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Research Article

Geostatistical Framework for Estimating Dependence of Multivariate Structures

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Abstract: Geostatistics is a subdomain of statistics that deals with phenomena which have spatial or spatiotemporal distribution. This paper proposes an extension of properties of the multivariate F-alpha variogram within a spatial framework and in a max-stable random field with unit Fréchet margins. We use the copula associated with the distribution of the fields to model the multivariate F-alpha variogram while important properties of this dependence measure are discussed for spatial extreme events. The hypothesis of normality and the consistency of two non-natural estimators are investigated.

***Keywords*:** max stable processes, variogram, extreme values, spatial dependence, copulas

MSC: 65D17, 86A32, 97K60, 62H05, 62G32

1. Introduction

Historically, geostatistics was used for statistical modeling in the mining sector. Its applications then spread to other fields such as geomorphology, cartography, hydrogeology, soil science, the petroleum industry, finance, and climate science [1]; variograms help to select the most appropriate. This approach is inappropriate for modeling the extremes of stochastic processes. Extreme values theory (EVT) is a vast domain whose goal is to study uncommon events. As for extreme value copulas, they are a practical choice for modelling extreme events as well as positively dependent data. For example, the extreme-value copulas are essential in the dependence structure between exceptional events because they provide appropriate models. Copulas are increasingly used in hydrology but significant results are yet to be discovered.

The first-order variogram was introduced by Matheron [2] to overcome the deficiency of the variogram, namely the extreme observations. It determines the strength of the relationship between random variables represented in the different observations. While modeling spatial extreme variability of an isotropic and max-stable field, Cooley et al. [3] introduced the F-madogram as a substitute which extends the madogram. So, for a random field Z and for $x_1, x_2 \in \mathbb{R}$.

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$$V_F(x_1 - x_2) = \frac{1}{2}\mathbb{E}\left[\mid F_{x_1}\{Z(x_1)\} - F_{x_2}\{Z(x_2)\} \mid \right]. \tag{1}$$

where F_x denotes the cumulative distribution function of the random variable $Z(x)$. The bivariate extrema coefficient can be estimated from the first-order variogram to demonstrate the link between geostatistics and EVT. In the same way, Naveau et al. [4] proposed the λ-madogram while modeling the pairwise dependence of maxima in space. Recently, Fonseca et al. [5] introduced a new measure, the multivariate F-alpha variogram which is used to evaluate the dependence between extreme observations located in two separated regions of locations of $\mathbb{R}^2$. Copulas and max-stables processes have been the subject of several studies including those by Lazoglou et al. [6], Diakarya et al. [7], and Ribatet et al. [8] who achieved excellent results. Indeed, while studying geostatistical analysis with conditional extremal copulas, Barro et al. [9] characterized the λ-madogram of process distribution under a hypothesis of distortion by using spatial extreme value copulas. After that, the generalisation of madogram was proposed by Boulin et al. [10] by defining this quantity:

$$\nu(w) = \mathbb{E}\left[\bigvee_{j=1}^{d} \{F_j(X_j)\}^{1/w_j} - \frac{1}{d}\sum_{j=1}^{d} \{F_j(X_j)\}^{1/w_j} \right], \tag{2}$$

if $w_j = 0$ and $0 < u < 1$, then $u^{1/w_j} = 0$. The variogram shows shortcomings when the observations are extreme. Indeed, when the random field is max-stable with unitary Fréchet marginal laws, the variogram does not theoretically exist. Furthermore, the other dependence measures such as madogram and λ-madogram do not allow the analysis of the dependence between extremes in two different localities. It then becomes necessary to introduce a new tool.

In this paper, we are particularly interested in the class of distributions of extreme values in a spatial context. Max-stable processes are ideal candidates for modeling spatial extremes. While analyzing and characterizing the dependence of the spatial structure of observations, some tools are introduced such as variogram, correlogram, and madogram. In this study, the major contribution is the modeling of the multivariate F-alpha variogram and an estimation of its parameters using a max-stable random field with unit Fréchet margin. Our work will be structured as follows. In Section 2, we briefly present some spatial geostatistical tools, useful in modeling. Section 3 presents the main results of our study. We model the multivariate F-alpha variogram using an underlying copula function. Moreover, we discuss the dependence of extreme spatial events via the multivariate F-alpha variogram.

2. Preliminaries

In this section, we present some important geostatistical tools using in spatial dependence. Furthermore, an overview of max-stable processes is proposed.

