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Research Article

Weak Contractions via λ-sequences

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Abstract: In this note, we discuss a common fixed point for a family of self-mapping defined on a metric-type space and satisfying a weakly contractive condition. In our development, we make use of the λ-sequence approach and of a certain class of real-valued maps. We derive some implications for self-mappings on quasi-pseudometric type spaces.

Keywords: metric-type space, common fixed point, λ-sequence

MSC: 47H10, 54H25

1. Introduction and preliminaries

Weakly C-contractive maps have been introduced by Choudhury [1] as a modified version of the contractive conditions proposed by Chatterjee [2] and Kannan [3-4]. All these different types of contractive conditions have been shown to be independent of one another (see [5]) and most of these results have been discussed in considering a single self-map on a metric space. We intend, in the present paper, to extend the result of Choudhury [1] by considering a family of self-maps on a metric-type space. The paper by Jovanović et al. [6] is an interesting reference regarding the common fixed point in metric-type spaces.

The following is the definition in a metric space of a weakly C-contractive map as it appears in Choudhury [1, Definition 1.3].

Definition 1.1. A mapping $T : (X, m) \to (X, m)$, where (X, m) is a metric space is said to be weakly C-contractive or is a weak C-contraction if for all $x, y \in X$, the following inequality holds.

$$m(Tx,Ty) \leq \frac{1}{2}[m(x,Ty)+m(y,Tx)]-\psi(m(x,Tx),m(y,Ty)),$$

where $\psi : [0, +\infty)^2 \to [0, +\infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

It is important to point out that weak C-contractions constitute a very large class of mappings that contains the above-mentioned ones.

Next, we recall the definition of a metric-type space.

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Definition 1.2. (See [7, Definition 1.1]) Let X be a nonempty set, and let the function $D : X \times X \to [0, +\infty)$ satisfy the following properties.

(D1) $D(x, x) = 0$ for any $x \in X$;

(D2) $D(x, y) = D(y, x)$ for any $x, y \in X$;

(D3) $D(x, y) \le K(D(x, z_1) + D(z_1, z_2) + \ldots + D(z_n, y))$ for any points $x, y, z_i \in X$, $i = 1, 2, \ldots, n$ where $n \ge 1$ is a fixed natural number and some constant $K \ge 1$.

The triplet (X, D, K) is called a **metric-type space**.

The class of metric-type spaces is strictly larger than that of metric spaces (see [8]). The concepts of Cauchy sequence, convergence for a sequence, and completeness in a metric-type space are defined in the same way as defined for a metric space. We also recall the definition of a λ-sequence.

Definition 1.3. (See [7, Definition 2.1]) A sequence $(x_n)_{n\ge1}$ in a metric-type space (X, D, K) is a λ-sequence if there exist $0 < \lambda < 1$ and $n(\lambda) \in \mathbb{N}$ such that

$$\sum_{i=1}^{L} D(x_i, x_{i+1}) \le \lambda L \text{ for each } L \ge n(\lambda).$$

Finally, we denote by Φ be the class of continuous, non-decreasing, sub-additive, and homogeneous functions $F : [0, +\infty) \to [0, +\infty)$ such that $F^{-1}(0) = \{0\}$. We can now state our fixed point results.

2. Main results

2.1. *Common fixed point theorems (Kannan-Choudhury case)*

In this section, we prove the existence of a unique common fixed point for a family of contractive-type self-maps on a complete metric-type space by using the Kannan contractive condition as a base.

Theorem 2.1. Let (X, D, K) be a complete metric-type space and $\{T_n\}$ be a sequence of self-mappings on X. Assume that there exist two sequences (a_n) and (b_n) of elements of X such that

$$F(D(T_i(x), T_j(y))) \le F(\delta_{i,j}[D(x, T_i(x)) + D(y, T_j(y))]) - F(\gamma_{i,j}\psi[D(x, T_i(x)), D(y, T_j(y))]) \tag{1}$$

for $x, y \in X$ with $x \ne y$, $0 \le \delta_{i,j}, \gamma_{i,j} < 1$, $i, j = 1, 2, \ldots$, and for some $F \in \Phi$ homogeneous with degree s, where $\delta_{i,j} = D(a_i, a_j)$, $\gamma_{i,j} = D(b_i, b_j)$, and $\psi : [0, +\infty)^2 \to [0, +\infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$. If the sequence (s_n) where $s_i = \dfrac{\delta_{i,i+1}^s}{1-\delta_{i,i+1}^s}$ is a non-increasing λ-sequence of elements of $\mathbb{R}^+ = [0, +\infty)$ endowed with the max metric, then $\{T_n\}$ has a unique common fixed point in X. The max metric m refers to $m(x, x) = 0$ and $m(x, y) = \max\{x, y\}$ if $x \ne y$ for any $x, y \in [0, +\infty)$.

