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Research Article

On a Matrix Trace Inequality for Hermitian Matrices

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Abstract: Da-Wei Chang, obtained the matrix trace inequality for Hermitian matrices $\operatorname{tr}((AB)^{2^k}) \leq \operatorname{tr}(A^{2^k}B^{2^k})$ for any integer k. In this paper, we give an equality condition for this inequality by using the weak majorization of eigenvalue.

***Keywords*:** trace function, trace inequality, the weak majorization

MSC: 14A18

1. Introduction

Let $M_n(\mathbb{C})$ be the set of all $n \times n$ matrices over the complex number field $\mathbb{C}$. The eigenvalues of $A \in M_n(\mathbb{C})$ are $\lambda_1(A),\ \lambda_2(A),\ \ldots,\ \lambda_n(A)$, with $|\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots \geq |\lambda_n(A)|$. The singular values of $A \in M_n(\mathbb{C})$, denoted by $\sigma_1(A),\ \sigma_2(A),\ \ldots,\ \sigma_n(A)$, are the eigenvalues of the matrix $|A| = (AA^*)^{1/2}$ arranged in such a way that $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A)$. Note that $\sigma_i^2(A) = \lambda_i(AA^*) = \lambda_i(A^*A)$, so for a positive semidefinite matrix A, we have $\sigma_i(A) = \lambda_i(A)$ for all $i = 1,\ 2,\ \ldots,\ n$.

Given two real vectors $x = (x_1,\ x_2,\ \ldots,\ x_n)$ and $y = (y_1,\ y_2,\ \ldots,\ y_n)$ in decreasing order we say that x is weakly log majorized by y and we write $\log x \prec_w \log y$ if $\prod_{i=1}^k x_i \leq \prod_{i=1}^k y_i$, for every $k = 1,\ 2,\ \ldots,\ n$ but if equality occurs at $k = n$ we write it as $\log x \prec \log y$. We say that x is weakly majorized by y and we write $x \prec_w y$ if $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$ for every $k = 1,\ 2,\ \ldots,\ n$. We say that x is majorized by y and we write $x \prec y$ if

$$x \prec_w y \text{ and } \sum_{i=1}^n x_i = \sum_{i=1}^n y_i.$$

A normal matrix A can be decomposed as $A = UDU^*$ using the spectral decomposition, where $D = \operatorname{diag}\{\lambda_1(A), \lambda_2(A),\ \ldots, \lambda_n(A)\}$ and U is a unitary matrix.

It is well know that for all positive semidefinite matrices A and B,

$$0 \leq \operatorname{tr}(AB) \leq \operatorname{tr}(A)\operatorname{tr}(B).$$

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Lieb and Thirring [1] showed that

$$\mathrm{tr}((AB)^{\alpha}) \leq \mathrm{tr}(A^{\alpha}B^{\alpha}) \tag{1}$$

for all positive operators A, B on a separable Hilbert space and any real number $\alpha \geq 1$. In the case where $\alpha = m$ is a positive integer, some upper and lower bounds for the inequality (1) were obtained by Marcus [2], Le Couteur [3] and proved again by Bushell and Trustrum [4] as

$$\sum_{i=1}^{n} \lambda_i^m(A)\lambda_{n-i+1}^m(B) \leq \mathrm{tr}((AB)^m) \leq \mathrm{tr}(A^mB^m) \leq \sum_{i=1}^{n} \lambda_i^m(A)\lambda_i^m(B). \tag{2}$$

Da-Wei Chang [5] obtained the inequality

$$\mathrm{tr}((AB)^{2^k}) \leq \mathrm{tr}(A^{2^k}B^{2^k}), \tag{3}$$

where k is a positive integer, A and B are hermitian. Wang and Gong [6] generalized the inequality (2) in terms of majorization and proved

$$\log\lambda^{1/\alpha}(A^{\alpha}B^{\alpha}) \prec \log\lambda^{1/\beta}(A^{\beta}B^{\beta}), \qquad 0 < \alpha \leq \beta, \tag{4}$$

$$\lambda^{1/\alpha}(A^{\alpha}B^{\alpha}) \prec_{\omega} \lambda^{1/\beta}(A^{\beta}B^{\beta}), \qquad 0 < \alpha \leq \beta, \tag{5}$$

$$\lambda^{1/\beta}(A^{\beta}B^{\beta}) \prec_{\omega} \lambda^{1/\alpha}(A^{\alpha}B^{\alpha}), \qquad \alpha \leq \beta < 0, \tag{6}$$

$$\lambda^{\alpha}(AB) \prec_{\omega} \lambda(A^{\alpha}B^{\alpha}), \qquad |\alpha| \geq 1, \tag{7}$$

$$\lambda(A^{\alpha}B^{\alpha}) \prec_{\omega} \lambda^{\alpha}(AB), \qquad |\alpha| \leq 1. \tag{8}$$

In 1999 Chang [5] proved $\mathrm{tr}((AB)^{2^k}) \leq \mathrm{tr}(A^{2^k}B^{2^k})$ for Hermitian matrices A, B. In this paper we show that equality holds if and only if A, B commute by using the weak majorization of eigenvalue.