2.1 *A survey of geostatistical tools for spatial dependence*

In geostatistics studies, the variogram is a good tool to measure the spatial dependence between two realizations of a random field. It is used mainly to display the variability between data points as a function of distance and is given by:

$$\gamma(h) = \frac{1}{2}Var\big(Z(s+h) - Z(s)\big),\ s,\ (s+h) \in \chi. \tag{3}$$

F-madogram, is one of the better tools important when the univariate marginal probability laws of a stationary max-stable random field are of the unit Fréchet. It is related to the extremal coefficient function θ by

$$\theta(\| s_1 - s_2 \|) = \frac{1 + V_F(\| s_1 - s_2 \|)}{1 - V_F(\| s_1 - s_2 \|)}. \tag{4}$$

The tool which provides an extension of λ-madogram [5] is multivariate F-alpha variogram, defined as for all sites x and y like

$$\nu^{\alpha,\beta}(x, y) = \frac{1}{2}\mathbb{E}\left[| F^{\alpha}\left(M(x)\right) - F^{\beta}\left(M(y)\right) | \right],\ \alpha > 0,\ \beta > 0; \tag{5}$$

where

$$M(x) = \max_{1 \le i \le k}\left[Z(x_i)\right].$$

The max-stable processes are ideal candidates for the modeling of spatial extremes. Specifically, max-stable $Z = \{Z_x : x \in \mathbb{R}^d\}$ is the limit process of maxima of independent and identically distributed (i.i.d) random fields $Y_x^{(i)}$, $x \in \mathbb{R}^d$, $i = 1, \ldots, n$. Particularly, the distribution of $Z(t)$ has for some $a(t) > 0$ and $b(t)$ with $t \in [0, 1]$, $\gamma(t) \in \mathbb{R}$ a Von Mises representation [11].

$$\mathbb{P}\left(\frac{Z(t) - b(t)}{a(t)} \le x\right) = F_{\gamma(t)}(x),\ \gamma x \ge -1, \tag{6}$$

$\gamma \in \mathbb{R}$, where for suitable $a_n(x) > 0$ and $b_n(x) \in \mathbb{R}$,

$$Z_x = \lim_{n\to\infty}\frac{\max_{1\le i\le n}(Y_x^i) - b_n(x)}{a_n(x)},\ x \in \mathbb{R}^d, \tag{7}$$

and where,

$$F_{\gamma(t)}(x) = \exp\left(-\left(1+\gamma(t)x\right)^{-1/\gamma(t)}\right) = \begin{cases} 0 & \gamma(t) > 0 \quad x \le -1/\gamma(t), \\ 1 & \gamma(t) < 0 \quad x \ge -1/\gamma(t), \\ exp\left(-exp(-x)\right) & \gamma(t) = 0 \quad x \in \mathbb{R}. \end{cases}$$

Ribatet et al. [8] made a number of statements when he wanted to build parametric max-stable models from spectral characterisation. He gave two-dimensional marginal probability of respectively the gaussian extreme value model, Schlather model and the processus extremal-t of η in two sites $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$ which are given, for any $z_1 \in \mathbb{R}_+^*$ and $z_2 \in \mathbb{R}_+^*$ by

$$\mathbb{P}\left(\eta(s_1) \le z_1,\ \eta(s_2) \le z_2\right) = \begin{cases} \exp\left\{-\left[\frac{1}{z_1}\Phi\left(\frac{a}{2}+\frac{1}{a}log\left(\frac{z_2}{z_1}\right)\right)+\frac{1}{z_2}\Phi\left(\frac{a}{2}+\frac{1}{a}log\left(\frac{z_1}{z_2}\right)\right)\right]\right\} \\ \exp\left\{-\frac{1}{2}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)\left(1+\sqrt{1-2\left(\rho(h)+1\right)\frac{z_1 z_2}{(z_1+z_2)^2}}\right)\right\} \\ \exp\left[-\frac{1}{z_1}T_{\nu+1}(\tilde{z}_1)-\frac{1}{z_2}T_{\nu+1}(\tilde{z}_2)\right] \end{cases} \tag{8}$$

where $\tilde{z}_i = -\frac{\rho(s_1 - s_2)}{b} + \frac{1}{b}\left(\frac{z_j}{z_i}\right)^{1/\nu}$ for i, j = 1, 2 with $i \neq j$, $h = \| s_1 - s_2 \| \in \mathbb{R}^+$, Φ is the distribution function of a law $N(0; 1)$, $a^2 = (s_1 - s_2)^T \Sigma^{-1}(s_1 - s_2)$ the distance from Mahalanobis, T_ν is the distribution function of a student with ν degrees of freedom and $b^2 = \left\{1 - \rho(s_1 - s_2)^2\right\} / (\nu + 1)$.

