d-\kappa}(W_{d-\kappa}) \sim U(0,\ 1), \tag{54}$$

$K_{d-\kappa}$ is a one-dimensional description of $C_{d-\kappa}$. We also have

$$K_i(U_i) = \mathbb{P}(U_i \le u_i) = u_i. \tag{55}$$

K_i and $K_{d-\kappa}$ being continuous. There is a unique two-dimensional copula C_2 corresponding to the distribution function of the vector $(U_i, V_{d-\kappa})$. Thus,

$$\frac{C_d(u_i, u_{\kappa+1}, \cdots, u_d)}{C_{d-\kappa}(u_{\kappa+1}, \cdots, u_d)} = \frac{C_2(u_i, v_{d-\kappa})}{v_{d-\kappa}}. \tag{56}$$

Therefore,

$$f_{X_i|X_{(d-\kappa)}}(x_i, x_{d-\kappa}) = f_i(x_i)\partial_{d-\kappa}\frac{\partial_i C_2(u_i, v_{d-\kappa})}{v_{d-\kappa}}. \tag{57}$$

Then,

$$f_{X_i|X_{(d-\kappa)}}(x_i, x_{d-\kappa}) = f_i(x_i)c_2(u_i\,|v_{d-\kappa})\big|_{v_{d-\kappa}=\alpha} \tag{58}$$

where $c_2(u_i\,|v_{d-\kappa})$ is the density of the conditional copula $C_2(u_i\,|v_{d-\kappa})$.

On the basis of relation (48), we obtain

$$\zeta^L_{\kappa,i}(\alpha) = \int_{F_i^{-1}(\alpha)}^{\infty} x_i c_2(u_i\,|\,\alpha) f_i(x_i)dx_i, \tag{59}$$

where $c_2(u_i\,|\,\alpha) = \dfrac{\partial c_2(u_i\,|\,\alpha)}{\partial\alpha}$.

Finally, the expected lower tail function is given by,

$$\zeta^L_{\kappa,i}(\alpha) = \frac{\partial}{\partial\alpha}\int_{F_i^{-1}(\alpha)}^{1} \mathrm{VaR}_{u_i}(X_i)c_2(u_i,\alpha)du_i. \tag{60}$$

So, the result is proved as disserted.

(ii) Similarly, the conditional distribution of $(X_i \geq .\,|X_{\kappa+1} \geq ., \cdots X_d \geq .)$ is given for each marginal expected tail function of the upper tail $\zeta^U_{\kappa,i}(\alpha)$, $i = 1, \cdots, \kappa$ by,

$$\overline{F}_{X_i|X_{(d-\kappa)}}(x_i, x_{\kappa+1}, \cdots, x_d) = \frac{\overline{C}_d\left(1-F_i(x_i), 1-F_{\kappa+1}(x_{\kappa+1}), \cdots, 1-F_d(x_d)\right)}{\overline{C}_{d-\kappa}\left(1-F_{\kappa+1}(x_{\kappa+1}), \cdots, 1-F_{X_d}(x_d)\right)}. \tag{61}$$

In the same way as above, using Sklar theorem, we have,

$$\overline{F}_{X_i|X_{(d-\kappa)}}(x_i, x_{\kappa+1}, \cdots, x_d) = \frac{\overline{C}_d\left(1-F_i(x_i), C_{d-h}(1-F_{\kappa+1}(x_{\kappa+1}), \cdots, 1-F_d(x_d)\right)}{\overline{C}_{d-\kappa}\left(1-F_{\kappa+1}(x_{\kappa+1}), \cdots, 1-F_d(x_d)\right)}. \tag{62}$$

Using the one-dimensional summary of $\overline{C}_{d-k}$, the corresponding density $\overline{f}_{X_i|X_{(d-\kappa)}}$ is given by

$$\overline{f}_{X_i|X_{(d-\kappa)}}(x_i, x_{d-\kappa}) = -\,\overline{c}(u_i\,|1-v_{d-\kappa})f_i(x_i)\big|_{v_{d-\kappa}=\alpha}. \tag{63}$$

It follows that,

$$\zeta^U_{\kappa,i}(\alpha) = \int_0^{1-\alpha} \mathrm{VaR}_{1-\alpha}(X_i)\overline{c}(u_i\,|1-\alpha)du_i, \tag{64}$$

then, finally,

$$\zeta^U_{\kappa,i}(\alpha) = \frac{\partial}{\partial\overline{v}}\int_0^{1-\alpha} \mathrm{VaR}_{1-\alpha}(X_i)\overline{c}(u_i,\overline{v})du_i\big|_{\overline{v}=1-\alpha}. \tag{65}$$

The conclusion that follows is a result of Proposition 5.