Proof. For any $x_0 \in X$, we construct the Picard type sequence (x_n) by setting $x_n = T_n(x_{n-1})$, $n = 1, 2, \ldots$. Using equation (1) and the homogeneity of F, we obtain

$$\begin{aligned} F(D(x_1, x_2)) &= F(D(T_1(x_0), T_2(x_1))) \\ &\le \delta_{1,2}^s F([D(x_0, T_1(x_0))) + D(x_1, T_2(x_1))]) - \gamma_{1,2}^s F(\psi[D(x_0, T_1(x_0))), D(x_1, T_2(x_1))]) \\ &= \delta_{1,2}^s F([D(x_0, x_1) + D(x_1, x_2)]) - \gamma_{1,2}^s F(\psi[D(x_0, x_1), D(x_1, x_2)]) \\ &\le \delta_{1,2}^s F([D(x_0, x_1) + D(x_1, x_2)]). \end{aligned}$$

Therefore, using the sub-additivity of F, we deduce that

$$(1-\delta_{1,2}^s) F(D(x_1, x_2)) \le (\delta_{1,2}^s) F(D(x_0, x_1)),$$

i.e.,

$$F(D(x_1, x_2)) \leq \left(\frac{\delta_{1,2}^s}{1-\delta_{1,2}^s} \right) F(D(x_0, x_1)).$$

Also, we get

$$\begin{aligned}
F(D(x_2, x_3)) &= F(D(T_2(x_1), T_3(x_2))) \\
&\leq \left(\frac{\delta_{2,3}^s}{1-\delta_{2,3}^s} \right) F(D(x_1, x_2)) \\
&\leq \left(\frac{\delta_{2,3}^s}{1-\delta_{2,3}^s} \right) \left(\frac{\delta_{1,2}^s}{1-\delta_{1,2}^s} \right) F(D(x_0, x_1)).
\end{aligned}$$

By repeating the above process, we have

$$F(D(x_n, x_{n+1})) \leq \prod_{i=1}^{n} \left(\frac{\delta_{i,i+1}^s}{1-\delta_{i,i+1}^s} \right) F(D(x_0, x_1)). \tag{2}$$

Hence, we derive, by making use of the triangle inequality and the properties of F, that for $p > 0$

$$\begin{aligned}
F(D(x_n, x_{n+p})) &\leq K^s [F(D(x_n, x_{n+1})) + F(D(x_{n+1}, x_{n+2})) + \ldots + F(D(x_{n+p-1}, x_{n+p}))] \\
&\leq K^s \left[\prod_{i=1}^{n} \left(\frac{\delta_{i,i+1}^s}{1-\delta_{i,i+1}^s} \right) F(D(x_0, x_1)) + \prod_{i=1}^{n+1} \left(\frac{\delta_{i,i+1}^s}{1-\delta_{i,i+1}^s} \right) F(D(x_0, x_1)) + \ldots + \prod_{i=1}^{n+p-1} \left(\frac{\delta_{i,i+1}^s}{1-\delta_{i,i+1}^s} \right) F(D(x_0, x_1)) \right] \\
&= K^s \left[\sum_{k=0}^{p-1} \prod_{i=1}^{n+k} \left(\frac{\delta_{i,i+1}^s}{1-\delta_{i,i+1}^s} \right) F(D(x_0, x_1)) \right] \\
&= K^s \left[\sum_{k=n}^{n+p-1} \prod_{i=1}^{k} \left(\frac{\delta_{i,i+1}^s}{1-\delta_{i,i+1}^s} \right) F(D(x_0, x_1)) \right].
\end{aligned}$$

Now, let λ and $n(\lambda)$ as in Definition 1.3, then for $n \geq n(\lambda)$ and using the fact that the geometric mean of non-negative real numbers is at most their arithmetic mean, it follows that

$$\begin{aligned}
F(D(x_{\ddot{u}}, x_{\ +})) &\leq K^s \left[\sum_{\ddot{u}=}^{n+p-1} \left[\frac{1}{k} \sum_{=1}^{k} \left(\frac{\delta_{i,i+1}^s}{1-\delta_{i,i+1}^s} \right) \right]^k F(D(x_0, x_1)) \right] \\
&\leq K^s \left[\left(\sum_{k=n}^{n+p-1} \lambda^k \right) F(D(x_0, x_1)) \right] \\
&\leq K^s \frac{\lambda^n}{1-\lambda} F(D(x_0, x_1)).
\end{aligned} \tag{3}$$

Letting $n \to +\infty$ and since $F^{-1}(0) = \{0\}$ and F is continuous, we deduce that $D(x_n, x_{n+p}) \to 0$. Thus (x_n) is a Cauchy sequence and, by completeness of X, converges to say $x^* \in X$. Moreover, for any natural number $m \neq 0$, we have

$$\begin{aligned}
F(D(x_n, T_m(x^*))) &= F(D(T_n(x_{n-1}), T_m(x^*))) \\
&\leq \delta_{n,m}^s [F(D(x_{n-1}, x_n)) + F(D(x^*, T_m(x^*)))] - \gamma_{n,m}^s F(\psi[D(x_{n-1}, x_n), D(x^*, T_m(x^*)]).
\end{aligned}$$

Again, letting $n \to +\infty$, we get

$$F(D(x^*, T_m(x^*))) \leq \delta_{n,m}^s [F(D(x^*, x^*)) + F(D(x^*, T_m(x^*)))] - \gamma_{n,m}^s F(\psi[0, D(x^*, T_m(x^*)])$$
$$\leq \delta_{n,m}^s F(D(x^*, T_m(x^*)),$$

and since $0 \leq \delta_{n,m} < 1$, it follows that $F(D(x^*, T_m(x^*))) = 0$, i.e., $T_m(x^*) = x^*$.

