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Sequential Kannan Type Contractions in Partial *b*-Metric Spaces

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Abstract: This paper presents the concept of the "sequential condition" in the context of fixed point results for self-maps in partial b-metric spaces. We demonstrate the existence of a unique fixed point when this condition is satisfied. Additionally, we provide illustrative examples to showcase the application and effectiveness of our findings.

***Keywords*:** sequential Kannan type, fixed point, partial b-metrics

MSC: 47H05, 47H09, 47H10

1. Introduction and preliminaries

The exploration of fixed points in b-metric spaces offers a fresh perspective that diverges from traditional metric spaces, yielding notable advantages. Rooted in b-metric spaces is a versatile framework for modeling complex systems, with fixed points playing a pivotal role in analyzing the stability of iterative processes across disciplines such as computer science, physics, and economics. Extending this paradigm to b-metric spaces broadens our toolkit, enabling a more robust investigation into the convergence and uniqueness of fixed points across various systems. This extension unveils novel insights and outcomes that may elude traditional metric spaces, driven by the adaptable conditions inherent in b-metric spaces, thereby facilitating innovative applications and solutions.

The study of fixed points in b-metric spaces marks a compelling intersection of theory and application, offering a lens through which to view complex dynamics more holistically. It underscores the interconnectedness of mathematics and real-world scenarios, enriching our comprehension of intricate phenomena while providing fertile ground for innovative problem-solving approaches.

Partial metrics, a broader category than metric spaces, extend the framework by allowing non-zero self-distances between points. These structures prove particularly advantageous for representing partially defined information, an essential consideration within the realm of computer science. The study of self-maps on partial metric spaces has prompted the formulation of fixed point theorems, as highlighted in references [1] and [2]. In 2014, Satish introduced the notion of partial b-metric spaces [3], introducing the Banach-type contraction concept. In this work, we propose a novel variation known as the modified Kannan-type contraction, establishing the existence of fixed points for such contractions. Collectively termed the "sequential condition", these various approaches are explored alongside illustrative examples. To

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pave the way, we will revisit the definitions pertinent to b-metric and partial metric spaces, as elucidated in [3]. This research encapsulates an intersection of mathematical theory and practical application, capitalizing on the flexibility of partial metrics to address real-world situations with incomplete information. By introducing a novel contraction framework and delving into fixed point theorems, the paper contributes to the evolving landscape of partial b-metric spaces, providing a deeper understanding of their properties and potential applications.

Definition 1 (Compare [3, Definition 3]) A partial metric type on a set X is a function $p : X \times X \to [0, \infty)$ such that:

(pm1) $x = y$ iff $p(x, x) = p(x, y) = p(y, y)$ whenever $x, y \in X$,

(pm2) $0 \leq p(x, x) \leq p(x, y)$ whenever $x, y \in X$,

(pm3) $p(x, y) = p(y, x)$ whenever $x, y \in X$,

(pm4) There exists a real number $s \geq 1$ such that

$$p(x, y) + p(z, z) \leq s[p(x, z) + p(z, y)]$$

for any points $x, y, z \in X$.

The triple (X, p, s) is called a partial b-metric space.

It is clear that, if $p(x, y) = 0$, then, from (pm1) and (pm2), $x = y$. The family B' of sets

$$B'_p(x, \varepsilon) := \{y \in X : p(x, y) < \varepsilon + p(x, x)\},\ x \in X, \varepsilon > 0,$$

is a basis for a topology $\tau(p)$ on X. The topology $\tau(p)$ is T_0. One of the direct implications is that the limit of a convergent sequence in a partial b-metric space need not be unique. This was illustrated in [3, Example 2].

Definition 2 Let (X, p) be a partial b-metric space. Let $(x_n)_{n\geq 1}$ be any sequence in X and $x \in X$.