The space-time max-stable process $Z(s, t)$ is characterized by the variogram. In geostatistics, the dependence function γ is given by

$$\gamma(s_1 - s_2, t_1 - t_2) = \frac{1}{2}\mathbb{V}ar\left(\varepsilon(s_1, t_1) - \varepsilon(s_2, t_2)\right).$$

According to [12], let Φ be the standard normal distribution function.

For x_1, $x_2 > 0$, the bivariate cumulative distributive function ($F_{h,l}$) of $(Z(s_1, t_1), Z(s_2, t_2))$ in the stationary case is given by:

$$-\log F_{h,l}(x_1, x_2) = \frac{1}{x_1}\Phi\left(\sqrt{\frac{\gamma(h,l)}{2}} + \frac{\log\left(\frac{x_2}{x_1}\right)}{\sqrt{2\gamma(h,l)}}\right) + \frac{1}{x_2}\Phi\left(\sqrt{\frac{\gamma(h,l)}{2}} + \frac{\log\left(\frac{x_1}{x_2}\right)}{\sqrt{2\gamma(h,l)}}\right). \tag{9}$$

3. The main results of the study

The results reported in this section are other variants of the multivariate F-alpha variogram which are very useful for the spatial dependence of extreme events.

3.1 *New characterization of the multivariate F-alpha variogram*

Let $Z = \{Z_x\}_{x \in \mathbb{R}^2}$ be a max-stable random field with unit Fréchet margins and $x = \{x_i\}_{i=1,\ldots,k}$ and $y = \{y_j\}_{j=1,\ldots,s}$ be the two disjoint regions of $\mathbb{R}^2$. $\hat{U}_{x_i} = \frac{-1}{\log\left(\hat{F}_{x_i}(Z_{x_i}^t)\right)}$ as the empirical Fréchet normalization of the variables and $W_1, \ldots, W_T$ as independent copies.

Proposition 1 Z being a max-stable random field with unit Fréchet margins and F a marginal distribution, if $\alpha + \beta = 1$, then the multivariate F-alpha variogram is equivalent to λ-madogram.

Proof. From the definition of the multivariate F-alpha variogram we have

$$\nu^{\alpha,\beta}(x, y) = \frac{1}{2}\mathbb{E}\left[\left|F^\alpha\left(M(x)\right) - F^\beta\left(M(y)\right)\right|\right].$$

For $\beta = 1 - \alpha$, $\alpha \in (0, 1)$, we obtain:

$$\nu^{\alpha,1-\alpha}(x, y) = \frac{1}{2}\mathbb{E}\left[\left|F^\alpha\left(M(x)\right) - F^{1-\alpha}\left(M(y)\right)\right|\right].$$

In addition, $M(x) = \bigvee_{i=1}^k Z_{x_i}$ and $M(y) = \bigvee_{j=1}^s Z_{y_j}$. Let's take $k = s = 1$. Then $M(x) = \bigvee_{i=1}^1 Z_{x_i} = Z_{x_1} = Z_x$ and $M(y) = \bigvee_{j=1}^1 Z_{y_j} = Z_{y_1} = Z_y$. It follows that

$$\nu^{\alpha,1-\alpha}(x,y)=\frac{1}{2}\mathbb{E}\left[\,|\,F^{\alpha}\left(Z(x)\right)-F^{1-\alpha}\left(Z(y)\right)|\,\right].$$

Hence $\nu^{\alpha,1-\alpha}(x,y)$ is a λ-madogram.

The multivariate F-alpha variogram is a new dependency measurement that extends the existing λ-madogram concept. The following result motivates the definition of natural estimators.

Proposition 2 The multivariate F-alpha variogram $\nu^{\alpha,\beta}$ associated to two regions x and y is obtained as follows:

$$\nu^{\alpha,\beta}(x,y)=\frac{1}{2}\mathbb{E}\left[\,|\,F_{\alpha}^{*}(x_i)-F_{\beta}^{*}(y_i)\,|\,\right]. \tag{10}$$

Proof. It comes that,

$$\nu^{\alpha,\beta}(x,y)=\frac{1}{2}\mathbb{E}\left[\,|\,F^{\alpha}\left(M(x)\right)-F^{\beta}\left(M(y)\right)|\,\right].$$

