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Research Article

New Type of Characterizations of Pre-Open (Closed and Continuous) Mappings

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Abstract: In this paper, new characterizations of pre-open (pre-closed, pre-continuous) maps will be obtained using the map $\eta^{\#}$ induced via a map η between any two topological spaces. Further, these maps are characterized in terms of saturated sets under some sufficient conditions.

***Keywords*:** pre-open, pre-closed, pre-continuous, pre-closure, saturated

MSC: 54C05, 54C08, 54C10

1. Introduction

In [1], Mashhour et al. introduced pre-open sets and obtained some of its properties. Also various characterizations of these pre-open sets in terms of other type of open sets viz. semi-open, regular-open, α-open sets etc. were given by them. After introducing the notion of pre-open sets, they brought in the concept of pre-open mappings and pre-continuous mappings between topological spaces along with its several characterizations. In [2], El-Deeb et al. introduced pre-closure of a set and obtained some related properties. Pre-closed maps were also introduced and characterized by them using the induced image map. Therefore, investigation of various topological concepts using pre-open and pre-closed maps has been done by various authors viz. [3-6] etc. Further, the theory of induced mappings is of great significance in many spaces of topology. Therefore, apart from inducing the usual image and inverse image maps between power sets there was another induced map introduced denoted by $\eta^{\#}$ corresponding to any map $\eta : U \rightarrow W$ [7] defined on $\wp(U) \rightarrow \wp(W)$.

In this paper, new characterizations of pre-open maps will be obtained in terms of closure operator rather than interior operator using the induced map $\eta^{\#}$ via a map η between any two topological spaces (Theorem 2.1 below) and utilize it to show that inverse image of pre-dense set is dense (Corollary 2.4 below). This motivates to characterize pre-closed maps in terms of interior rather than in terms of closure operator (Theorem 2.5 below). Further, pre-continuous maps are characterized in terms of interior operator using the induced map $\eta^{\#}$ (Theorem 2.8) and in terms of pre-open as well as pre-closed sets under the sufficient condition of surjectivity (Theorem 2.10 below). Relationship between closure and pre-closure of a set using the induced map $\eta^{\#}$ under pre-continuous injective maps is also given (Theorem 2.11 below) and utilize it to show that # image of pre-dense set is dense (Corollary 2.14 below). Finally, these maps

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are characterized in terms of saturated sets under some sufficient conditions. To show the necessity of some of the conditions assumed counterexamples are also provided.

"Given a topological space (U, σ), for a subset M of U, M is pre-open [1] subset of U if $M \subseteq int(cl(M))$ and complement of M is said to be pre-closed [2] subset of U. The family of all pre-open sets is denoted by $PO(U)$. Further, the intersection of all pre-closed sets containing M is known as pre-closure [2] of M denoted by $pcl(M)$ and the union of all pre-open sets contained in M is known as pre-interior [1] of M, denoted by $pint(M)$. Also a subset M is said to be dense (pre-dense) subset of U if $cl(M) = U(pcl(M) = U)$."

We will also make use of the following results :

Properties 1.1 Let (U, σ) be a topological space and M be any subset of U. Then

(a) $pint(M) = M \cap int(cl(M))$ [1].

(b) $pcl(M) = M \cup cl(int(M))$ [2].

(c) $pcl(U - M) = U - pint(M)$ [3].

Definition 1.2 [1] A mapping $\eta : (U, \sigma) \to (W, v)$ is called pre-open (pre-continuous) if image (inverse image) of open set in $U(W)$ is pre-open in $W(U)$.

Definition 1.3 [2] A mapping $\eta : (U, \sigma) \to (W, v)$ is called pre-closed if image of closed set in U is pre-closed in W.

Theorem 1.4 [1] A mapping $\eta : (U, \sigma) \to (W, v)$ is pre-open if and only if $\eta(int(M)) \subseteq pint(\eta(M))$ for any subset M of U.