1. The sequence $(x_n)_{n\geq 1}$ is said to be convergent with respect to $\tau(p)$ (or $\tau(p)$-convergent) and converges to x, if $\lim_{n\to\infty} p(x, x_n) = p(x, x)$. We write

$$x_n \xrightarrow{p} x.$$

2. The sequence $(x_n)_{n\geq 1}$ is said to be a p-Cauchy sequence if

$$\lim_{n\to\infty,\ m\to\infty} p(x_n, x_m)$$

exists and is finite.

(X, p) is said to be complete if, for every p-Cauchy sequence $(x_n)_{n\geq 1} \subseteq X$, there exists $x \in X$ such that:

$$\lim_{n\to\infty,\ m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x, x_n) = p(x, x).$$

The above definition of completeness gives rise to an interesting notion, in the setting of partial metric spaces, namely:

Definition 3 One says that a sequence (x_n) in a partial metric space *converges properly* to $x \in X$ if

$$\lim_{n\to\infty} p(x, x_n) = p(x, x) = \lim_{n\to\infty} p(x_n, x_n).$$

In that case, we shall write

$$x_n \xrightarrow{ppr} x.$$

Alternatively, we can say that (x_n) converges properly to x if it converges to x under the topology induced by the p-metric $\tau(p)$ and furthermore,

$$\lim_{n \to \infty} p(x_n, x_n) = p(x, x).$$

Definition 4 (Compare [4, Definition 3.10]) Suppose (X, p) and (X', p') are partial b-metric spaces. $f : (X, p) \to (X', p')$ is continuous if, and only if,

$$f(x_n) \xrightarrow{ppr} f(x^*) \text{ whenever } x_n \xrightarrow{ppr} x^*.$$

We conclude this introductory part with the definition of the Φ-class:

Definition 5 (See [5]) Set the class Φ consisting of continuous functions $F : [0, \infty) \to [0, \infty)$ that are non-decreasing, sub-additive, homogeneous, and satisfy the property $F^{-1}(0) = \{0\}$.

Now, we can state our main results.

2. Main results

2.1 *Unique map fixed point*

We begin this first part of the main result on a fixed point for a single map.

Theorem 1 Consider a Hausdorff complete partial b-metric space (X, p, K), where X is the underlying set, p is a partial b-metric on X, and K is a nonempty set. Let T be a continuous self-mapping on X satisfying the following condition:

$$p(T^n x, T^n y) \leq a_n [p(x, Tx) + p(y, Ty)] \tag{1}$$

for any $x, y, z \in X$, satisfying the conditions $a_n > 0$ for all $n \geq 1$ (independent of x, y, z) and $0 \leq a_1 < \frac{1}{2}$, and if the series $\sum a_n$ (or alternatively, if the sequence (a_n) converges to 0), then the self-mapping T possesses a unique fixed point in X.

Proof. Let $x_0 \in X$. We consider the sequence of iterates $x_n = T^n x_0, n = 1, 2, 3, \cdots$. Then for $n \geq 1$

$$p(T^n x_0, T^{n+1} x_0) \leq a_n \left[p(x_0, Tx_0) + p(Tx_0, T^2 x_0) \right]$$

Again

$$p(Tx_0, T^2 x_0) \leq a_1 \left[p(x_0, Tx_0) + p(Tx_0, T^2 x_0) \right].$$

Therefore

$$p(T^n x_0,\ T^{n+1} x_0) \leq a_n \left[1 + \frac{2a_1}{1-2a_1}\right] p(x_0,\ Tx_0). \tag{2}$$

Using property (pm4), we can write:

$$\begin{aligned} p(x_n,\ x_{n+m}) &= p(T^n x_0,\ T^{n+m} x_0) \\ &\leq K\big[p(T^n x_0,\ T^{n+1} x_0) + p(T^{n+1} x_0,\ T^{n+2} x_0) \\ &\quad + \cdots + p(T^{n+m-1} x_0,\ T^{n+m} x_0)\big]. \end{aligned}$$

So using (2), we get

$$p(x_n,\ x_{n+m}) \leq [a_n + a_{n+1} + \cdots + a_{n+m-1}] \left[1 + \frac{2a_1}{1-2a_1}\right] p(x_0,\ Tx_0).$$