Equivalently,

$$\nu^{\alpha,\beta}(x,y)=\frac{1}{2}\mathbb{E}\left[\,|\,F^{\alpha}\left(\bigvee_{i=1}^{k}Z_{x_i}\right)-F^{\beta}\left(\bigvee_{j=1}^{s}Z_{y_j}\right)|\,\right].$$

and otherwise using one of the properties of the maximum, it follows that

$$\nu^{\alpha,\beta}(x,y)=\frac{1}{2}\mathbb{E}\left[\,|\bigvee_{i=1}^{k}F^{\alpha}(Z_{x_i})-\bigvee_{j=1}^{s}F^{\beta}(Z_{y_j})\,|\,\right].$$

Hence, one obtains

$$\nu^{\alpha,\beta}\left(x,y\right)=\frac{1}{2}\mathbb{E}\left[\,|\bigvee_{i=1}^{k}F\left(\frac{Z_{x_i}}{\alpha}\right)-\bigvee_{j=1}^{s}F\left(\frac{Z_{y_j}}{\beta}\right)|\,\right].$$

F being a marginal distribution, we have $F^{\alpha}(Z_{x_i})=F\left(\frac{Z_{x_i}}{\alpha}\right)$. This means that,

$$\nu^{\alpha,\beta}\left(x,y\right)=\frac{1}{2}\mathbb{E}\left[\,|\bigvee_{i=1}^{k}F\left(\frac{Z_{x_i}}{\alpha}\right)-\bigvee_{j=1}^{s}F\left(\frac{Z_{y_j}}{\beta}\right)|\,\right].$$

So, by setting $F_{\alpha}^{*}(x_i)=\bigvee_{i=1}^{k}F\left(\frac{Z_{x_i}}{\alpha}\right)$ and $F_{\beta}^{*}(y_i)=\bigvee_{j=1}^{s}F\left(\frac{Z_{y_j}}{\beta}\right)$ it comes,

$$\nu^{\alpha,\beta}(x,y)=\frac{1}{2}\mathbb{E}\left[\,|\,F_{\alpha}^{*}(x_i)-F_{\beta}^{*}(y_i)\,|\,\right].$$

Extremal coefficient function, $\theta(.)$ comes from the extremal coefficient notion. The multivariate F-alpha variogram function can be used in geostatistics to characterize the spatial structure of a process, $Z(.)$. The following result provides a relationship between θ and ν and is a useful property in modeling the dependence of spatial extreme events.

Proposition 3 Let θ be an extremal coefficient and $\nu^{\alpha,\beta}$ the multivariate F-alpha variogram. It follows that:

$$\nu^{\alpha,\alpha}(x,y)=\frac{\theta_{x\cup y}}{\alpha+\theta_{x\cup y}}-\frac{1}{2}h_{\theta}(\alpha), \tag{11}$$

where h is a two-variable function with parameter α.

Proof. It's easy to show that, for all $m, n\in\mathbb{R}$, $|m-n|=2\max(m,n)-m-n$

$$|F^{\alpha}(M(x))-F^{\alpha}(M(y))|=2\max\left(F^{\alpha}(M(x)),F^{\alpha}(M(y))\right)-F^{\alpha}(M(x))-F^{\alpha}(M(y)).$$

Let's move on to the mathematical expectation, it comes that

$$\mathbb{E}\left[|F^{\alpha}(M(x))-F^{\alpha}(M(y))|\right]=\mathbb{E}\left[2\max\left(F^{\alpha}(M(x)),F^{\alpha}(M(y))\right)-F^{\alpha}(M(x))-F^{\alpha}(M(y))\right].$$

Equivalently, one has

$$\mathbb{E}\left[|F^{\alpha}(M(x))-F^{\alpha}(M(y))|\right]=2\mathbb{E}\left[\max\left(F^{\alpha}(M(x)),F^{\alpha}(M(y))\right)\right]-\mathbb{E}\left[F^{\alpha}(M(x))\right]-\mathbb{E}\left[F^{\alpha}(M(y))\right].$$

By replacing F, one has

$$\mathbb{E}\left[|F^{\alpha}(M(x))-F^{\alpha}(M(y))|\right]=2\mathbb{E}\left[F\left(\max\left(\frac{M(x)}{\alpha},\frac{M(y)}{\alpha}\right)\right)\right]-\mathbb{E}\left[F\left(\frac{M(x)}{\alpha}\right)\right]-\mathbb{E}\left[F\left(\frac{M(y)}{\alpha}\right)\right].$$

Moreover, since one has

$$\nu^{\alpha,\beta}(x,y)=\frac{1}{2}\mathbb{E}\left[|F^{\alpha}(M(x))-F^{\beta}(M(y))|\right].$$

Then, it comes that

$$\nu^{\alpha,\alpha}(x,y)=\frac{1}{2}\mathbb{E}\left[|F^{\alpha}(M(x))-F^{\alpha}(M(y))|\right].$$