2. Main results

Throughout this section, we work with square matrices. First of all we have to show that for any Hermitian matrices A and B, $\mathrm{tr}((AB)^2) = \mathrm{tr}(A^2B^2)$ if and only if $AB = BA$.

Lemma 1 Let $A \in M_n(\mathbb{R})$ be a diagonal matrix and $B \in M_n(\mathbb{C})$ a Hermitian matrix. If $\mathrm{tr}((AB)^2) = \mathrm{tr}(A^2B^2)$, then $AB = BA$.

Proof. Suppose that $\mathrm{tr}((AB)^2) = \mathrm{tr}(A^2B^2)$, while

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \overline{b}_{12} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{b}_{1n} & \overline{b}_{2n} & \cdots & b_{nn} \end{bmatrix}$$

in which $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $b_{11}, b_{22}, \ldots, b_{nn} \in \mathbb{R}$. Then we have

$$\begin{aligned} \operatorname{tr}((AB)^2) = &(a_1b_{11}a_1b_{11} + a_1b_{12}a_2\overline{b}_{12} + \cdots + a_1b_{1n}a_n\overline{b}_{1n}) \\ &+ (a_2\overline{b}_{12}a_1b_{12} + a_2b_{22}a_2b_{22} + \cdots + a_2b_{2n}a_n\overline{b}_{2n}) \\ &+ \cdots + (a_n\overline{b}_{1n}a_1b_{1n} + a_n\overline{b}_{2n}a_2b_{2n} + \cdots + a_nb_{nn}a_nb_{nn}) \\ = &\sum_{i=1}^{n}\sum_{j=1}^{n} a_ia_j|b_{ij}|^2 \end{aligned}$$

and

$$\begin{aligned} \operatorname{tr}(A^2B^2) = &(a_1a_1b_{11}b_{11} + a_1a_1b_{12}\overline{b}_{12} + \cdots + a_1a_1b_{1n}\overline{b}_{1n}) \\ &+ (a_2a_2\overline{b}_{12}b_{12} + a_2a_2b_{22}b_{22} + \cdots + a_2a_2b_{2n}\overline{b}_{2n}) \\ &+ \cdots + (a_na_n\overline{b}_{1n}b_{1n} + a_na_n\overline{b}_{2n}b_{2n} + \cdots + a_na_nb_{nn}b_{nn}) \\ = &\sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2|b_{ij}|^2. \end{aligned}$$

Thus

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_ia_j|b_{ij}|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2|b_{ij}|^2.$$

Now

$$AB - BA = \begin{bmatrix} 0 & (a_1-a_2)b_{12} & \cdots & (a_1-a_n)b_{1n} \\ (a_2-a_1)\overline{b}_{12} & 0 & \cdots & (a_2-a_n)b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (a_n-a_1)\overline{b}_{1n} & (a_n-a_2)\overline{b}_{2n} & \cdots & 0 \end{bmatrix}$$

.

But since $|b_{ij}| = |b_{ji}|$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i-a_j)^2|b_{ij}|^2=\sum_{i=1}^{n}\sum_{j=1}^{n}a_i^2|b_{ij}|^2+\sum_{i=1}^{n}\sum_{j=1}^{n}a_j^2|b_{ij}|^2-2\sum_{i=1}^{n}\sum_{j=1}^{n}a_ia_j|b_{ij}|^2$$

$$=2\Big\{\sum_{i=1}^{n}\sum_{j=1}^{n}a_i^2|b_{ij}|^2-\sum_{i=1}^{n}\sum_{j=1}^{n}a_ia_j|b_{ij}|^2\Big\}.$$

Thus $AB-BA = 0$. □

Theorem 2 Let A and B be Hermitian matrices. Then $AB = BA$ if and only if $\mathrm{tr}((AB)^2)=\mathrm{tr}(A^2B^2)$.

Proof. If $AB = BA$, then it is easy to see that $\mathrm{tr}((AB)^2)=\mathrm{tr}(A^2B^2)$. Next suppose that $\mathrm{tr}((AB)^2)=\mathrm{tr}(A^2B^2)$. Since A is Hermitian, $A=UDU^*$ where U is a unitary matrix and D is a diagonal matrix whose diagonal entries are eignevalues of A. Thus

$$\mathrm{tr}((AB)^2)=\mathrm{tr}[(UDU^*)B(UDU^*)B]=\mathrm{tr}[D(U^*BU)D(U^*BU)]$$

$$=\mathrm{tr}\left([D(U^*BU)]^2\right) \tag{9}$$

and

$$\mathrm{tr}(A^2B^2)=\mathrm{tr}[(UDU^*)(UDU^*)B^2]=\mathrm{tr}[(UD^2U^*)B^2]$$

$$=\mathrm{tr}[D^2(U^*B^2U)]. \tag{10}$$

From equality of (9) and (10), Lemma 1 implies that $AB=BA$. □

Next we will give an equality condition for the inequality (3) in case $k \geq 1$.