Theorem 1.5 [1] A mapping $\eta : (U, \sigma) \to (W, v)$ is pre-continuous if and only if $\eta(pcl(M)) \subseteq cl(\eta(M))$ for any subset M of U.

Theorem 1.6 [2] A mapping $\eta : (U, \sigma) \to (W, v)$ is pre-closed if and only if $pcl(\eta(M)) \subseteq \eta(cl(M))$ for any subset M of U.

Definition 1.7 [7] For any two sets U and W, let $\eta : U \to W$ be any map and M be any subset of U. Then # image of a set M in U, denoted by $\eta^{\#}(M)$ is a set in W defined by $\eta^{\#}(M) = \{p \in W : \eta^{-1}(p) \subseteq M\}$. Therefore, for given any map $\eta : U \to W$, # image of a set in U gives rise to an induced map $\eta^{\#} : \wp(U) \to \wp(W)$.

Definition 1.8 [7] For any two sets U and W, let $\eta : U \to W$ be any map and M be any subset of U. Then M is called saturated subset of U if $M = \eta^{-1}(N)$ for some subset N of W.

Note: If M is saturated subset of U then $M^c = U - M$ is also saturated subset of U. Since $M^c = (\eta^{-1}(N))^c = \eta^{-1}(N^c)$ and hence M^c is saturated subset of U.

Properties 1.9 [7] For any two sets U and W, let $\eta : U \to W$ be any map and M be any subset of U. Then

(a) $\eta^{\#}(M^c) = (\eta(M))^c$ and so $\eta^{\#}(M) = (\eta(M^c))^c$ and $\eta(M) = (\eta^{\#}(M^c))^c$.

(b) $\eta^{-1}(\eta^{\#}(M)) = M^{\#} = \{\eta^{-1}(p) : p \in W$ and $\eta^{-1}(p) \subseteq M\} \subseteq M$.

(c) M is saturated if and only if $M = M^{\#}$.

(d) $N \subseteq \eta^{\#}(\eta^{-1}(N))$ for any subset N of W.

(e) $\eta^{\#}(U) = W$.

2. Results

In [1], Mashhour et al. gave the characterization of pre-open maps in terms of interior operator using the induced image map. But the following theorem gives the characterization of pre-open maps using induced map $\eta^{\#}$ under given any map $\eta : U \to W$ in terms of closure operator.

Theorem 2.1 Let $\eta : (U, \sigma) \to (W, v)$ be any map. Then the following conditions are equivalent:

(a) η is pre-open map.

(b) $pcl(\eta^{\#}(M)) \subseteq \eta^{\#}(cl(M))$ for every subset M of U.

(c) $cl(int(\eta^{\#}(M))) \subseteq \eta^{\#}(cl(M))$ for every subset M of U.

(d) For each closed subset K of U, $\eta^{\#}(K)$ is pre-closed in W.

(e) $\eta^{-1}(pcl(N)) \subseteq cl(\eta^{-1}(N))$ for every subset N of W.

Proof. $(a) \Rightarrow (b)$: For any subset M of U, consider an element $p \in pcl(\eta^{\#}(M))$. We have to prove that $p \in \eta^{\#}(cl(M))$ which means by definition of $\eta^{\#}$, the fiber $\eta^{-1}(p) \subseteq cl(M)$. For this, consider an element $e \in \eta^{-1}(p)$ and an open set G

containing e in U then by (a), $\eta(G)$ is pre-open set containing $\eta(e) = p$. Therefore, $p \in pcl(\eta^{\#}(M))$ implies that $\eta(G) \cap \eta^{\#}(M) \neq \emptyset$ and so for an element $q \in \eta(G) \cap \eta^{\#}(M)$, we have $q \in \eta(G)$ and $\eta^{-1}(q) \subseteq M$. This implies that for an element $g \in G$ where $\eta(g) = q$, $G \cap M \neq \emptyset$, since $g \in G \cap M$ and so $e \in cl(M)$. Thus, the fiber $\eta^{-1}(p) \subseteq cl(M)$ and hence (b) holds.