So, we have

$$\nu^{\alpha,\alpha}(x,y)=\mathbb{E}\left[F\left(\max\left(\frac{M(x)}{\alpha},\frac{M(y)}{\alpha}\right)\right)\right]-\frac{1}{2}\left(\mathbb{E}\left[F\left(\frac{M(x)}{\alpha}\right)\right]+\mathbb{E}\left[F\left(\frac{M(y)}{\alpha}\right)\right]\right).$$

Let's express the three esperances,

$$\mathbb{P}\left(\frac{M(x)}{\alpha}\leq z\right)=\mathbb{P}\left[M(x)\leq\alpha z\right].$$

If G_x is the extreme value model describing the asymptotic behaviour of M, it comes that

$$lim_{n\to+\infty}\mathbb{P}\left(\frac{M(x)}{\alpha}\leq z\right)=G_x\left(\alpha z,\ldots,\alpha z\right).$$

We already know from the multivariate extreme value theory that if G_x is an extreme value distribution with Fréchet unit margin then, one has

$$\mathbb{P}\left(\frac{M(x)}{\alpha}\leq z\right)=\exp\left\{-z^{-1}V_x(\alpha,\ldots,\alpha)\right\},\, z>0.$$

where V is a homogeneous function of order -1, i.e, $V(nu_1,\ldots,nu_k)=n^{-1}V(u_1,\ldots,u_k)$ for all $n > 0$, and is known as the exponent function. Hence, it gives

$$\mathbb{E}\left[F\left(\frac{M(x)}{\alpha}\right)\right]=\int_0^{+\infty}F(z)\exp\left(-V_x(\alpha z,\ldots,\alpha z)\right)\frac{d}{dz}\left(-V_x(\alpha z,\ldots,\alpha z)\right);$$

which provides

$$\mathbb{E}\left[F\left(\frac{M(x)}{\alpha}\right)\right]=\int_0^{+\infty}\exp\left(-z^{-1}-z^{-1}V_x(\alpha,\ldots,\alpha)\right)z^{-2}V_x(\alpha,\ldots,\alpha)dz;$$

thus,

$$\mathbb{E}\left[F\left(\frac{M(x)}{\alpha}\right)\right]=\frac{V_x(\alpha,\ldots,\alpha)}{1+V_x(\alpha,\ldots,\alpha)};$$

as a result,

$$\mathbb{E}\left[F\left(\frac{M(x)}{\alpha}\right)\right]=\frac{V_x(1,\ldots,1)}{\alpha+V_x(1,\ldots,1)}.$$

In a similar way,

$$\mathbb{E}\left[F\left(\frac{M(y)}{\alpha}\right)\right]=\frac{V_y(1,\ldots,1)}{\alpha+V_y(1,\ldots,1)};$$

and one also has,

$$\mathbb{E}\left[F\left(\max\left(\frac{M(x)}{\alpha},\frac{M(y)}{\alpha}\right)\right)\right]=\frac{V_{x,y}(\alpha,\ldots,\alpha)}{1+V_{x,y}(\alpha,\ldots,\alpha,\alpha,\ldots,\alpha)}.$$

either,

$$\mathbb{E}\left[F\left(\max\left(\frac{M(x)}{\alpha},\frac{M(y)}{\alpha}\right)\right)\right]=\frac{V_{x,y}(1,\ldots,1,1,\ldots,1)}{1+V_{x,y}(1,\ldots,1)}.$$

Extremal coefficient is independent of z by homogeneity of V_x [9], i.e

$$\theta_x = V_x(1,\ldots,1), \theta_y = V_y(1,\ldots,1),$$

and

$$\theta_{x\cup y} = V_{x,y}(1,\ldots,1).$$

Thus,

$$\mathbb{E}\left[F\left(\max\left(\frac{M(x)}{\alpha},\frac{M(y)}{\alpha}\right)\right)\right]=\frac{\theta_{x\cup y}}{\alpha+\theta_{x\cup y}},\ \mathbb{E}\left[F\left(\frac{M(x)}{\alpha}\right)\right]=\frac{\theta_x}{\alpha+\theta_x},$$

and

$$E\left[F\left(\frac{M(y)}{\alpha}\right)\right]=\frac{\theta_y}{\alpha+\theta_y}.$$

Let's set

$$h_\theta(\alpha)=\frac{\theta_x}{\alpha+\theta_x}+\frac{\theta_y}{\alpha+\theta_y}.$$

It comes then that,

$$\nu^{\alpha,\alpha}(x,y)=\frac{\theta_{x\cup y}}{\alpha+\theta_{x\cup y}}-\frac{1}{2}h_\theta(\alpha),$$

as disserted.