Then, x^* is a common-fixed point of $\{T_m\}_{m\geq 1}$.

To prove the uniqueness of x^*, let us suppose that y^* is a common-fixed point of $\{T_m\}_{m\geq 1}$, that is $T_m(y^*) = y^*$ for any $m \geq 1$. Then, by equation (1), we have

$$\begin{aligned} F(D(x^*, y^*)) &\leq F(D(T_m(x^*), T_m(y^*))) \\ &\leq \delta_{n,m}^s [F(D(x^*, T_m(x^*)) + F(D(y^*, T_m(y^*))] - \gamma_{n,m}^s F(\psi[D(x^*, T_m(x^*), D(y^*, T_m(y^*))]) \\ &= \delta_{n,m}^s [F(0) + F(0)] - \gamma_{n,m}^s F(\psi[0,0]) \\ &= 0. \end{aligned}$$

So, x^* is the unique common-fixed point of $\{T_m\}$.

As particular cases of Theorem 2.1, we have the following two corollaries.

Corollary 2.2. Let (X, D, K) be a complete metric-type space and $\{T_n\}$ be a sequence of self-mappings on X. Assume that there exist two sequences (a_n) and (b_n) of elements of X such

$$D(T_i(x), T_j(y))) \leq \delta_{i,j}[D(x, T_i(x)) + D(y, T_j(y))] - \gamma_{i,j}\psi[D(x, T_i(x)), D(y, T_j(y))] \tag{4}$$

for $x, y \in X$ with $x \neq y$, $0 \leq \delta_{i,j}, \gamma_{i,j} < 1$, $i, j = 1, 2, \ldots$, where $\delta_{i,j} = D(a_i, a_j)$, $\gamma_{i,j} = D(b_i, b_j)$, and $\psi : [0, +\infty)^2 \to [0, +\infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$. If the sequence (s_n) where $s_i = \frac{\delta_{i,i+1}^s}{1 - \delta_{i,i+1}^s}$ is a non-increasing λ-sequence of elements of $\mathbb{R}^+ = [0, +\infty)$ endowed with the max metric, then $\{T_n\}$ has a unique common fixed point in X.

Proof. Apply Theorem 2.1 by putting $F = I_{[0,+\infty)}$, the identity map.

Corollary 2.3. Let (X, D, K) be a complete metric-type space and $\{T_n\}$ be a sequence of self mappings on X. Assume that there exists a sequence (a_n) of elements of X such

$$F(D(T_i(x), T_j(y))) \leq F(\delta_{i,j}[D(x, T_i(x)) + D(y, T_j(y))]) \tag{5}$$

for $x, y \in X$ with $x \neq y$, $0 \leq \delta_{i,j}, \gamma_{i,j} < 1$, $i, j = 1, 2, \ldots$, and for some $F \in \Phi$ homogeneous with degree s, where $\delta_{i,j} = D(a_i, a_j)$. If the sequence (s_n) where $s_i = \frac{\delta_{i,i+1}^s}{1 - \delta_{i,i+1}^s}$ is a non-increasing λ-sequence of elements of $[0, +\infty)$ endowed with the max metric, then $\{T_n\}$ has a unique common fixed point in X.

Proof. Apply Theorem 2.1 by putting $\gamma_{i,j} = 0$. In this case, we can choose (b_n) to be any constant sequence of elements of X.

A more general (natural in some sense) weak contraction could also be considered as we write our next result.

Theorem 2.4. Let (X, D, K) be a complete metric-type space and $\{T_n\}$ be a sequence of self mappings on X. Assume that there exist two sequences (a_n) and (b_n) of elements of X such

$$\begin{aligned} F(D(T_i(x), T_j(y))) &\leq F(\delta_{i,j}[D(x, T_i(x)) + D(y, T_j(y)) + D(x, y)]) \\ &\quad - F(\gamma_{i,j}\psi[D(x, T_i(x)), D(y, T_j(y)), D(x, y)]) \end{aligned} \tag{6}$$

for $x, y \in X$ with $x \neq y$, $0 \leq \delta_{i,j}, \gamma_{i,j} < 1$, $i, j = 1, 2, \ldots$, and for some $F \in \Phi$ homogeneous with degree s, where $\delta_{i,j} = D(a_i, a_j)$, $\gamma_{i,j} = D(b_i, b_j)$, and $\psi : [0, +\infty)^3 \to [0, +\infty)$ is a continuous mapping such that $\psi(x, y, z) = 0$ if and only if

$x = y = z = 0$. If the sequence (s_n) where $s_i = \dfrac{2^s \delta^s_{i,i+1}}{1-\delta^s_{i,i+1}}$ where is a non-increasing λ-sequence of elements of $\mathbb{R}^+ = [0, +\infty)$ endowed with the max metric, then $\{T_n\}$ has a unique common fixed point in X.

We shall omit the proof, as it is merely a copy of that of Theorem 2.1. Then, we also give, without proof, the following corollary.