(b) $\Rightarrow$ (c) : Since, for any subset M of U, $pcl(M) = M \cup cl(int(M))$. Therefore, by ($b$), $\eta^{\#}(M) \cup cl(int(\eta^{\#}(M))) = pcl(\eta^{\#}(M)) \subseteq \eta^{\#}(cl(M))$ and so $cl(int(\eta^{\#}(M))) \subseteq \eta^{\#}(cl(M))$. Hence ($c$) holds.

(c) $\Rightarrow$ (d) : For any closed subset K of U, by (c), it follows that $cl(int(\eta^{\#}(K))) \subseteq \eta^{\#}(cl(K)) = \eta^{\#}(K)$ and hence (d) holds by definition of pre-closed sets.

(d) $\Rightarrow$ (e) : For any subset N of W, consider an element $e \in \eta^{-1}(pcl(N))$. We have to prove that $e \in cl(\eta^{-1}(N))$. For this, consider an open set G containing e in U and so $K = U - G$ is closed subset of U then by (d), $\eta^{\#}(K)$ is pre-closed subset of W not containing $\eta(e)$. Therefore, $(\eta^{\#}(K))^c$ is pre-open set containing $\eta(e)$ in W and so $e \in \eta^{-1}(pcl(N))$ i.e. $\eta(e) \in pcl(N)$ implies that $(\eta^{\#}(K))^c \cap N \neq \emptyset$. Thus, for an element $p \in (\eta^{\#}(K))^c \cap N$, we have $p \in (\eta^{\#}(K))^c$ and $p \in N$. Now, $p \notin \eta^{\#}(K)$ implies that $\eta^{-1}(p) \not\subseteq K$ and so $\eta^{-1}(p) \cap K^c \neq \emptyset$ i.e. $\eta^{-1}(p) \cap G \neq \emptyset$. Further, $\eta^{-1}(p) \subseteq \eta^{-1}(N)$ implies that $\eta^{-1}(N) \cap G \neq \emptyset$ and hence (e) holds.

(e) $\Rightarrow$ (a) : For any subset G open in U, consider the subset $N = W - \eta(G)$ of W in (e). Therefore, $\eta^{-1}(pcl(W - \eta(G))) \subseteq cl(\eta^{-1}(W - \eta(G)))$ and so $\eta^{-1}(W - pint(\eta(G))) \subseteq cl(\eta^{-1}(\eta(G)))^c$, since $pcl(W - N) = W - pint(N)$. This implies that $(\eta^{-1}(pint(\eta(G))))^c \subseteq cl(\eta^{-1}(\eta(G)))^c \subseteq cl(G^c) = (int(G))^c$ and so $G = int(G) \subseteq \eta^{-1}(pint(\eta(G)))$. Further, $\eta(G) \subseteq \eta(\eta^{-1}(pint(\eta(G))) \subseteq pint(\eta(G))$ and so $\eta(G)$ is pre-open subset of W. Hence (a) holds. □

The following Corollary gives the characterization of pre-open maps under some sufficient condition.

Corollary 2.2 Let $\eta : (U, \sigma) \rightarrow (W, v)$ be a surjective map. Then η is pre-open if and only if for every closed subset M of U, $\eta(M^{\#})$ is pre-closed subset of W.

Proof. For any subset M of U, by definition of $M^{\#}$ (Property 1.9(b)), it follows that $\eta(M^{\#}) = \eta(\eta^{-1}(\eta^{\#}(M)))$. Further, η is surjective implies $\eta(\eta^{-1}(\eta^{\#}(M))) = \eta^{\#}(M)$ and so $\eta(M^{\#}) = \eta^{\#}(M)$. Hence the result holds by the equivalence of Theorem 2.1 (a) & (d). □

The following corollary gives the characterization of pre-open maps for saturated closed sets under some sufficient condition.

