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Research Article

Classes of Ordinary Differential Equations for the Alpha Power Lomax Distribution

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Abstract: This paper focuses on the ordinary differential equations (ODEs) of the alpha-power Lomax distribution. The ODEs are derived from the alpha-power Lomax (APL) distribution for the density, quantile, survival and hazard functions. These ODEs helps to study the shape of various functions that describe the probability distribution. Under necessary conditions of the parameters, the existence of the ODEs is possible and obtained.

***Keywords*:** Alpha power transformation, calculus, Lomax distribution, ordinary differential equations

MSC: 34, 15A04, 15A06

1. Introduction

Several continuous life time distributions have been used to deal with the reliability modeling of various fields like engineering, medical science, business, finance, insurance, etc. The Lomax distribution is one of them that were originally introduced by Lomax [1] for the modeling the failure data in business and engineering. Harris [2] used the Lomax distribution for income and wealth data. Bryson [3] considered this distribution while modeling the heavy tailed data. Its applications were discussed in detail by Hassan and Ghamdi [4] in reliability modeling and life tests experiments. Besides, due to the widespread applications of the Lomax distribution, it was further modified so as to increase the flexibility of the data. For example, Poisson-Lomax distribution by Al-Awadhi and Ghitany [5], double Lomax distribution by Punathumparamhath [6]. For more detail study, we refer to see [7-20]. However, these generalizations are not adequate since their flexibilities are limited.

Several mathematical disciplines are significantly utilised probability distributions like Calculus, differential equations, algebra, measure theory, etc. Ordinary differential equations (ODE) have primarily been used up to now for parameter estimation and approximation. Approximation of quantile function features prominently in the use of ODE in approximation [21]. The ODEs of the distributions including beta distribution, Lomax distribution, beta prime distribution, Laplace distribution and raised cosine distribution can be found in literature [22]. However, the ODEs have not been studied before neither for the alpha-power Lomax distribution or other extensions of the Lomax distribution. Finally, it is noteworthy that spectral element method [23-27] due to its high accuracy as a numerical method can be

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utilized in modeling of similar problems.

The novelty of this paper to examine some classes of ODE, s for the APL distribution.

2. Ordinary differential equations of the alpha power lomax distribution

The cumulative distribution function (cdf) of APT is given by

$$F_{APT}(x)=\begin{cases}\dfrac{\alpha^{F(x)}-1}{\alpha-1} & \text{if } \alpha>0,\ \alpha\neq 1\\ F(x) & \text{if } \alpha=1\end{cases} \tag{1}$$

the probability density function (pdf) is given as

$$f_{APT}(x)=\begin{cases}\dfrac{\log(\alpha)f(x)\alpha^{F(x)}}{\alpha-1} & \text{if } \alpha>0,\ \alpha\neq 1\\ f(x) & \text{if } \alpha=1\end{cases} \tag{2}$$

Using the cdf of the Lomax distribution eq (2) yields the pdf of APL distribution

$$f(x)=\frac{\log(\alpha)\left(ab^{a}\right)\left(\alpha^{1-b^{a}(x+b)^{-a}}\right)}{(\alpha-1)(x+b)^{a+1}} \tag{3}$$

2.1 *ODEs for the PDF*

The first order ODEs Eq. (1) provides

$$f'(x)=f(x)\left[\frac{\log(\alpha)\left(ab^{a}\right)}{(x+b)^{a+1}}-\frac{(a+1)}{(x+b)}\right] \tag{4}$$

For the solution of Eq (4), it is required a, b, α , $x>0$ and $\alpha\neq 1$. The simplified form of Eq. (4) is

$$f'(x)=f(x)\left[\frac{(\log\alpha)\left(ab^{a}\right)}{(x+b)^{a+1}}-\frac{(a+1)}{(x+b)}\right] \tag{5}$$

Table 1. ODEs for the PDF with different parameter values

a	b	α	ODEs
1	1	2	$(x+1)f'(x)-1.8f(x)=0$
2	1	2	$(x+1)^3 f'(x)-\left(0.6-2(x+1)^2\right)f(x)=0$
1	2	2	$(x+2)^2 f'(x)-(0.6-2(x+2))f(x)=0$
2	2	2	$(x+2)^3 f'(x)-\left(2.4-3(x+2)^2\right)f(x)=0$

The ODEs for Eq. (5) can be solved only when the parameter values are specified. Table 1 describes a class of ODEs in respect of various parameter values.