Corollary 2.5. Let (X, D, K) be a complete metric-type space and $\{T_n\}$ be a sequence of self mappings on X. Assume that there exists a sequence (a_n) of elements of X such

$$F(D(T_i(x),T_j(y))) \le F(\delta_{i,j}[D(x,T_i(x))+D(y,T_j(y))+D(x,y)]), \tag{7}$$

for $x, y \in X$ with $x \neq y$, $0 \le \delta_{i,j}, \gamma_{i,j} < 1$, $i, j = 1, 2, \ldots$, where $\delta_{i,j} = D(a_i, a_j)$, and $\psi : [0, +\infty)^3 \to [0, +\infty)$ is a continuous mapping such that $\psi(x, y, z) = 0$ if and only if $x = y = z = 0$. If the sequence (s_n) where $s_i = \dfrac{2^s \delta^s_{i,i+1}}{1-\delta^s_{i,i+1}}$ is a non-increasing λ-sequence of elements of $\mathbb{R}^+ = [0, +\infty)$ endowed with the max metric, then $\{T_n\}$ has a unique common fixed point in X.

Example 2.6. Let $X = [0, 1]$ and D be the metric defined by $D(x, x) = 0$ and $D(x, y) = \max\{x, y\}$ if $x \neq y$. Clearly, (X, D) is a complete metric space. Just observe that any Cauchy sequence in (X, D) converges to 0. Following the notation in Theorem 2.1, we set $a_i = \left(\dfrac{1}{1+2^i}\right)^2$ so that $\delta_{i,j} = \left(\dfrac{1}{1+2^\eta}\right)^2$ where $\eta = \min\{i, j\}$. We also define $T_i(x) = \dfrac{x}{16^i}$ for all $x \in X$ and $i = 1, 2, \ldots$ and $F : [0, +\infty) \to [0, +\infty)$, $x \mapsto \sqrt{x}$. Then, F is continuous, non-decreasing, sub-additive and homogeneous with degree $s = \dfrac{1}{2}$ and $F^{-1}(0) = \{0\}$.

Assume $i < j$ and $x > y$. Hence, we have

$$F(D(T_i(x),T_j(y))) = \sqrt{\frac{x}{16^i}}$$

and

$$F(\delta_{i,j}[D(x,T_i(x))+D(y,T_j(y))+D(x,y)]) = \sqrt{\left(\frac{1}{1+2^i}\right)^2 (2x+y)}.$$

Therefore, condition (7) is satisfied for all $x, y \in$ X with $x \neq y$. Moreover, since F is homogeneous with degree $s = \dfrac{1}{2}$, the sequence

$$s_i = \frac{2^s \delta^s_{i,i+1}}{1-\delta^s_{i,i+1}} = \frac{\sqrt{2}}{2^i}$$

is a λ-sequence with $\lambda = \dfrac{\sqrt{2}}{2}$. Then, by Corollary 2.5, $\{T_n\}$ has a common fixed point, which is this case $x^* = 0$.

The contractive condition in Theorem 2.1 can be relaxed, and we obtain the next result:

Theorem 2.7. Let (X, D, K) be a complete metric-type space and $\{T_n\}$ be a sequence of self mappings on X such that

$$F(D(T_i(x),T_j(y))) \le F(\delta_{i,j}[D(x,T_i(x))+D(y,T_j(y))]) - F(\gamma_{i,j}\psi[D(x,T_i(x)),D(y,T_j(y))]) \tag{8}$$

for $x, y \in X$ with $x \neq y$, $0 \le \delta_{i,j}, \gamma_{i,j} < 1$, $i, j = 1, 2, \ldots$, for some $F \in \Phi$ homogeneous with degree s and $\psi : [0,+\infty)^2 \to [0,+\infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$. If

(i) for each j, $\limsup_{i\to+\infty} \delta_{i,j}^s < 1$,

(ii) $\sum_{n=1}^{+\infty} C_n < +\infty$, where $C_n = \prod_{i=1}^{n} \frac{\delta_{i,i+1}^s}{1-\delta_{i,i+1}^s}$,

then $\{T_n\}$ has a unique common fixed point in X.

Proof. Just observe that the Picard type sequence $(x_n) = (T_n(x_n-1))$ for any initial $x_0 \in X$ is such that

$$F(D(x_n, x_{n+1})) \le \prod_{i=1}^{n} \left(\frac{\delta_{i,i+1}^s}{1-\delta_{i,i+1}^s} \right) F(D(x_0, x_1)) =: C_n F(D(x_0, x_1)), \tag{9}$$

and so for $p > 1$,

$$F(D(x_n, x_{n+p})) \le K^s \left[\sum_{k=n}^{n+p-1} C_k \right] F(D(x_0, x_1)).$$

Letting $n \to +\infty$, we deduce that $D(x_n, x_{n+p}) \to 0$. Thus (x_n) is a Cauchy sequence and, by completeness of X, converges to say $x^* \in X$. It is easy to see that x^* is the unique common fixed point of $\{T_m\}$.