Corollary 2.3 Let $\eta : (U, \sigma) \rightarrow (W, v)$ be a surjective map. Then η is pre-open if and only if for every closed saturated subset M of U, $\eta(M)$ is pre-closed subset of W.

Proof. For any saturated subset M of U, by Property 1.9(c), it follows that $\eta(M) = \eta(M^{\#})$. Hence the result holds by Corollary 2.2. □

The following corollary shows that inverse image of pre-dense set is dense under pre-open maps.

Corollary 2.4 Let $\eta : (U, \sigma) \rightarrow (W, v)$ be pre-open map. Then inverse image of pre-dense subset of W is dense subset in U.

Proof. Let N be any pre-dense subset of W then $pcl(N) = W$. Therefore, by the equivalence of Theorem 2.1 (a) & (e), it follows that $\eta^{-1}(pcl(N)) \subseteq cl(\eta^{-1}(N))$ and so $U = \eta^{-1}(W) \subseteq cl(\eta^{-1}(N))$. This implies $cl(\eta^{-1}(N)) = U$ and hence $\eta^{-1}(N)$ is dense in U. □

Characterization of pre-open maps in terms of closure operator also motivates to characterize pre-closed maps in terms of interior operator rather than closure operator already introduced in literature. Therefore, the following theorem gives the characterization of pre-closed maps using induced map $\eta^{\#}$ under given any map $\eta : U \rightarrow W$ in terms of interior operator.

Theorem 2.5 Let $\eta : (U, \sigma) \rightarrow (W, v)$ be any map. Then the following conditions are equivalent:

(a) η is pre-closed map.

(b) $\eta^{\#}(int(M)) \subseteq pint(\eta^{\#}(M))$ for every subset M of U.

(c) $\eta^{\#}(int(M)) \subseteq int(cl(\eta^{\#}(M)))$ for every subset M of U.

(d) For each open subset G of U, $\eta^{\#}(G)$ is pre-open in W.

Proof. (a) $\Rightarrow$ (b) : For any subset M of U, consider an element $p \in \eta^{\#}(int(M))$. We have to prove that $p \in pint(\eta^{\#}(M))$ which means by definition of pre-interior, there exists a pre-open set V in W such that $p \in V \subseteq \eta^{\#}(M)$, which further mean by definition of $\eta^{\#}$, we have to prove that $\eta^{-1}(V) \subseteq M$. Now, $p \in \eta^{\#}(int(M))$ implies that the fiber $\eta^{-1}(p) \subseteq$

$int(M)$. Therefore, there exists an open set G in U such that $\eta^{-1}(p) \subseteq G \subseteq M$ and so by (a), $\eta(G^c)$ is pre-closed and so $(\eta(G^c))^c$ is pre-open set containing p in W. Further, consider the pre-open set $V = (\eta(G^c))^c$ then $\eta^{-1}(V) = \eta^{-1}((\eta(G^c))^c) = (\eta^{-1}(\eta(G^c)))^c \subseteq (G^c)^c = G \subseteq M$. Thus, there exists a pre-open set V such that $\eta^{-1}(V) \subseteq M$ and hence (b) holds.

(b) $\Rightarrow$ (c) : Since, for any subset M of U, $pint(M) = M \cap int(cl(M))$. Therefore, by ($b$), $\eta^\#(int(M)) \subseteq pint(\eta^\#(M)) = \eta^\#(M) \cap int(cl(\eta^\#(M))) \subseteq int(cl(\eta^\#(M)))$ and so $\eta^\#(int(M)) \subseteq int(cl(\eta^\#(M)))$. Hence ($c$) holds.

(c) $\Rightarrow$ (d) : For any open subset G of U, by (c), it follows that $\eta^\#(int(G)) \subseteq int(cl(\eta^\#(G)))$ and so $\eta^\#(G) \subseteq int(cl(\eta^\#(G)))$. Hence ($d$) holds by definition of pre-open sets.