Theorem 3 Let A and B be Hermitian matrices. Then for any natural number $k \geq 1$, $AB = BA$ if and only if $\mathrm{tr}((AB)^{2^k})=\mathrm{tr}(A^{2^k}B^{2^k})$.

Proof. If A and B commute the results is trivial. To prove sufficiency, we will show that if $\mathrm{tr}((AB)^{2^k})=\mathrm{tr}(A^{2^k}B^{2^k})$, $k \geq 1$ then $\mathrm{tr}((AB)^2)=\mathrm{tr}(A^2B^2)$ which implies $AB=BA$. Suppose that $\mathrm{tr}((AB)^2)\neq\mathrm{tr}(A^2B^2)$, so from the inequalily (3) we have that

$$\sum_{i=1}^{n}\lambda_i^2(AB)<\sum_{i=1}^{n}\lambda_i(A^2B^2). \tag{11}$$

Thus

$$\mathrm{tr}((AB)^{2^k})=\sum_{i=1}^{n}\lambda_i^{2^{k-1}2}\Big((AB)\Big)<\sum_{i=1}^{n}\lambda_i^{2^{k-1}}\Big(A^2B^2\Big). \tag{12}$$

Since A and B are Hermitioan, A^2 and B^2 are positive semidefinite. Therefore by (7) we have that

$$\sum_{i=1}^{n} \lambda_i^{2^{k-1}} \left(A^2 B^2\right) \leq \sum_{i=1}^{n} \lambda_i (A^{2^k} B^{2^k}). \tag{13}$$

We can therefore conclude that if $\mathrm{tr}((AB)^2) < \mathrm{tr}(A^2B^2)$ then $\mathrm{tr}((AB)^{2^k}) < \mathrm{tr}(A^{2^k}B^{2^k})$. Hence by the inequality (3) we have, if $\mathrm{tr}((AB)^{2^k}) = \mathrm{tr}(A^{2^k}B^{2^k})$ then $\mathrm{tr}((AB)^2) = \mathrm{tr}(A^2B^2)$ which implies $AB = BA$. □

Next we will give an equality condition for the inequality (3) in case $k < 0$. First we will consider $k = -1$, after which any $k \leq 1$ can be considered. To be able to find the square root of any matrix, we need to consider positive semidefinite Hermitian matrices.

Lemma 4 Let A be a positive semidefinite diagonal matrix and $B \in M_n(\mathbb{C})$ a positive semidefinite matrix. If $\mathrm{tr}((AB)^{1/2}) = \mathrm{tr}(A^{1/2}B^{1/2})$, then $AB = BA$.

Proof. Since B is positive semidefinite, $B = UDU^*$ where U is a unitary matrix and D is a diagonal matrix whose diagonal entries are non-negative eignevalues of B. Then

$$\mathrm{tr}((AB)^{1/2}) = \mathrm{tr}(A^{1/2}B^{1/2}) = \mathrm{tr}(A'B'), \tag{14}$$

where $A' = U^*A^{1/2}U$ and $B' = D^{1/2}$. We also have that

$$\begin{aligned}
\mathrm{tr}((A'^2B'^2)^{1/2}) &= \sum_{i=1}^{n} \lambda_i^{1/2}(A'^2B'^2) \\
&= \sum_{i=1}^{n} \lambda_i^{1/2}(U^*AUD) \\
&= \sum_{i=1}^{n} \lambda_i^{1/2}(AUDU^*) \\
&= \mathrm{tr}((AB)^{1/2}).
\end{aligned} \tag{15}$$

From equation (14) and (15) we have that $\mathrm{tr}(A'B') = \mathrm{tr}((A'^2B'^2)^{1/2})$. But

$$\mathrm{tr}((A'^2B'^2)^{1/2}) = \sum_{i=1}^{n} \lambda_i^{1/2}(A'^2B'^2) = \sum_{i=1}^{n} \lambda_i^{1/2}\left((A'B')(A'B')^*\right) = \sum_{i=1}^{n} \sigma_i(A'B').$$

Thus $\sum_{i=1}^{n} \lambda_i(A'B') = \mathrm{tr}(A'B') = \mathrm{tr}((A'^2B'^2)^{1/2}) = \sum_{i=1}^{n} \sigma_i(A'B')$.