From a statistical point of view, the following proposition is interesting because it suggests an obvious and nice estimator.

3.2 *Estimation of multivariate F-alpha variogram values*

Some estimators of madogram have been studied in the literature. In this section, we give two estimators of the new dependence measure.

Definition 1 Let $(Z_{x_1}^{(t)},\ldots,Z_{x_k}^{(t)}), t=1,\ldots,T$ and F a marginal distribution. An estimator of multivariate F-alpha variogram is given by:

$$\hat{\nu}^{\alpha,\beta}(x,y)=\frac{1}{2T}\sum_{i=1}^{T}|F^\alpha\left(M_i(x)\right)-F^\alpha\left(M_i(y)\right)|,\ \alpha,\beta\in\mathbb{R}^+ \tag{12}$$

where $M_i(x) = \bigvee_{i=1}^{k} Z_{x_i}^{(t)}$ for $t = 1,\ldots,T$ are random samples of $M(x)$ and $M(y)$.

The following is a consequence of the previous result.

Corollary 1 Based on the assumption of Proposition 3.1 in [5], the following convergences hold

$$\frac{\sqrt{T}\left(\hat{\nu}^{\alpha,\alpha}(x,y) - \nu^{\alpha,\alpha}(x,y)\right)}{\sigma} \to N(0,1). \tag{13}$$

And

$$\frac{\sqrt{T}\left(\hat{\hat{\nu}}^{\alpha,\beta}(x,y) - \hat{\nu}^{\alpha,\beta}(x,y)\right)}{\sigma} \to N(0,1), \tag{14}$$

where $\sigma^2 = \frac{1}{2}\gamma_F^{\alpha,\alpha}(x,y) - \left(\nu^{\alpha,\alpha}(x,y)\right)^2$ and $\gamma_F^{\alpha,\alpha}(x,y) = \frac{1}{2}E\left[\left(F^{\alpha}\left(M(x)\right) - F^{\alpha}\left(M(y)\right)\right)^2\right]$.

Remark 1 $\hat{\nu}^{\alpha,\alpha}(x,y)$ converges almost surely to $\nu^{\alpha,\alpha}(x,y)$.

Now consider distribution of Z_{x_i} and F_{x_i} is unknown.

Definition 2 Let $\hat{U}_{x_i}$ be the Fréchet empirical normalization of variables where $\hat{F}_{x_i}(Z_{x_i}^t)$ is the empirical distribution function, the modified estimator is:

$$\hat{\hat{\nu}}^{\alpha,\beta}(x,y) = \frac{1}{2T}\sum_{i=1}^{T} \left| \bigvee_{i=1}^{k} \hat{F}_{x_i}^{\alpha}(Z_{x_i}^{(t)}) - \bigvee_{j=1}^{s} \hat{F}_{y_j}^{\beta}(Z_{y_j}^{(t)}) \right|, \ \alpha,\beta \in \mathbb{R}^+. \tag{15}$$

where $\hat{F}_{x_i}(u) = \frac{1}{T}\sum_{t=1}^{T} \mathbf{1I}_{\left\{Z_{x_i}^{(t)} \le u\right\}}$.

Proof. Let $W_1,\ldots,W_T$ be independent copies of $W = \frac{1}{2T}\ |F^{\alpha}\left(M_i(x)\right) - F^{\beta}(M_i(y))|$.

It comes then that,

$$\frac{\sqrt{T}(\bar{W} - \mu_W)}{\sigma_W} \to N(0,1),$$

where $\mu_W = \frac{1}{2T}E\left|F^{\alpha}\left(M_i(x)\right) - F^{\beta}\left(M_i(y)\right)\right| = \hat{\nu}^{\alpha,\beta}(x,y)$ and $\sigma_W^2 = \frac{1}{2}\gamma_F^{\alpha,\beta}(x,y) - \left(\hat{\nu}^{\alpha,\beta}(x,y)\right)^2$.

The following result shows the strong consistency under unknown marginal distribution.

Proposition 4 Let's consider two estimators of the multivariate F-alpha variogram obtained previously, one has $\hat{\hat{\nu}}^{\alpha,\alpha}(x,y)$ converges almost surely to $\hat{\nu}^{\alpha,\alpha}(x,y)$.