(d) $\Rightarrow$ (a) : For any closed subset K of U, consider the subset $G = U - K$ open in U then by (d), $\eta^\#(G)$ is pre-open in W i.e. $\eta^\#(K^c)$ is pre-open in W. Therefore, by Property 1.9(a), it follows that $(\eta(K))^c$ is pre-open in W and so $\eta(K)$ is pre-closed in W. Hence η is pre-closed map. □

The following Corollary gives the characterization of pre-closed maps in terms of pre-open sets using the induced image surjective maps.

Corollary 2.6 Let $\eta : (U, \sigma) \to (W, v)$ be a surjective map. Then η is pre-closed if and only if for every open subset M of U, $\eta(M^\#)$ is pre-open subset of W.

Proof. For any subset M of U, by definition of $M^\#$ (Property 1.9(b)), it follows that $\eta(M^\#) = \eta(\eta^{-1}(\eta^\#(M)))$. Further, η is surjective implies $\eta(\eta^{-1}(\eta^\#(M))) = \eta^\#(M)$ and so $\eta(M^\#) = \eta^\#(M)$. Hence the result holds by the equivalence of Theorem 2.5 (a) & (d). □

The following Corollary gives the characterization of pre-closed maps for saturated open sets under some sufficient condition.

Corollary 2.7 Let $\eta : (U, \sigma) \to (W, v)$ be a surjective map. Then η is pre-closed if and only if for every open saturated subset M of U, $\eta(M)$ is pre-open subset of W.

Proof. For any saturated subset M of U, by Property 1.9(c), it follows that $\eta(M) = \eta(M^\#)$. Hence the result holds by Corollary 2.6. □

Next we characterize pre-continuous maps in terms of interior operator using the induced map $\eta^\#$.

Theorem 2.8 Let $\eta : (U, \sigma) \to (W, v)$ be any map. Then the following conditions are equivalent:

(a) η is pre-continuous map.

(b) $int(\eta^\#(M)) \subseteq \eta^\#(pint(M))$ for every subset M of U.

(c) $int(\eta^\#(M)) \subseteq \eta^\#(int(cl(M)))$ for every subset M of U.

Proof. (a) $\Rightarrow$ (b) : For any subset M of U, consider an element $p \in int(\eta^\#(M))$. We have to prove that $p \in \eta^\#(pint(M))$ which means by definition of $\eta^\#$, we have to prove that the fiber $\eta^{-1}(p) \subseteq pint(M)$ i.e. by definition of pre-interior, we have to prove that there exists a pre-open set V such that $\eta^{-1}(p) \subseteq V \subseteq M$. Now, $p \in int(\eta^\#(M))$ implies that there exists an open set G in W such that $p \in G \subseteq \eta^\#(M)$ and so by (a), $\eta^{-1}(G)$ is pre-open set containing $\eta^{-1}(p)$ in U such that $\eta^{-1}(p) \subseteq \eta^{-1}(G) \subseteq \eta^{-1}(\eta^\#(M)) \subseteq M$. Therefore, for the pre-open set $V = \eta^{-1}(G)$, $\eta^{-1}(p) \subseteq V \subseteq M$. Hence ($b$) holds.

(b) $\Rightarrow$ (c) : Since, for any subset M of U, $pint(M) = M \cap int(cl(M))$. Therefore, by ($b$), $int(\eta^\#(M)) \subseteq \eta^\#(pint(M)) = \eta^\#(M \cap int(cl(M)) \subseteq \eta^\#(int(cl(M)))$ and so $int(\eta^\#(M)) \subseteq \eta^\#(int(cl(M)))$. Hence ($c$) holds.