We differentiate again Eq. (5) to get the second order derivative as

$$f''(x) = f(x)\left[(\log\alpha)\left(ab^a\right)(-a-1)(x+b)^{-(a+2)} + (a+1)(x+b)^{-2}\right] + \left[(\log\alpha)\left(ab^a\right)(x+b)^{-(a+1)} - (a+1)(x+b)^{-1}\right]f'(x) \tag{6}$$

Eq. (6) can be solved if a, b, α, $x > 0$ and $\alpha \neq 1$

$$f''(x) = \frac{(a+1)f(x)}{(x+b)^2} - \frac{(a+1)\left(ab^a\right)(\log\alpha)f(x)}{(x+b)^{a+2}} + \frac{\left(ab^a\right)(\log\alpha)f'(x)}{(x+b)^{a+1}} - \frac{(a+1)f'(x)}{(x+b)} \tag{7}$$

The simplified form of Eq. (7) becomes

$$f''(x) = \left[\frac{\left(ab^a\right)(\log\alpha)}{(x+b)^{a+1}} - \frac{(a+1)}{(x+b)}\right]f'(x) + \left[\frac{(a+1)}{(x+b)^2} - \frac{(a+1)\left(ab^a\right)(\log\alpha)}{(x+b)^{a+2}}\right]f(x) \tag{8}$$

Eq. (4) can simplify as

$$\frac{f'(x)}{f(x)} = \left[\frac{(\log\alpha)\left(ab^a\right)}{(x+b)^{a+1}} - \frac{(a+1)}{(x+b)}\right] \tag{9}$$

Substitute Eq. (9) into Eq. (8), we get

$$f''(x) = \left[\frac{f'(x)}{f(x)}\right]f'(x) + \left[\frac{(a+1)}{(x+b)^2} - \frac{(a+1)\left(ab^a\right)(\log\alpha)}{(x+b)^{a+2}}\right]f(x) \tag{10}$$

2.2 *ODEs for the quantile function (QF)*

The QF of APL distribution is obtained as

$$Q_n = \frac{1}{\alpha-1}\left[\alpha^{1-b^a(x+b)^{-a}} - 1\right] \tag{11}$$

By differentiating Eq. (11), we get the first order ODEs

$$Q'_{(n)} = \frac{\left(ab^a\right)(\log\alpha)\alpha^{1-b^a(x+b)^{-a}}(x+b)^{-(a+1)}}{\alpha-1} \tag{12}$$

The solution of Eq. (12) is possible when a, b, α, $x > 0$, $\alpha \neq 1$. The simplified form of Eq. (11) is given by

$$\alpha^{1-b^a(x+b)^{-a}} = (\alpha-1)Q_{(n)} + 1 \tag{13}$$

Putting Eq. (13) in Eq. (12), we obtain

$$(\alpha-1)Q'_{(n)} = (\alpha-1)\left(ab^a\right)(\log\alpha)(x+b)^{-(a+1)}Q_{(n)} + \left(ab^a\right)(\log\alpha)(x+b)^{-(a+1)} \tag{14}$$

Eq. (14) can be solved for the unique values of the parameters. Table 2 reflects the ODEs with their specified parameter values.