Proof. Pretty proof by its simplicity.

$$\left|\hat{\hat{\nu}}^{\alpha,\alpha}(x,y) - \hat{\nu}^{\alpha,\alpha}(x,y)\right| = \left|\frac{1}{2}\frac{1}{T}\sum_{i=1}^{T}\left|\bigvee_{i=1}^{k}\hat{F}_{x_i}^{\alpha}(Z_{x_i}^{(t)}) - \bigvee_{j=1}^{s}\hat{F}_{y_j}^{\alpha}(Z_{y_j}^{(t)})\right| - \frac{1}{2}\frac{1}{T}\sum_{i=1}^{T}\left|F^{\alpha}\left(M_i(x)\right) - F^{\alpha}\left(M_i(x)\right)\right|\right|.$$

By definition of the small absolute value,

$$\left|\hat{\hat{\nu}}^{\alpha,\alpha}(x,y) - \hat{\nu}^{\alpha,\alpha}(x,y)\right| \le \frac{1}{2}\frac{1}{T}\sum_{i=1}^{T}\bigvee_{i=1}^{k}\hat{F}_{x_i}^{\alpha}(Z_{x_i}^{(t)}) - \bigvee_{j=1}^{s}\hat{F}_{y_j}^{\alpha}(Z_{y_j}^{(t)}) - \left|\bigvee_{i=1}^{k}F^{\alpha}(Z_{x_i}^{(t)}) - \bigvee_{j=1}^{s}F^{\alpha}(Z_{y_j}^{(t)})\right|\|.$$

In an equivalent way, it follows that

$$|\hat{\hat{v}}^{\alpha,\alpha}(x,y)-\hat{v}^{\alpha,\alpha}(x,y)|\leq\frac{1}{2}\frac{1}{T}\sum_{i=1}^{T}\bigvee_{i=1}^{k}\hat{F}^{\alpha}_{x_i}(Z^{(t)}_{x_i})-\bigvee_{j=1}^{s}\hat{F}^{\alpha}_{y_j}(Z^{(t)}_{y_j})-\bigvee_{i=1}^{k}F^{\alpha}(Z^{(t)}_{x_i})+\bigvee_{j=1}^{s}F^{\alpha}(Z^{(t)}_{y_j})|.$$

which gives

$$|\hat{\hat{v}}^{\alpha,\alpha}(x,y)-\hat{v}^{\alpha,\alpha}(x,y)|\leq\frac{1}{2}\frac{1}{T}\sum_{i=1}^{T}\left[\bigvee_{i=1}^{k}|\hat{F}^{\alpha}_{x_i}(Z^{(t)}_{x_i})-F^{\alpha}(Z^{(t)}_{x_i})|+\bigvee_{j=1}^{s}|\hat{F}^{\alpha}_{y_j}(Z^{(t)}_{y_j})-F^{\alpha}(Z^{(t)}_{y_j})|\right].$$

So that,

$$|\hat{\hat{v}}^{\alpha,\alpha}(x,y)-\hat{v}^{\alpha,\alpha}(x,y)|\leq\frac{1}{2}\frac{1}{T}\sum_{i=1}^{T}\left[\sum_{i=1}^{k}\hat{F}^{\alpha}_{x_i}(Z^{(t)}_{x_i})-F^{\alpha}(Z^{(t)}_{x_i})|+\sum_{j=1}^{s}\hat{F}^{\alpha}_{y_j}(Z^{(t)}_{y_j})-F^{\alpha}(Z^{(t)}_{y_j})|\right].$$

It is known that in one hand, $\frac{1}{T}\sum_{i=1}^{T}X_i$ converges almost surely to $E|X_1|$ where X_i, t = 1,...,T, are i.i.d random variables with $E|X_1| < \infty$. And in the other hand, the empirical distribution function is strong consistent. So, the quantity $|\hat{\hat{v}}^{\alpha,\alpha}(x,y)-\hat{v}^{\alpha,\alpha}(x,y)|$ converges almost surely zero.

3.3 *Copulas multivariate F-alpha variogram modeling*

A copula function is a tool of stochastic modeling which can be used to describe the dependence of variables or used for spatial interpolation, by coupling marginal distributions and forming their joint cumulative distribution function [13]. More specifically, every continuous multivariate distribution $F = (F_1,...,F_n)$ can be canonically parametrized by a copula C such as for all $x=(x_1,\dots,x_n)\in\mathbb{R}^n$,

$$F(x_1,\dots,x_n)=C\left(F_1^*(x_1),\dots,F_n^*(x_n)\right).$$

Or inversely, for all $(u_1,\dots,u_n)\in[0,1]^n$;

$$C(u_1,\dots,u_n)=F\left[F_1^{-1}(u_1),F_2^{-1}(u_2),\dots,F_n^{-1}(u_n)\right].$$

The following result provides a relationship between the multivariate F-alpha variogram [5] and the copula function.