Given that the series $\sum a_n$ is convergent, it follows that as n approaches infinity, $p(x_n,\ x_{n+m})$ tends to zero. This implies that the sequence (x_n) is p-Cauchy. Furthermore, as X is complete, there exists an element $x^* \in X$ such that the sequence $(x_n)_{n\geq 1}$ converges properly to x^*, meaning that:

$$\lim_{n\to\infty,\ m\to\infty} p(x_n,\ x_m) = \lim_{n\to\infty} p(x^*,\ x_n) = p(x^*,\ x^*) = 0.$$

Next, we will demonstrate that x^* serves as a fixed point of T.

As T is continuous and the sequence (x_n) converges properly to x^*, we can conclude that $(x_{n+1}) = (Tx_n)$ also converges properly to $Tx^* = x^*$. This result is possible because $(X,\ p,\ K)$ is a Hausdorff space, ensuring the uniqueness of the limit.

Assume that u and v are fixed points of T. By utilizing Equation (1), we can deduce that:

$$p(u,\ v) = p(T^n u,\ T^n v) \leq a_n[p(u,\ u) + p(v,\ v)] \to 0 \text{ as } n \to +\infty.$$

Consequently, we can conclude that $u = v$, indicating that T possesses a unique fixed point. □

We present an intriguing result that provides a generalization of Theorem 1.

Theorem 2 Let $(X,\ p,\ K)$ be a Hausdorff complete partial b-metric space and let T be a continuous self-mapping on X such that

$$F(p(T^n x,\ T^n y)) \leq F\big(a_n\big[p(x,\ Tx) + p(y,\ Ty)\big]\big) \tag{3}$$

for all $x,\ y,\ z \in X$, under the conditions that $a_n(>0)$ is independent of $x,\ y,\ z,\ 0 \le a_1 < \frac{1}{2}$, and $F \in \Phi$ is homogeneous with degree s, we assert the following notable result. If the series $\sum a_n$ is convergent, then T possesses a unique fixed point in X.

Remark 1 If we set $F = Id_{[0,\ \infty)}$ in Theorem 2, we obtain the result of Theorem 1.

Proof. Let $x_0 \in X$. We consider the sequence of iterates $x_n = T^n x_0,\ n = 1,\ 2,\ 3,\ \cdots$. Then for $n \ge 1$

$$F(p(T^n x_0,\ T^{n+1}x_0)) \le F(a_n[p(x_0,\ Tx_0) + p(Tx_0,\ T^2x_0)])$$

$$\le a_n^s F(p(x_0,\ Tx_0)) + a_n^s F(p(Tx_0,\ T^2x_0))$$

Again

$$F(p(Tx_0,\ T^2x_0)) \le F(a_1[p(x_0,\ Tx_0) + p(Tx_0,\ T^2x_0)])$$

$$\le a_1^s F(p(x_0,\ Tx_0)) + (a_1)^s F(p(Tx_0,\ T^2x_0)).$$

which gives

$$F(p(Tx_0,\ T^2x_0)) \le \frac{a_1^s}{1 - a_1^s} F(p(x_0,\ Tx_0))$$

Therefore

$$F(p(T^n x_0,\ T^{n+1}x_0)) \le a_n^s \left[1 + \frac{(a_1)^s}{1-(a_1)^s}\right] F(p(x_0,\ Tx_0)). \tag{4}$$

Using property (pm4) and (4), we can write:

$$F(p(x_n,\ x_{n+m})) \le [a_n^s + a_{n+1}^s + \cdots + a_{n+m-1}^s] \left[1 + \frac{(a_1)^s}{1-(a_1)^s}\right] F(p(x_0,\ Tx_0)).$$

As n tends to infinity, due to the properties $F^{-1}(0) = \{0\}$ and the continuity of F, it follows that $p(x_n,\ x_{n+m})$ tends to zero. This implies that the sequence (x_n) is a p-Cauchy sequence. Moreover, since X is a complete space and T is continuous, there exists an element $x^* \in X$ such that (x_n) converges properly to x^*, and (x_{n+1}) converges properly to $Tx^* = x^*$ because $(X,\ p,\ K)$ is a Hausdorff space.