Table 2. ODEs for the QF with different parameter values

a	b	α	ODEs
1	1	2	$(x+1)^2 Q'_{(n)} - 0.3Q_{(n)} - 0.3 = 0$
1	2	2	$(x+2)^2 Q'_{(n)} - 0.6Q_{(n)} - 0.6 = 0$
2	1	2	$(x+1)^3 Q'_{(n)} - 0.6Q_{(n)} - 0.6 = 0$
2	2	2	$(x+2)^3 Q'_{(n)} - 2.4Q_{(n)} - 2.4 = 0$

Differentiate Eq. (15) to get the second order derivative

$$(\alpha-1)Q''_{(n)} = \frac{-(\alpha-1)(a+1)\left(ab^a\right)(\log\alpha)Q_{(n)}}{(x+b)^{a+2}} + \frac{(\alpha-1)\left(ab^a\right)(\log\alpha)Q'(n)}{(x+b)^{a+1}} - \frac{(a+1)\left(ab^a\right)(\log\alpha)}{(x+b)^{a+2}} \tag{15}$$

Then, we have

$$\begin{aligned}&(\alpha-1)(x+b)^{a+2}Q''_{(n)} - (\alpha-1)\left(ab^a\right)(\log\alpha)(x+b)^{-1}Q'_{(n)}\\ &+(\alpha-1)(a+1)\ \left(ab^a\right)(\log\alpha)Q_{(n)} + (a+1)\left(ab^a\right)(\log\alpha) = 0\end{aligned} \tag{16}$$

2.3 *ODEs for the survival function (SF)*

The SF of APL distribution is given by

$$S_{(x)} = 1 - \frac{\alpha^{1-b^a(x+b)^{-a}} - 1}{\alpha-1} \tag{17}$$

By differentiating Eq. (17), we get the first order ODEs

$$S'_{(x)} = \frac{-\left(ab^a\right)(\log\alpha)(x+b)^{-(a+1)}\alpha^{1-b^a(x+b)^{-a}}}{\alpha-1} \tag{18}$$

Eq. (17) can be rewritten as

$$\alpha^{1-b^{a}(x+b)^{-a}} = \alpha - (\alpha - 1)S_{(x)} \tag{19}$$

Putting Eq. (19) in Eq. (18), we get

$$S'_{(x)} = \frac{-\left(ab^{a}\right)(\log\alpha)(x+b)^{-(a+1)}\left(\alpha - (\alpha-1)S_{(x)}\right)}{\alpha - 1} \tag{20}$$

Finally, we obtain

$$(\alpha - 1)S'(x) = \frac{-(\alpha)\left(ab^{a}\right)(\log\alpha)}{(x+b)^{a+1}} + \frac{(\alpha-1)\left(ab^{a}\right)(\log\alpha)S_{(x)}}{(x+b)^{a+1}} \tag{21}$$

The ODEs for Eq. (21) can be obtained only when the parameter values are specified. Table 3 reflects the numerical results against specified parameter values.

Table 3. ODEs for the SF with different parameter values

a	b	α	ODEs
1	1	2	$(x+1)^{2} S'_{(x)} - 0.3S_{(x)} + 0.6 = 0$
1	2	2	$(x+2)^{2} S'_{(x)} - 0.6S_{(x)} + 1.2 = 0$
2	1	2	$(x+1)^{3} S'_{(x)} - 0.6S_{(x)} + 1.2 = 0$
2	2	2	$(x+2)^{3} S'_{(x)} - 2.4S_{(x)} + 4.8 = 0$

Again differentiating Eq. (22)

$$(\alpha-1)S''(x) = \frac{-\alpha(a+1)\left(ab^{a}\right)(\log\alpha)}{(x+b)^{a+2}} + \frac{(\alpha-1)\left(ab^{a}\right)(\log\alpha)S'(x)}{(x+b)^{a+1}} - \frac{(\alpha-1)(a+1)\left(ab^{a}\right)(\log\alpha)S_{(x)}}{(x+b)^{a+1}} \tag{22}$$

Finally, we obtain

$$\begin{aligned}&(\alpha-1)(x+b)^{a+2} S''_{(x)} - (\alpha-1)\left(ab^{a}\right)(\alpha)S'_{(x)} \\ &+(\alpha-1)(a+1)\left(ab^{a}\right)(\log\alpha)S_{(x)} + \alpha(a+1)\left(ab^{a}\right)(\log\alpha) = 0\end{aligned} \tag{23}$$

2.4 *ODEs for the hazard function (HF)*

The HF of APL distribution is defined by

$$h_{(x)} = \frac{(\log\alpha)\left(ab^a\right)\alpha^{-b^a(x+b)^{-a}}}{\left(1-\alpha^{-b^a(x+b)^{-a}}\right)(x+b)^{a+1}} \tag{24}$$