Corollary 2.8. Let (X, D, K) be a complete metric-type space and $\{T_n\}$ be a sequence of self mappings on X such that

$$F(D(T_i(x), T_j(y))) \le F(\delta_{i,j}[D(x, T_i(x)) + D(y, T_j(y))]) \tag{10}$$

for $x, y \in X$ with $x \ne y$, $0 \le \delta_{i,j} < 1$, $i, j = 1, 2, \ldots$, for some $F \in \Phi$ homogeneous with degree s and $\psi : [0, +\infty)^2 \to [0, +\infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$. If

(i) for each j, $\lim_{i\to+\infty}$ ü $\delta_{i,j}^s < 1$,

(ii) $\sum_{n=1}^{+\infty} C_n < +\infty$, where $C_n = \prod_{i=1}^{n} \frac{\delta_{i,i+1}^s}{1-\delta_{i,i+1}^s}$,

then $\{T_n\}$ has a unique common fixed point in X.

Proof. Apply Theorem 2.7 by putting $\gamma_{i,j} = 0$.

Example 2.9. Let $X = [0,1]$ and $D(x, y) = |x - y|$ whenever $x, y \in [0, 1]$. Clearly, (X, G) is a complete metric space.

Following the notation in Theorem 2.7, we set $\delta_{i,j} = \left(\frac{1}{1+2^i}\right)^2$. We also define $T_i(x) = \frac{x}{4^i}$ for all $x \in X$ and $i = 1, 2, \ldots$ and $F : [0, +\infty) \to [0, +\infty)$, $x \mapsto \sqrt{x}$. Then F is continuous, non-decreasing, sub-additive and homogeneous of degree $s = \frac{1}{2}$ and $F^{-1}(0) = \{0\}$. Assume $i < j$ and $x > y \ge z$. Hence, we have

$$F(D(T_i(x), T_j(y)) = \sqrt{\left| \frac{x}{4^i} - \frac{y}{4^j} \right|}$$

and

$$F(\delta_{i,j}[D(x, T_i(x)) + D(y, T_j(y))]) = \sqrt{\left(\frac{1}{1+2^i}\right)^2 \left(\left| x - \frac{x}{4^i} \right| + \left| y - \frac{y}{4^j} \right| \right)}.$$

Therefore, condition (10) is satisfied for $x, y \in X$ with $x \ne y$. Moreover, since F is homogeneous of degree $s = \frac{1}{2}$, we have

(i) $\limsup_{i\to+\infty} \delta_{i,j}^s < 1$,

(ii) $C_n = \prod_{i=1}^{n} \frac{\delta_{i,i+1}^s}{1-\delta_{i,i+1}^s} = \prod_{i=1}^{n} \frac{1}{2^i} = \frac{1}{2^{\frac{n(n+1)}{2}}} \le \frac{1}{2^n}$.

The conditions of Corollary 2.8 are satisfied, $\{T_n\}$ has a common fixed point, which is in this case $x^* = 0$.

An interesting direction to look into is that of the quasi-pseudometric type spaces which were investigated by Kazeem et al. [9]. In [9], we can read the following definition for a quasi-pseudometric type space:

Definition 2.10. (Compare [9, Definition 29]) Let X be a nonempty set, and let the function $D : X \times X \to [0, +\infty)$ satisfy the following properties:

(q1) $D(x, x) = 0$ for any $x \in X$;

(q2) $D(x, y) \le K(D(x, z_1) + D(z_1, z_2) + \ldots + D(z_n, y))$ for any points $x, y, z_i \in X$, $i = 1, 2, ..., n$ where $n \ge 1$ is a fixed natural number and some constant $K \ge 0$. The triplet (X, D, K) is called a **quasi-pseudometric type space**.

Moreover, if

$$D(x, y) = 0 = D(y, x) \Rightarrow x = y,$$

then D is said to be a T_0-quasi-metric type.

Remark 2.11. For a given T_0-quasi-metric type D, it is easy to verify that the function $D^s(x, y) = \max\{D(x, y), D(y, x)\}$ is metric type. Moreover, a T_0-quasi-metric type space (X, D) will be called **bicomplete** if the metric-type space (X, D^s) is complete.

Theorem 2.1 and Theorem 2.4 can be reformulated in the asymmetric setting respectively as:

Theorem 2.12. (Compare Theorem 2.1) Let (X, D, K) be a bicomplete quasi-metric type space and $\{T_n\}$ be a sequence of self-mappings on X. Assume that there exist two sequences (a_n) and (b_n) of elements of X such

$$F(D(T_i(x), T_j(y))) \le F(\delta_{i,j}[D(x, T_i(x)) + D(T_j(y), y)]) - F(\gamma_{i,j}\psi[D^s(x, T_i(x)), D^s(y, T_j(y))]) \tag{11}$$

for $x, y \in X$ with $x \ne y$, $0 \le \delta_{i,j}, \gamma_{i,j} < 1$, $i, j = 1, 2, \ldots$, and for some $F \in \Phi$ homogeneous with degree t, where $\delta_{i,j} = D^s(a_i, a_j)$, $\gamma_{i,j} = D^s(b_i, b_j)$, and $\psi : [0, +\infty)^2 \to [0, +\infty)$ is a symmetric continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$. If the sequence (s_n) where $s_i = \frac{\delta_{i,i+1}^t}{1-\delta_{i,i+1}^t}$ is a non-increasing λ-sequence of elements of $\mathbb{R}^+ = [0, +\infty)$ endowed with the max metric, then $\{T_n\}$ has a unique common fixed point in X.