The uniqueness of x^* is guaranteed by the condition stated in Equation (3). □

Example 1 Let $X = [0,\ 1]$ and $p(x,\ y) = \max\{x,\ y\}$ whenever $x,\ y \in [0,\ 1]$. Clearly, $(X,\ p,\ 1)$ is a complete partial metric space. Following the notation in Theorem 2, we set $a_n = \left(\frac{1}{1+2^n}\right)^2$. We also define $T(x) = \frac{x}{16}$ for all $x \in [0,\ 1]$ and let F be defined as $F: [0,\ \infty) \to [0,\ \infty),\ x \mapsto \sqrt{x}$. Then F is continuous, nondecreasing, subadditive and homogeneous of degree $s = \frac{1}{2}$ and $F^{-1}(0) = \{0\}$. Assume $x > y$. Hence, we have

$$F(p(T^n x,\ T^n y)) = \sqrt{\frac{x^n}{16^n}} \leq \sqrt{\frac{x}{16^n}},$$

and

$$F(a_n[p(x,\ Tx)+p(y,\ Ty)]) = \sqrt{\left(\frac{1}{1+2^n}\right)^2 (x+y)}\ .$$

Observe that $\sum a_n \leq \sum \frac{1}{n^2} < \infty$ and $a_1 = \frac{1}{9} < \frac{1}{2}$. The conditions of Theorem 2 are satisfied, so T has a unique fixed point, which in this case is $x^* = 0$.

2.2 *Common fixed point*

In this next section, we explore the concept of a common fixed point for a family of self-maps.

We commence by introducing a lemma that extends property (pm4) from Definition 1.

Lemma 1 Let $(X,\ p,\ K)$ be a partial b-metric space. Then

$$p(x,\ y) \leq K^n[p(x,\ z_1)+p(z_1,\ z_2)+\cdots+p(z_n,\ y)]$$

for any points $x,\ y,\ z_i \in X,\ i = 1,\ 2,\ \ldots,\ n$ where $n \geq 1$.

Proof. The proof is trivial and shall be omitted. □

Theorem 3 Let $(X,\ p,\ K)$ be a complete partial b-metric space and $\{T_n\}$ be a sequence of self-mappings on X such that

$$F(p(T_i(x),\ T_j(y)) \leq F(\delta_{i,\ j}[p(x,\ T_i(x))+p(y,\ T_j(y))]) + F(\gamma_{i,\ j}\, p(x,\ y)) \tag{5}$$

for $x,\ y,\ z \in X$ with $x \neq y$, $0 \leq \delta_{i,\ j},\ \gamma_{i,\ j} < 1,\ i,\ j = 1,\ 2,\ \cdots$, and for some $F \in \Phi$ homogeneous with degree s. If

(i) for each, j, $\limsup_{i\to\infty} \delta^s_{i,\ j} < 1$,

(ii)

$$\sum_{n=1}^{\infty} C_n < \infty \ \text{ where }\ C_n = \prod_{i=1}^{n} \frac{\delta^s_{i,\ i+1}+\gamma^s_{i,\ i+1}}{1-\delta^s_{i,\ i+1}},$$

then $\{T_n\}$ has a unique common fixed point in X.

Proof. For any $x_0 \in X$, we construct the sequence (x_n) as $x_n = T_n(x_{n-1})$, $n = 1,\ 2,\ \cdots$. Using (5) and the homogeneity of F, we obtain

$$
\begin{aligned}
F(p(x_1,\ x_2)) &= F(p(T_1(x_0),\ T_2(x_1))) \\
&\leq \delta_{1,\ 2}^s F([p(x_0,\ T_1(x_0)) + p(x_1,\ T_2(x_1))]) \\
&\quad + \gamma_{1,\ 2}^s F(p(x_0,\ x_1)) \\
&= \delta_{1,\ 2}^s F([p(x_0,\ x_1) + p(x_1,\ x_2)]) + \gamma_{1,\ 2}^s F(p(x_0,\ x_1)).
\end{aligned}
$$