Corollary 6 Let $\mathbf{X} = (X_1, \cdots, X_d)$ be a random vector, with distribution F and corresponding Archimedean copula C_F. For all $i = 1, \cdots, \kappa$ with $\kappa \leq d$, then,

(i) if C_F is an Archimedean copula and $F(\mathbf{X})$ distributed as $K_d(\alpha)$, $\alpha \in [0, 1]$,

$$F_{X_i|F(\mathbf{X})}(x_i, \alpha) = F_{X_i|X_{(\kappa+1)}}(x_i, x_{\kappa+1}, \cdots, x_d), \tag{66}$$

(ii) for an Archimedean survival copula $\bar{C}_{\bar{F}}$ and distribution function $K_{\bar{C}}(1-\alpha)$ of $\bar{F}(\mathbf{X})$,

$$\bar{F}_{X_i|\bar{F}(\mathbf{X})}(x_i, 1-\alpha) = \bar{F}_{X_i|X_{(\kappa+1)}}(x_i, x_{\kappa+1}, \cdots, x_d). \tag{67}$$

Proof.

(i) Let $\mathbf{X} = (X_1, \cdots, X_d)$ be a random vector with marginal distributions F_i, $i = 1, \cdots, d$ and distribution F. $F(\mathbf{X})$ is the multivariate integral transformation of vector $\mathbf{X}$ with distribution function $K_d(\alpha)$, $\alpha \in [0, 1]$ and C is the Archimedean copula associated to distribution F. Then, for all $i = 1, \cdots, \kappa$ with $\kappa \le d$, the conditional distribution associated to $(X_i | X_{\kappa+1}, \cdots, X_d)$ is given by:

$$F_{X_i|X_{(\kappa+1)}}(x_i, x_{\kappa+1}, \cdots, x_d) = \frac{\mathbb{P}(X_i \le x_i, X_{\kappa+1} \le x_{\kappa+1}, \cdots, X_d \le x_d)}{\mathbb{P}(X_{\kappa+1} \le x_{\kappa+1}, \cdots, X_d \le x_d)}, \tag{68}$$

Hence, it follows from Sklar's theorem that

$$F_{X_i|X_{(\kappa+1)}}(x_i, x_{\kappa+1}, \cdots, x_d) = \frac{C_d\left(F_i(x_i), F_{\kappa+1}(x_{\kappa+1}), \cdots, F_d(x_d)\right)}{C_{d-\kappa}\left(F_{\kappa+1}(x_{\kappa+1}), \cdots, F_d(x_d)\right)} \tag{69}$$

as C is an Archimedean copula, by using the representation of the theorem in [15], it follows that,

$$C\left(F_{X_i}(x_i), F_{\kappa+1}(x_{\kappa+1}), \cdots, F_d(x_d)\right) = C(u_1, u_2) \tag{70}$$

where $u_1 = F_{X_i}(x_i)$, and $u_2 = C_{d-\kappa}\left(F_{\kappa+1}(x_{\kappa+1}), \cdots, F_d(x_d)\right)$.

Suppose that, $C_{d-\kappa}\left(F_{\kappa+1}(x_{\kappa+1}), \cdots, F_d(x_d)\right) \le \alpha$. Therefore, $u_2 = K_C(\alpha)$.

Finally,

$$F_{X_i|X_{(\kappa+1)}} = F_{X_i|F(\mathbf{X})}. \tag{71}$$

(ii) For the conditional distribution of $(X_i | X_{(\kappa+1)} \ge ., \cdot, X_d \ge .)$, it comes that,

$$\bar{C}\left(1-F_i(x_i), 1-F_{\kappa+1}(x_{\kappa+1}), \cdots, 1-F_d(x_d)\right) = \bar{C}(\bar{u}_1, \bar{u}_2), \tag{72}$$

where $\bar{u}_1 = 1 - F_i(x_i)$ and $\bar{u}_2 = \bar{C}_{d-\kappa}\left(1-F_{\kappa+1}(x_{\kappa+1}), \cdots, 1-F_d(x_d)\right)$, we suppose $\bar{C}_{d-\kappa}\left(1-F_{\kappa+1}(x_{\kappa+1}), \cdots, 1-F_d(x_d)\right) \le \alpha$. So it comes that $\bar{u}_2 = K_{\hat{C}}(1-\alpha)$.

And finally, it comes that

$$\bar{F}_{X_i|X_{(\kappa+1)}} = \bar{F}_{X_i|F(\bar{X})},$$

as disserted.

4. Conclusion and discussion

In a multivariate framework, stochastic financial modelling of quantile risk measures requires to consider information on the type of extremal dependence structures. Taking into consideration the extreme dependence structure can help to avoid extreme losses or to reduce their magnitudes. This paper contributed to extreme risk modelling. Thus, by using Kendall distribution to model extremal dependence structure of losses, two versions of multidimensional VaR proposed are similar to the versions that exist in the univariate frameworks. The lower tail of VaR is constructed from level sets of multivariate distribution functions whereas the VaR of upper tail is constructed from level sets of multivariate survival functions. Some properties of these risk measures have been analysed. In particular, the conditional distribution of loss is given in terms of copulas. We have also shown that a distribution function of loss random vector is a decreasing function of the probability level of VaR. Due to the representation theorem of Archimedean copulas, a connection is established between the two versions of multidimensional VaR and tails dependence functions.

The proposed VaR measures can be used in risk management applications. In particular, considering the literature in modelling risk using Gaussian distribution functions and whole hyperspace, the current work presents a bridge between these two.

Conflict of interest

There is no conflict of interest in this study.

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