By differentiating Eq. (24), we get the following first order ODEs

$$\begin{aligned} h'_{(x)} = &\frac{\alpha^{-b^a(x+b)^{-a}}}{\left(1-\alpha^{-b^a(x+b)^{-a}}\right)(x+b)^{a+1}}\left[\left[(ab^a)(\log\alpha)\left(1-\alpha^{-b^a(x+b)^{-a}}\right)\right]\right. \\ &\left.-\left[(a+1)(x+b)^a\left(1-\alpha^{-b^a(x+b)^{-a}}\right)\right]\frac{-\left[(ab^a)(\log\alpha)(x+b)^{-(a+1)}\left(\alpha^{-b^a(x+b)^{-a}}\right)\right]}{\left(1-\alpha^{-b^a(x+b)^{-a}}\right)(x+b)^{a+1}}\right] \end{aligned} \tag{25}$$

$$h'_{(x)} = h_{(x)}\left[\left(ab^a\right)(\log\alpha)(x+b)^{-(a+1)} - (a+1)(x+b)^{-1} + \left(ab^a\right)(\log\alpha)(x+b)^{-(a+1)}h_{(x)}\right] \tag{26}$$

Table 4 reflects the ODEs for the HF of APL distribution with the specified parameter values.

Table 4. ODEs for the HF with different parameter values

a	b	α	ODEs
1	1	2	$h'_{(x)} - h_{(x)}\left[0.3(x+1)^{-2} + 2(x+1)^{-1} - 0.3(x+1)^{-2}h_{(x)}\right] = 0$
1	2	2	$h'_{(x)} - h_{(x)}\left[0.6(x+2)^{-2} + 2(x+2)^{-1} - 0.6(x+2)^{-2}h_{(x)}\right] = 0$
2	1	2	$h'_{(x)} - h_{(x)}\left[0.6(x+1)^{-3} + 3(x+1)^{-1} - 0.6(x+1)^{-3}h_{(x)}\right] = 0$
2	2	2	$h'_{(x)} - h_{(x)}\left[2.4(x+2)^{-3} + 3(x+2)^{-1} - 2.4(x+2)^{-3}h_{(x)}\right] = 0$

Now differentiating Eq. (26), we get

$$\begin{aligned} h''(x)(x+b)^{a+2} = &\, h(x)\left((a+1)(x+b)^a - (a+1)ab^a\log(x) + (a+1)h(x) + h'(x)(x+b)\right) \\ &+h'(x)\left(\left(ab^a\log(x)\right) - (a+1)(x+b)^a - h(x)\right) \end{aligned} \tag{27}$$

2.5 *Application*

The data set defines the mortality rate of COVID-19 in Pakistan which was taken from the URL https://github.com/owid/covid-19-data.

Table 5. Descriptive statistics

Min	Max	Mean	Median	SD
0.009	47.358	22.051	27.709	13.90364

Table 6. MLEs and standard errors

Parameter	MLE	Standard error
a	4.464472	0.7106576
b	51.128516	9.7869292
c	8.772719	2.8485851

Table 5 present some descriptive analysis while Table 6 defines the mles and their corresponding standard errors that will be used in the computation of ODEs.

The solution of the ODEs in each case of the functions can be determine by substituting the estimates of the parameters. For instance, using the above estimates into Eq (17), and (20), we get the following results

$$S'_{(x)} + 5,299,708(x+b)^{-5.46}\left(8.77 - 7.77S_{(x)}\right) = 0$$

and

$$S_{(1)} = 0.97$$

The solution defines the survival function of the Alpha Power Lomax distribution.

3. Results

In this paper, the first and second order ODEs for the density, quantile, survival and hazard functions of the APL distribution is derived. Under some restrictions on the parameters of APL distribution, various ODEs are obtained. The same study may be conducted for some other probability distributions including Gull Alpha Power Weibull, flexible Lomax and some others.

Conflicts of interest

The authors declare no competing financial interest.

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