By the sub-additivity of F, we have

$$
(1-\delta_{1,\ 2}^s) F(p(x_1,\ x_2)) \leq (\delta_{1,\ 2}^s + \gamma_{1,\ 2}^s) F(p(x_0,\ x_1)),
$$

that is,

$$
F(p(x_1,\ x_2)) \leq \left(\frac{\delta_{1,\ 2}^s + \gamma_{1,\ 2}^s}{1-\delta_{1,\ 2}^s}\right) F(p(x_0,\ x_1)).
$$

Also, we get

$$
\begin{aligned}
F(p(x_2,\ x_3)) &= F(p(T_2(x_1),\ T_3(x_2))) \\
&\leq \left(\frac{\delta_{2,\ 3}^s + \gamma_{2,\ 3}^s}{1-\delta_{2,\ 3}^s}\right) F(p(x_1,\ x_2)) \\
&\leq \left(\frac{\delta_{2,\ 3}^s + \gamma_{2,\ 3}^s}{1-\delta_{2,\ 3}^s}\right) \left(\frac{\delta_{1,\ 2}^s + \gamma_{1,\ 2}^s}{1-\delta_{1,\ 2}^s}\right) F(p(x_0,\ x_1)).
\end{aligned}
$$

By repeating the above process, we have

$$
F(p(x_n,\ x_{n+1})) \leq \prod_{i=1}^{n} \left(\frac{\delta_{i,\ i+1}^s + \gamma_{i,\ i+1}^s}{1-\delta_{i,\ i+1}^s}\right) F(p(x_0,\ x_1)) =: C_n F(p(x_0,\ x_1)).
$$

Hence, we derive, by making use of Lemma 1 and the properties of F, that for, $k \in \mathbb{N},\ k > 0$

$$
p(x_n,\ x_{n+k}) \leq K^{p-1}[p(x_n,\ x_{n+1}) + \cdots + p(x_{n+k-1},\ x_{n+k})]
$$

and so

$$F(p(x_n,\, x_{n+k})) \leq K^{(k-1)s}[F(p(x_n,\, x_{n+1})) + F(p(x_{n+1},\, x_{n+2}))$$

$$+\cdots + F(p(x_{n+k-1},\, x_{n+k}))]$$

$$\leq K^{(k-1)s}[C_n F(p(x_0,\, x_1)) + C_{n+1} F(p(x_0,\, x_1))$$

$$+\cdots + C_{n+k-1} F(p(x_0,\, x_1))]$$

$$= K^{(k-1)s}\left[\sum_{k=0}^{k-1} C_{n+k} F(p(x_0,\, x_1))\right]$$

$$= K^{(k-1)s}\left[\sum_{k=n}^{n+k-1} C_k\right] F(p(x_0,\, x_1)).$$

By taking the limit as n approaches infinity, and considering that $F^{-1}(0) = \{0\}$ and F is continuous, we can conclude that $p(x_n,\, x_{n+k})$ tends to zero. Therefore, the sequence (x_n) is a p-Cauchy sequence and, due to the completeness of X, it converges to an element denoted as $x^* \in X$.

Furthermore, for any nonzero natural number m, we have the following result:

$$F(p(x_n,\, T_m(x^*))) = F(p(T_n(x_{n-1}),\, T_m(x^*)))$$

$$\leq \delta^s_{n,\, m}[F(p(x_{n-1},\, x_n)) + F(p(x^*,\, T_m(x^*)))]$$

$$+ \gamma^s_{n,\, m} F(p(x_{n-1},\, x^*)).$$

Taking $\limsup$ as $n \to \infty$, we get

$$F(p(x^*,\, T_m(x^*))) = \delta^s_{n,\, m} F(p(x^*,\, T_m(x^*))),$$

and since $\limsup\limits_{i\to\infty} \delta^s_{i,\, j} < 1$, it follows that $F(p(x^*,\, T_m(x^*)) = 0$, that is, $T_m(x^*) = x^*$. Then x^* is a common fixed point of $\{T_m\}_{m\geq 1}$.