Proof. Just observe that for $x, y \in X$ with $x \ne y$ condition (11) gives

$$\begin{aligned}
F(D^{-1}(T_i(x), T_j(y))) &= F(D(T_j(y)), T_i(x)) \\
&\le F(\delta_{j,i}[D(y, T_j(y)) + D(T_i(x), x)]) - F(\gamma_{j,i}\psi[D^s(y, T_j(y)), D^s(x, T_i(x))]) \\
&= F(\delta_{i,j}[D^{-1}(x, T_i(x)) + D^{-1}(T_j(y), y)]) - F(\gamma_{i,j}\psi[D^s(x, T_i(x)), D^s(y, T_j(y))]),
\end{aligned}$$

which implies that

$$F(D^s(T_i(x), T_j(y))) \le F(\aleph_{i,j}[D^s(T_i(x), x) + D(y, T_j(y))]) - F(\quad_{i,j}\quad[D^s(x, T_i(x)), D^s(y, T_j(y))]),$$

i.e., the conditions of Theorem 2.1 are fulfilled. This completes the proof.

Similarly, we have

Theorem 2.13. (Compare Theorem 2.4) Let (X, D, K) be a bicomplete quasi-metric type space and $\{T_n\}$ be a sequence of self-mappings on X. Assume that there exist two sequences (a_n) and (b_n) of elements of X such

$$\begin{aligned}
F(D(T_i(x), T_j(y))) \le F(\quad_{i,j}[D(x, T_i(x)) + D(T_j(y), y) + D(x, y)]) \\
- F(\gamma_{i,j}\psi[D^s(x, T_i(x)), D^s(y, T_j(y)), D^s(x, y)])
\end{aligned} \tag{12}$$

for $x, y \in X$ with $x \neq y$, $0 \leq \delta_{i,j}, \gamma_{i,j} < 1, i, j = 1, 2, \ldots$, and for some $F \in \Phi$ homogeneous with degree t, where $\delta_{i,j} = D^s(a_i, a_j)$, $\gamma_{i,j} = D^s(b_i, b_j)$, and $\psi : [0, +\infty)^3 \to [0, +\infty)$ is a symmetric continuous mapping such that $\psi(x, y, z) = 0$ if and only if $x = y = z = 0$. If the sequence (s_n) where $s_i = \dfrac{2^t \delta^t_{i,i+1}}{1-\delta^t_{i,i+1}}$ is a non-increasing λ-sequence of elements of $\mathbb{R}^+ = [0, +\infty)$ endowed with the max metric, then $\{T_n\}$ has a unique common fixed point in X.

2.2 *Common fixed point theorems (Chatterjea-Choudhury case)*

In this section, we prove the existence of a unique common fixed point for a family of contractive-type self-maps on a complete metric-type space by using the Chatterjea contractive condition as a base.

Theorem 2.14. Let (X, D, K) be a complete metric-type space and $\{T_n\}$ be a sequence of self-mappings on X. Assume that there exist two sequences (a_n) and (b_n) of elements of X such that

$$F(D(T_i(x), T_j(y))) \leq F(\delta_{i,j}[D(x, T_j(y)) + D(y, T_i(x))]) - F(\gamma_{i,j}\psi[D(x, T_j(y)), D(y, T_i(x))]) \tag{13}$$

for $x, y \in X$ with $x \neq y$, $0 \leq \delta_{i,j}, \gamma_{i,j} < 1, i, j = 1, 2, \ldots$, and for some $F \in \Phi$ homogeneous with degree s, where $\delta_{i,j} = D(a_i, a_j)$, $\gamma_{i,j} = D(b_i, b_j)$, and $\psi : [0, +\infty)^2 \to [0, +\infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$. If the sequence (s_n) where $s_i = \dfrac{\delta^s_{i,i+1}}{1-\delta^s_{i,i+1}}$ is a non-increasing λ-sequence of elements of $\mathbb{R}^+ = [0, +\infty)$ endowed with the max metric, then $\{T_n\}$ has a unique common fixed point in X.

Proof. The proof follows exactly the same steps as the proof of Theorem 2.1.

Theorem 2.15. Let (X, D, K) be a complete metric-type space and $\{T_n\}$ be a sequence of self-mappings on X. Assume that there exist two sequences (a_n) and (b_n) of elements of X such that

$$\begin{aligned} F(D(T_i(x), T_j(y))) &\leq F(\delta_{i,j}[D(x, T_j(x)) + D(y, T_i(y)) + D(x, y)]) \\ &\quad - F(\gamma_{i,j}\psi[D(x, T_j(y)), D(y, T_i(x)), D(x, y)]) \end{aligned} \tag{14}$$

for $x, y \in X$ with $x \neq y$, $0 \leq \delta_{i,j}, \gamma_{i,j} < 1, i, j = 1, 2, \ldots$, and for some $F \in \Phi$ homogeneous with degree s, where $\delta_{i,j} = D(a_i, a_j)$, $\gamma_{i,j} = D(b_i, b_j)$, and $\psi : [0, +\infty)^3 \to [0, +\infty)$ is a continuous mapping such that $\psi(x, y, z) = 0$ if and only if $x = y = z = 0$. If the sequence (s_n) where is $s_i = \dfrac{2^s \delta^s_{i,i+1}}{1-\delta^s_{i,i+1}}$ a non-increasing λ-sequence of elements of $\mathbb{R}^+ = [0, +\infty)$ endowed with the max metric, then $\{T_n\}$ has a unique common fixed point in X.