(c) $\Rightarrow$ (a) : Consider an open subset N in W then for the subset $M = \eta^{-1}(N)$ in U, (c) implies that $int(\eta^\#(\eta^{-1}(N))) \subseteq \eta^\#(int(cl(\eta^{-1}(N))))$. Therefore, by Property 1.9(d), it follows that $int(N) \subseteq int(\eta^\#(\eta^{-1}(N))) \subseteq \eta^\#(int(cl(\eta^{-1}(N))))$ and so $N \subseteq \eta^\#(int(cl(\eta^{-1}(N))))$, since N is open in W. Further, $\eta^{-1}(N) \subseteq \eta^{-1}(\eta^\#(int(cl(\eta^{-1}(N))))) \subseteq int(cl(\eta^{-1}(N)))$ using Property 1.9(b). Hence $\eta^{-1}(N)$ is pre-open subset in U and so η is pre-continuous map. □

The following Corollary gives the characterization of pre-continuous mappings under the assumption of surjectivity.

Corollary 2.9 Let $\eta : (U, \sigma) \to (W, v)$ be a surjective map. Then η is pre-continuous map if and only if $int(\eta(M^\#)) \subseteq \eta((pint(M))^\#)$ for every subset M of U.

Proof. For any subset M of U, by definition of $M^\#$ (Property 1.9(b)), it follows that $\eta(M^\#) = \eta(\eta^{-1}(\eta^\#(M)))$. Further, η is surjective implies $\eta(\eta^{-1}(\eta^\#(M))) = \eta^\#(M)$ and so $\eta(M^\#) = \eta^\#(M)$. Hence the result holds by the equivalence of Theorem 2.8 (a) & (b). □

The following theorem gives another characterizations of pre-continuous mappings under the assumption of surjectivity.

Theorem 2.10 Let $\eta : (U, \sigma) \to (W, v)$ be a surjective map. Then the following are equivalent.

(a) η is pre-continuous map.

(b) $M^{\#}$ is pre-open in U whenever $\eta(M^{\#})$ is open in W.

(c) $M^{\#}$ is pre-open in U whenever $\eta^{\#}(M)$ is open in W.

(d) for any saturated set M in U, M is pre-closed in U whenever $\eta(M)$ is closed in W.

(e) for any saturated set M in U, M is pre-open in U whenever $\eta^{\#}(M)$ is open in W.

(f) for any saturated set M in U, M is pre-open in U whenever $\eta(M)$ is open in W.

Proof. (a) $\Rightarrow$ (b) : Let $\eta(M^{\#})$ be open in W then by (a), $\eta^{-1}(\eta(M^{\#}))$ is pre-open in U. Therefore, by Property 1.9(b), it follows that $\eta^{-1}(\eta(\eta^{-1}(\eta^{\#}(M))))$ is pre-open in U and so $\eta^{-1}(\eta^{\#}(M))$ is pre-open in U, since $\eta^{-1} \circ \eta \circ \eta^{-1} = \eta^{-1}$. Hence $M^{\#} = \eta^{-1}(\eta^{\#}(M))$ is pre-open in U.

(b) $\Rightarrow$ (c) : Let $\eta^{\#}(M)$ be open in W then η is surjective implies $\eta^{\#}(M) = \eta(\eta^{-1}(\eta^{\#}(M))) = \eta(M^{\#})$ is open in W and so by (b), $M^{\#}$ is pre-open in U.

(c) $\Rightarrow$ (d) : Let $\eta(M)$ be closed subset of W, where M is saturated set in U then $(\eta(M))^c$ is open in W and so by Property 1.9(a), $\eta^{\#}(M^c)$ is open in W. Therefore by (c), it follows that $(M^c)^{\#}$ is pre-open in U and so by Property 1.9(c), M^c is pre-open in U, since M is saturated implies M^c is saturated. Hence M is pre-closed in U and so (d) holds.

(d) $\Rightarrow$ (e) : Let $\eta^{\#}(M)$ be open subset of W for some saturated subset M of U. Then $(\eta^{\#}(M))^c$ is closed in W and so $\eta(M^c)$ is closed in W by Property 1.9(a). Therefore, by (d), it follows that M^c is pre-closed subset of U and so M is pre-open in U. Hence (e) holds.