To prove the uniqueness of x^*, suppose that y^* is a common fixed point of $\{T_m\}_{m\geq 1}$, that is, $T_m(y^*) = y^*$ for any $m \geq 1$. Then, by (5), we have

$$
\begin{aligned}
F(p(x^*,\, y^*)) \leq\ & F(p(T_n(x^*),\, T_m(y^*))) \\
\leq\ & \delta^s_{n,\, m}[F(p(x^*,\, T_n(x^*)) + F(p(y^*,\, T_m(y^*))] + \gamma^s_{n,\, m} F(p(x^*,\, y^*)) \\
=\ & \delta^s_{n,\, m}[F(p(x^*,\, x^*) + F(p(y^*,\, y^*)] + \gamma^s_{n,\, m} F(p(x^*,\, y^*)) \\
=\ & \gamma^s_{n,\, m} F(p(x^*,\, y^*)).
\end{aligned}
$$

And again, since $0 \leq \gamma_{n,\, m} < 1$, $x^* = y^*$. So x^* is the unique common fixed point of $\{T_m\}$. □

As particular cases of Theorem 3, we have the following two corollaries.

Corollary 1 Let $(X,\, D,\, K)$ be a G-complete G-metric type space and $\{T_n\}$ be a sequence of self mappings on X such that

$$
\begin{aligned}
p(T_i(x),\, T_j(y)) \leq \delta_{i,\, j}[p(x,\, T_i(x)) + p(y,\, T_j(y))] \\
+ \gamma_{i,\, j}\, p(x,\, y),
\end{aligned}
$$

for $x,\, y \in X$ with $x \neq y$, $0 \leq \delta_{i,\, j},\, \gamma_{i,\, j} < 1,\ i,\ j = 1,\, 2, \cdots$. If

(i) for each j, $\limsup\limits_{i\to\infty} \delta_{i,\, j} < 1$,

(ii)

$$
\sum_{n=1}^{\infty} C_n < \infty \ \text{ where } \ C_n = \prod_{i=1}^{n} \frac{\delta_{i,\, i+1} + \gamma_{i,\, i+1}}{1 - \delta_{i,\, i+1}},
$$

then $\{T_n\}$ has a unique common fixed point in X.

Proof. Apply Theorem 3 by putting $F = I_{[0,\, \infty)}$, the identity map. □

Corollary 2 Let $(X,\, D,\, K)$ be a G-complete G-metric type space and $\{T_n\}$ be a sequence of self-mappings on X such that

$$
F(p(T_i(x),\, T_j(y))) \leq F(\delta_{i,\, j}[p(x,\, T_i(x)) + p(y,\, T_j(y))]) \tag{6}
$$

for $x,\, y,\, z \in X$ with $x \neq y$, $0 \leq \delta_{i,\, j} < 1,\ i\ , j = 1,\ 2,\ \cdots$, and for some $F \in \Phi$ homogeneous with degree s. If

(i) for each, j, $\limsup\limits_{i\to\infty} \delta^s_{i,\, j} < 1$,

(ii)

$$
\sum_{n=1}^{\infty} C_n < \infty \ \text{ where } \ C_n = \prod_{i=1}^{n} \frac{\delta^s_{i,\, i+1}}{1 - \delta^s_{i,\, i+1}},
$$

then $\{T_n\}$ has a unique common fixed point in X.