Proposition 5 Let Z be a max-stable random field with unit Fréchet margins, x and y are two disjoint regions of $\mathbb{R}^2$. Then, the multivariate F-alpha variogram is such that,

$$v^{\alpha,\beta}(x,y)=\int_0^1 udC(u^{1/\alpha},u^{1/\beta})+g_\sigma(\alpha,\beta),\quad \alpha>0,\beta>0, \tag{16}$$

where g_σ is a convex function in two variables and C a copula.

Proof. By applying to the definition of $v^{\alpha,\beta}(x,y)$, the relation $|m-n|/2=m\bigvee n-(a+b)/2$, where $m\bigvee n=\max(m,n)$ we get:

$$v^{\alpha,\beta}(x,y)=\mathbb{E}\left(\left[F\left(M(x)\right)\right]^{\alpha}\bigvee\left[F\left(M(y)\right)\right]^{\beta}\right)-\frac{1}{2}\left\{\mathbb{E}\left(\left[F\left(M(x)\right)\right]^{\alpha}\right)+E\left(\left[F\left(M(y)\right)\right]^{\beta}\right)\right\}.$$

Furthermore,

$$F_M^{\alpha,\beta}(u) = \mathbb{P}\left(\left[F\left(M(x)\right)\right]^{\alpha} \vee \left[F\left(M(y)\right)\right]^{\beta} \leq u\right) \text{ for } u > 0.$$

This equality is equivalent to:

$$F_M^{\alpha,\beta}(u) = \mathbb{P}\left(\left[F\left(M(x)\right)\right]^{\alpha} \leq u, \left[F\left(M(y)\right)\right]^{\beta} \leq u\right).$$

In a more interesting way, it comes then that,

$$F_M^{\alpha,\beta}(u) = \mathbb{P}\left(F\left(M(x)\right) \leq u^{\frac{1}{\alpha}}, F\left(M(y)\right) \leq u^{\frac{1}{\beta}}\right).$$

And clearly, it follows that,

$$F_M^{\alpha,\beta}(u) = C\left(u^{\frac{1}{\alpha}}, u^{\frac{1}{\beta}}\right),$$

which amounts to,

$$\mathbb{E}\left(\left[F\left(M(x)\right)\right]^{\alpha} \vee \left[F\left(M(y)\right)\right]^{\beta}\right) = \int_0^1 u dC\left(u^{\frac{1}{\alpha}}, u^{\frac{1}{\beta}}\right).$$

In the same way, we have

$$\mathbb{E}\left(\left[F\left(M(x)\right)\right]^{\alpha}\right) = \frac{\sigma_{M(x)}^2}{1+\alpha\sigma_{M(x)}^2} \text{ and } \mathbb{E}\left(\left[F\left(M(y)\right)\right]^{\beta}\right) = \frac{\sigma_{M(y)}^2}{1+\beta\sigma_{M(y)}^2}.$$

So, by setting

$$g_\sigma(\alpha, \beta) = \frac{1}{4}\left[\frac{\sigma_{M(y)}^2}{1+\beta\sigma_{M(y)}^2} - \frac{\sigma_{M(x)}^2}{1+\alpha\sigma_{M(x)}^2}\right],$$

we obtain a convex function in two variables. So that,

$$\nu^{\alpha,\beta}(x, y) = \int_0^1 u dC(u^{1/\alpha}, u^{1/\beta}) + \frac{1}{4}\left[\frac{\sigma_{M(y)}^2}{1+\beta\sigma_{M(y)}^2} - \frac{\sigma_{M(x)}^2}{1+\alpha\sigma_{M(x)}^2}\right].$$

So, we obtain (16) as disserted.

4. Conclusion and discussion

This research article suggested an expansion of the characteristics of the multivariate F-alpha variogram within a spatial context and in a max-stable random field with unit Fréchet margins. Firstly, we used a max-stable random field with unit Fréchet margins and an underlying function copula to model the stochastic spatial tool. Secondly, some important properties of this measure for the dependence of spatial extreme events have been discussed. Finally, the paper examines the normality assumption and investigates the consistency of two non-natural estimators. They are well adapted to modeling the annual amount precipitation.

However, there are some limitations due to lack of practical applications of results obtained on data and the baseline assumptions imposed in the modelling approach. It would therefore, be interesting in our future research to go in this direction and why not extend the dependence of the multivariate F-alpha variogram to several regions or to use an asymptotic independent block model.

Whatever the case, important results are yet to be discovered because this tool is limited when data include more extreme values.

Conflict of interest

There is no conflict of interest in this study.

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