(e) $\Rightarrow$ (f) : Let $\eta(M)$ be open subset of W for some saturated subset M of U then by Property 1.9(c), it follows that $\eta(M^{\#})$ is open in W. Further, η is surjective map implies that $\eta^{\#}(M) = \eta(M^{\#})$ is open in W. Hence by (e), M is pre-open subset of U and so (f) holds.

(f) $\Rightarrow$ (a) : Let N be any open subset of W. Then $N = \eta(\eta^{-1}(N))$, since η is surjective map. Therefore $\eta(\eta^{-1}(N))$ is open in W and so for the subset $M = \eta^{-1}(N)$ which is also saturated subset of U, (f) implies that $\eta^{-1}(N)$ is pre-open in U. Hence η is pre-continuous map and so (a) holds. □

In [1], Mashhour et al. gave the characterization of pre-continuity in terms of closure operator using the induced image map. The following theorem gives the relationship between closure and pre-closure of a set using the induced map $\eta^{\#}$ under pre-continuous injective maps.

Theorem 2.11 Let $\eta : (U, \sigma) \to (W, v)$ be any map and η be pre-continuous injective map. Then $\eta^{\#}(pcl(M)) \subseteq cl(\eta^{\#}(M))$ for any subset M of U.

Proof. Consider the subset M of U and an element $p \in \eta^{\#}(pcl(M))$. Then the fiber $\eta^{-1}(p) \subseteq pcl(M)$. We have to prove that $p \in cl(\eta^{\#}(M))$ which means for any open subset G containing p in W, $G \cap \eta^{\#}(M) \neq \emptyset$. Now, for the open subset G containing p in W, η is pre-continuous map implies that $\eta^{-1}(G)$ is pre-open subset containing $\eta^{-1}(p)$ in U. Therefore, $\eta^{-1}(G) \cap M \neq \emptyset$ and so $\eta(\eta^{-1}(G)) \cap \eta(M) \neq \emptyset$ and so $G \cap \eta(M) \neq \emptyset$. Further, η is injective implies $\eta(M) \subseteq \eta^{\#}(M)$. Thus, $G \cap \eta^{\#}(M) \neq \emptyset$ and hence $\eta^{\#}(pcl(M)) \subseteq cl(\eta^{\#}(M))$. □

The following Example shows that the condition is not sufficient for a map to be pre-continuous in Theorem 2.11 i.e. converse need not to be true.

Example 2.12 Let $U = \{e, p\}$, $\sigma = \{\emptyset, \{e\}, U\}$, $W = \{1, 2, 3\}$ and $v = \{\emptyset, \{1\}, \{1, 2\}, W\}$. Define $\eta : (U, \sigma) \to (W, v)$ by $\eta(e) = 3$, $\eta(p) = 2$. Now $1 \in \eta^{\#}(M)$ and $cl(\{1\}) = W$, so $cl(\eta^{\#}(M)) = W$ for every subset M of U. Therefore, $\eta^{\#}(pcl(M)) \subseteq cl(\eta^{\#}(M))$ for every subset M of U. But η is not pre-continuous map since $\{1, 2\}$ is open in W but $\eta^{-1}\{1, 2\} = \{p\}$ is not pre-open in U.

The following Example shows the necessity of injective map in Theorem 2.11.