Proof. The proof follows exactly the same steps as the proof of Theorem 2.4.

Like in the case of Theorem 2.1, the condition (13) can also be relaxed, and we obtain:

Theorem 2.16. Let (X, D, K) be a complete metric-type space and $\{T_n\}$ be a sequence of self mappings on X such that

$$F(D(T_i(x), T_j(y))) \leq F(\delta_{i,j}[D(x, T_j(y)) + D(y, T_i(x))]) - F(\gamma_{i,j}\psi[D(x, T_j(y)), D(y, T_i(x))]) \tag{15}$$

for $x, y \in X$ with $x \neq y$, $0 \leq \delta_{i,j}, \gamma_{i,j}, i, j = 1, 2, \ldots$, and for some $F \in \Phi$ homogeneous with degree s and $\psi : [0, +\infty)^2 \to [0, +\infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$. If

(i) for each j, $\limsup_{i \to +\infty} \delta^s_{i,j} < 1$,

(ii) $\sum_{n=1}^{+\infty} C_n < +\infty$ where $C_n = \prod_{i=1}^{n} \dfrac{\delta^s_{i,i+1}}{1-\delta^s_{i,i+1}}$,

then $\{T_n\}$ has a unique common fixed point in X.

Also, it is not necessary to point out that this relaxed condition can be applied to modify the hypotheses of Theorem 2.15.

Corollary 2.17. In addition to hypotheses of Theorem 2.16, suppose that for every $n \geq 1$, there exists $k(n) \geq 1$ such that $a_{n,k(n)} < \frac{1}{2}$; then every T_n has a unique fixed point in X.

Remark 2.18. This corollary emphasizes the subtle fact that in addition to the existence of a unique common fixed for the family $\{T_n\}$, every single map T_n has a unique fixed point, which happens to be the common one. Also, a similar corollary can be formulated (we shall suppose that for every $n \geq 1$, there exists a $k(n) \geq 1$ such that $a_{n,k(n)} < \frac{1}{3}$) in the relaxed form of Theorem 2.15.

Proof. From Theorem 2.16, we know that the family $\{T_n\}$ has a unique common fixed point $x^* \in X$. If y^* is a fixed point of a given, T_m then

$$\begin{aligned} D(x^*, y^*) &= D(T_{k(m)x^*}, T_m y^*) \\ &\leq a_{k(m),m}[D(x^*, T_m y^*) + D(y^*, T_{k(m)} x^*)] \\ &= a_{k(m),m}[D(x^*, y^*) + D(y^*, x^*)] \\ &< D(x^*, y^*), \end{aligned}$$

which implies $D(x^*,y^*) = 0$; which gives the desired result.

Example 2.19. We endow the set $X = [0,1]$ with the usual distance $d(x, y) = |x - y|$ for $x, y \in X$. Then, (X, d) is a complete metric space. We set $F(x) = x$ whenever $x \in X$, then $F \in \Phi$ homogeneous with degree 1. For $n \geq 1$, we define the family $\{T_n\}$ of self-mappings on X by

$$T_n = \begin{cases} 1 & : 0 < x \leq 1, \\ \frac{2}{3} + \frac{1}{n+2} & : x = 0. \end{cases}$$

By using the notation of Theorem 2.16, we use the family of reals

$$\delta_{i,j} = \frac{1}{3} + \frac{1}{|i-j|+6}; \qquad \gamma_{i,j} = 0.$$

Then, clearly for each j,

$$\limsup_{i \to +\infty} \delta_{i,j}^s < 1,$$

and

$$C_n = \prod_{i=1}^{n} \frac{\delta_{i,i+1}^s}{1 - \delta_{i,i+1}^s} = \left(\frac{10}{11}\right)^n,$$

which is a convergent series.

Also, it is a straightforward calculation to verify that,

$$F(D(T_i(x), T_j(y))) \leq F(\delta_{i,j}[D(x, T_j(y)) + D(y, T_i(x))]) - F(\gamma_{i,j}\psi[D(x, T_j(y)), D(y, T_i(x))]) \tag{16}$$

i.e., equivalently,

$$D(T_i(x), T_j(y)) \leq \delta_{i,j}[D(x, T_j(y)) + D(y, T_i(x))]. \tag{17}$$

One just has to consider the three possible cases:

- $x \in (0, 1]$ and $y \in (0, 1]$;
- $x \in (0, 1]$ and $y = 0$;
- $x = y = 0$ with $i \neq j$.

So, all the conditions of Theorem 2.16 are satisfied, and note that $x = 1$ is the only fixed point for the T_n's.

3. Conclusion and future work

Using the same idea as in the proofs of Theorem 2.1 and Theorem 2.4, one can establish the following results.