We know that

$$\left|\sum_{i=1}^{n} \lambda_i(A'B')\right| \leq \sum_{i=1}^{n} \left|\lambda_i(A'B')\right| \leq \sum_{i=1}^{n} \lambda_i(|A'B'|) = \sum_{i=1}^{n} \sigma_i(A'B')$$

so $\sum_{i=1}^{n} \lambda_i(A'B') = \sum_{i=1}^{n} \left|\lambda_i(A'B')\right| = \sum_{i=1}^{n} \sigma_i(A'B')$.

Therefore, we can conclude that $\lambda_i(A'B')$ is non-negative for all $i = 1, 2, \ldots, n$. Since the arrangement of eigenvalues and singular values, $\lambda_i(A'B') = \sigma_i(A'B')$ for all $i = 1, 2, \ldots, n$ which implies $A'B'$ is a positive semidefinite matrix. Then we have,

$$\sum_{i=1}^{n} \lambda_i((A'B')^2) = \sum_{i=1}^{n} \lambda_i\Big((A'B')(A'B')^*\Big)$$

$$= \sum_{i=1}^{n} \lambda_i(A'B'B'A')$$

$$= \sum_{i=1}^{n} \lambda_i(A'^2B'^2).$$

Thus by Lemma 1, $A'B' = B'A'$. Hence A' and B' are simultaneously unitarily diagonalizable and commute. Therefore we can conclude that $AB = BA$. □

Theorem 1 Let A and B be positive semidefinite matrices. Then $AB = BA$ if and only if $\operatorname{tr}((AB)^{1/2}) = \operatorname{tr}(A^{1/2}B^{1/2})$.

Proof. Since A and B are positive semidefinite matrices, A and B are simultaneously unitary diagonalizable. If $AB = BA$, then $(AB)^{1/2} = A^{1/2}B^{1/2}$. Thus $\operatorname{tr}((AB)^{1/2}) = \operatorname{tr}(A^{1/2}B^{1/2})$. Next assume that $\operatorname{tr}((AB)^{1/2}) = \operatorname{tr}(A^{1/2}B^{1/2})$. Assume that $A = UDU^*$ is the spectral decomposition of A. Thus

$$\operatorname{tr}(A^{1/2}B^{1/2}) = \operatorname{tr}(UD^{1/2}U^*B^{1/2}) = \operatorname{tr}(D^{1/2}U^*B^{1/2}U). \tag{16}$$

By the unitary invariance of eigenvalues we have that

$$\operatorname{tr}((AB)^{1/2}) = \operatorname{tr}((UDU^*B)^{1/2}) = \operatorname{tr}((DU^*BU)^{1/2}). \tag{17}$$

By Lemma 2 and the equations (16) and (17), we have that $DU^*BU = U^*BUD$, which implies that $AB = BA$. □

Theorem 6 Let A and B be positive semidefinite matrices and k a positive integer. Then $AB = BA$ if and only if $\operatorname{tr}((AB)^{1/2^k}) = \operatorname{tr}(A^{1/2^k}B^{1/2^k})$.

Proof. Necessity is obvious because A and B are normal and commute, so they are simultaneously unitary diagonalizable. Next we will prove the sufficiency case. In [6] Wang and Gong proved the weak majorization $\lambda(A^{\alpha}B^{\alpha}) \prec_w \lambda^{\alpha}(AB)$ for $|\alpha| \leq 1$, so we have that

$$\operatorname{tr}(A^{1/2^k}B^{1/2^k}) \leq \operatorname{tr}(AB)^{1/2^k}.$$

We will show that $\operatorname{tr}(A^{1/2^k}B^{1/2^k}) = \operatorname{tr}(AB)^{1/2^k}$ implies $\operatorname{tr}(A^{1/2}B^{1/2}) = \operatorname{tr}(AB)^{1/2}$. If $\operatorname{tr}(A^{1/2}B^{1/2}) \neq \operatorname{tr}(AB)^{1/2}$, then $\sum_{i=1}^{n} \lambda_i(A^{1/2}B^{1/2}) < \sum_{i=1}^{n} \lambda_i^{1/2}(AB)$. We apply (8) and get

$$\begin{aligned}\operatorname{tr}(A^{1/2^k}B^{1/2^k}) &= \sum_{i=1}^{n}\lambda_i(A^{1/2^k}B^{1/2^k})\\ &\leq \sum_{i=1}^{n}\lambda_i^{\frac{1/2^k}{1/2}}(A^{1/2}B^{1/2})\\ &< \sum_{i=1}^{n}\lambda_i^{1/2^k}(AB)\\ &= \operatorname{tr}(AB)^{1/2^k}\end{aligned}$$

a contradiction. Thus $\lambda(A^{1/2}B^{1/2}) = \lambda^{1/2}(AB)$, by Theorem 3 we have that $AB = BA$. □

Conflict of interest

Author declares there is no conflict of interest at any point with reference to research findings.

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