Proof. Apply Theorem 3 by putting $\gamma_{i,\ i+1} = 0$. □

Example 2 (Compare [5]) Let $X = [0,\ 1]$ and $p(x,\ y) = [\max\{x,\ y\}]^2$ whenever $x,\ y \in [0,\ 1]$. Clearly, $(X,\ p,\ 4)$ is a complete partial b-metric space. Following the notation in Theorem 3, we set $\delta_{i,\ j} = \left(\frac{1}{1+2^{\eta}}\right)^2$ where $\eta = \min\{i,\ j\}$. We also define $T_i(x) = \frac{x}{4^i}$ for all $x \in X$ and $i = 1,\ 2,\ \cdots$ and $F : [0,\ \infty) \to [0,\ \infty),\ x \mapsto \sqrt{x}$. Then F is continuous, non-decreasing, sub-additive and homogeneous of degree $s = \frac{1}{2}$ and $F^{-1}(0) = \{0\}$. Assume $i < j$ and $x > y$. Hence, we have

$$F(p(T_i(x),\ T_j(y))) = \sqrt{\left[\max\left\{\frac{x}{4^i},\ \frac{y}{4^j}\right\}\right]^2}$$

and

$$F(\delta_{i,\ j}[p(x,\ T_i(x)) + p(y,\ T_j(y))] = \sqrt{\left(\frac{1}{1+2^i}\right)^2\left(\left[\max\left\{\frac{x}{4^i},\ x\right\}\right]^2 + \left[\max\left\{\frac{y}{4^j},\ y\right\}\right]^2\right)}.$$

Therefore, condition (6) is satisfied for all $x,\ y \in X$ with $x \neq y$. Moreover, since F is homogeneous of degree $s = \frac{1}{2}$, we have

(i) $\limsup_{i\to\infty} \delta_{i,\ j} < 1$,

(ii)

$$C_n = \prod_{i=1}^{n} \frac{\delta^s_{i,\ i+1}}{1-\delta^s_{i,\ i+1}} = \prod_{i=1}^{n} \frac{1}{2^i} = \frac{1}{2^{\frac{n(n+1)}{2}}} \leq \frac{1}{2^n}.$$

The conditions of Corollary 2 are satisfied, $\{T_n\}$ has a common fixed point, which is this case $x^* = 0$.

Using the same idea as in the proof of Theorem 3, one can establish the following result.

Theorem 4 Let $(X,\ p,\ K)$ be a complete partial b-metric type space and $\{T_n\}$ be a sequence of self-mappings on X such that

$$F(p(T_i(x),\ T_j(y))) \leq F(\delta_{i,\ j}[p(x,\ T_j(y)) + p(y,\ T_i(x))])$$

for $x,\ y \in X$ with $x \neq y$, $0 \leq \delta_{i,\ j}$, $i,\ j = 1,\ 2,\ \cdots$, and for some $F \in \Phi$ homogeneous with degree s. If

(i) for each, j, $\limsup_{i\to\infty} \delta^s_{i,\ j} < 1$,

(ii)

$$\sum_{n=1}^{\infty} C_n < \infty \ \text{ where } \ C_n = \prod_{i=1}^{n} \frac{\delta^s_{i,\ i+1}}{1-\delta^s_{i,\ i+1}},$$

then $\{T_n\}$ has a unique common fixed point in X.

Corollary 3 In addition to the hypotheses of Theorem 4, supposed that for every $m \in \mathbb{N}$, there exists a $k_m \in \mathbb{N}$ such that $\delta^s_{m,\, k_m} < \frac{1}{2}$, then every T_m has a unique fixed point in X.

Proof. According to Theorem 4 $\{T_m\}$ has a unique common fixed point $u \in X$. If v is a fixed point for T_m, then

$$\begin{aligned} F(p(u,\, v)) &= F(p(T_{k_m}u,\, T_m v)) \\ &\leq F(\delta_{m,\, k_m}[p(u,\, T_m v) + p(v,\, T_{k_m}u)]) \\ &\leq F(\delta_{m,\, k_m}[p(u,\, v) + p(v,\, u)]) \\ &\leq \delta^s_{m,\, k_m}[F(p(u,\, v)) + F(p(u,\, v))] \\ &= 2\delta^s_{m,\, k_m} F(p(u,\, v)) \end{aligned}$$

which implies that $p(u,\, v) = 0$. Therefore $u = v$. □

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Conflict of interest

The authors declare no conflict of interest.

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