Example 2.13 Let $U = \{e, a, p, b\}$, $\sigma = \{\emptyset, \{a, b\}, \{e\}, \{a, b, e\}, U\}$, $W = \{1, 2, 3\}$ and $v = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, W\}$. Then $PO(U) = \{\emptyset, \{a\}, \{b\}, \{e\}, \{a, b\}, \{b, e\}, \{a, e\}, \{a, b, e\}, U\}$. Define $\eta : (U, \sigma) \to (W, v)$ by $\eta(a) = 2$, $\eta(b) = 1$, $\eta(e) = 3$, $\eta(p) = 3$. It can be easily checked that η is pre-continuous map. Now for the subset $M = \{e\}$, $\eta^{\#}(M) = \emptyset$ and so $cl(\eta^{\#}(M)) = \emptyset$. But $pcl(M) = \{e, p\}$ and $\eta^{\#}(pcl(M)) = \eta^{\#}(\{e, p\}) = \{3\}$. Therefore, the condition $\eta^{\#}(pcl(M)) \subseteq cl(\eta^{\#}(M))$ does not hold for every subset M of U and so the assumption of injectivity can not be dropped.

The following Corollary shows that # image of pre-dense set is dense under pre-continuous injective maps.

Corollary 2.14 Let $\eta : (U, \sigma) \rightarrow (W, \nu)$ be pre-continuous injective map. Then # image of pre-dense subset of U is dense subset in W.

Proof. Let M be any pre-dense subset of U then $pcl(M) = U$. Therefore, by Theorem 2.11, it follows that $\eta^{\#}(pcl(M)) = \eta^{\#}(U) \subseteq cl(\eta^{\#}(M))$ and so by Property 1.9(e), $W \subseteq cl(\eta^{\#}(M))$. Hence $cl(\eta^{\#}(M)) = W$ and so $\eta^{\#}(M)$ is dense in W. □

3. Conclusion

Topological spaces have many applications in real life which are not only related to mathematics field but also related to various other fields like Physics, Computer, Engineering, Chemistry etc. In real life applications of topological spaces, continuous maps as well as open and closed maps play an important role to study various topological properties. Therefore, due to so many applications of topological spaces various types of strong and weak form of continuous maps, open maps and closed maps were introduced by many authors in literature, where pre-continuous(open and closed) maps are one of them. Motivated by the applications of these mappings, new type of characteriza-tions of these maps have been introduced using the induced map $\eta^{\#}$ induced via a map η between any two topological spaces. Also these characterizations are utilized to study some topological properties. The next target is to study further topological properties using these characterizations and also to introduce another type of mappings using the induced map $\eta^{\#}$ and the theoretical study of these mappings.

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Authors' contributions

Nitakshi Goyal had ideas for the article, performed the literature search, drafted and critically revised the work.

Conflict of interest

The author declares no competing financial interest.

References

[1] Mashhour AS, El-Monsef ME, El-Deeb SN. On pre-continuous and weak pre-continuous mapping. *Proceedings of the Mathematical and Physical Society of Egypt*. 1982; 53: 47-53.

[2] El-Deeb SN, Hasanein IA, Mashhour AS, Noiri T. On p-regular spaces. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie*. 1983; 27(75): 311-315. Available from: https://www.jstor.org/stable/43680828.

[3] Jun YB, Jeong SW, Lee HJ, Lee JW. Applications of pre-open sets. *Applied General Topology*. 2008; 9(2): 213-228. Available from: https://doi.org/10.4995/agt.2008.1802.

[4] Mashhour AS, ABD El-Monsef ME, Hasanein IA. On pretopological spaces. *Bulletin mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie Nouvelle Série*. 1984; 28(76): 39-45. Available from: https://www.jstor.org/stable/43680862.

[5] Paul B, Bhattacharyya P. On p-normal spaces. *Soochow Journal of Mathematics*. 1995; 21(3): 273-289.

[6] Sen AK, Bhattacharyya P. On preclosed mapping. *Bulletin of the Calcutta Mathematical Society*. 1993; 85(5): 409-412.

[7] Arkhangle'skii AV, Ponomarev VI. *Fundamentals of General Topology: Problems and Ex-ercises, Mathematics*

and Its Applications. Hindustan Publishing Corporation, Delhi, India; 1984. Available from: https://link.springer.com/book/9789027713551.