Theorem 3.1. Let (X, D, K) be a complete metric-type space and $\{T_n\}$ be a sequence of self-mappings on X. Assume that there exist two sequences (a_n) and (b_n) of elements of X such

$$F(D(T_i^p(x), T_j^p(y))) \leq F(\delta_{i,j}[D(x, T_i^p(x)) + D(y, T_j^p(y))]) - F(\gamma_{i,j}\psi[D(x, T_i^p(x)), D(y, T_j^p(y))]) \tag{18}$$

for $x, y \in X$ with $x \neq y, p \geq 1,\ 0 \leq \delta_{i,j}, \gamma_{i,j} < 1, i, j = 1, 2, \ldots$, and for some $F \in \Phi$ homogeneous with degree s, where $\delta_{i,j} = D(a_i, a_j)$, $\gamma_{i,j} = D(b_i, b_j)$, and $\psi : [0, +\infty)^2 \to [0, +\infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$. If the sequence (s_n) where $s_i = \dfrac{\delta_{i,i+1}^s}{1-\delta_{i,i+1}^s}$ is a non-increasing λ-sequence of elements of $\mathbb{R}^+ = [0, +\infty)$ endowed with the max metric, then $\{T_n\}$ has a unique common fixed point in X.

Theorem 3.2. Let (X, D, K) be a complete metric-type space and $\{T_n\}$ be a sequence of self-mappings on X. Assume that there exist two sequences (a_n) and (b_n) of elements of X such

$$\begin{aligned} F(D(T_i^p(x), T_j^p(y))) &\leq F(\delta_{i,j}[D(x, T_i^p(x)) + D(y, T_j^p(y)) + D(x, y)]) \\ &\quad - F(\gamma_{i,j}\psi[D(x, T_i^p(x)), D(y, T_j^p(y)), D(x, y)]) \end{aligned} \tag{19}$$

for $x, y \in X$ with $x \neq y$, $p \geq 1$, $0 \leq \delta_{i,j}, \gamma_{i,j} < 1$, $i, j = 1, 2, \ldots$, and for some $F \in \Phi$ homogeneous with degree s, where $\delta_{i,j} = D(a_i, a_j)$, $\gamma_{i,j} = D(b_i, b_j)$, and $\psi : [0, +\infty)^3 \to [0, +\infty)$ is a continuous mapping such that $\psi(x, y, z) = 0$ if and only if $x = y = z = 0$. If the sequence (s_n) where $s_i = \dfrac{2^s\delta_{i,i+1}^s}{1-\delta_{i,i+1}^s}$ is a non-increasing λ-sequence of elements of $\mathbb{R}^+ = [0, +\infty)$ endowed with the max metric, then $\{T_n\}$ has a unique common fixed point in X.

Moreover, the above two generalizations also apply to Theorem 2.14 and Theorem 2.15 respectively.

Recently the so-called C-class functions, which were introduced by Ansari [10] in 2014 and cover a large class of contractive conditions, have been applied successfully in the generalization of certain contractive conditions. We read

Definition 3.3. [10] A continuous function $f : [0, +\infty)^2 \to \mathbb{R}$ is called C-class function if for any $s, t \in [0, +\infty)$, the following conditions hold:

(1) $f(s, t) \leq s$;

(2) $f(s, t) = s$ implies that either $s = 0$ or $t = 0$.

We shall denote by $\mathcal{C}$ the collection of C-class functions.

Example 3.4. [10] The following examples show that the class $\mathcal{C}$ is nonempty:

(1) $f(s, t) = s - t$.

(2) $f(s, t) = ms$, for some $m \in (0, 1)$.

(3) $f(s, t) = \dfrac{s}{(1+t)^r}$ for some $r \in (0, +\infty)$.

(4) $f(s, t) = \log(t + a^s)/(1 + t)$, for some $a > 1$.

Therefore, the authors plan to study, in another manuscript, the existence of common fixed points for a family of mappings $T_i : (X, D, K) \to (X, D, K)$, $i = 1, 2, \ldots$, defined on a metric-type space (X, D, K), which satisfy:

$$F(D(T_i(x),T_j(y))) \le f(F(\delta_{i,j}[D(x,T_i(x))+D(y,T_j(y))+D(x,y)]),$$
$$F(\gamma_{i,j}\psi[D(x,T_i(x)),D(y,T_j(y)),D(x,y)])) \tag{20}$$

for $x, y \in X$ with $x \neq y$ where

(1) $f \in \mathcal{C}$,

(2) (a_n) and (b_n) are sequences of elements of X and $\delta_{i,j} = D(a_i, a_j)$, $\gamma_{i,j} = D(b_i, b_j)$ with $0 \le \delta_{i,j}, \gamma_{i,j} < i, j = 1, 2, \ldots,$

(3) $F \in \Phi$ homogeneous with degree s,

(4) $\psi : [0, +\infty)^3 \to [0, +\infty)$ is a continuous mapping such that $\psi(x, y, z) = 0$ if and only if $x = y = z = 0$.

under the condition that $s_i = \dfrac{2^s \delta_{i,i+1}^s}{1-\delta_{i,i+1}^s}$ is a non-increasing λ-sequence of elements of $\mathbb{R}^+ = [0, +\infty)$ endowed with the max metric.

Moreover, if the existence of a common fixed point is established in that setting, therefore Theorems 2.1 and 2.4 become direct corollaries by just setting $f(s, t) = s - t$ and the same applies to their equivalent asymmetric formulations.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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