Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

The Generalized Odd Maxwell-Kumaraswamy Distribution: Its Properties and Applications

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Received: 20 April 2023; **Revised:** 18 December 2023; **Accepted:** 25 December 2023

Abstract: This study introduces a novel family of continuous probability distributions using Alzaatreh's technique to explore the Generalized Odd Maxwell-Generated (GOM-G) distribution family. Within the Generalized Odd Maxwell family, the GOM-Kumaraswamy distribution is presented as extended form of the Kumaraswamy distribution. The investigation thoroughly examines the cumulative distribution function, probability density function, hazard, and survival functions, as well as mixture representations of the Generalized Odd Maxwell-Kumaraswamy distribution. The suggested distribution's structural properties, including moments, skewness, kurtosis, probability-weighted moments, entropies, stress-strength models, and order statistics, are derived. Model parameters are estimated through the maximum likelihood method, and simulation studies evaluate the performance of maximum likelihood estimations using the quantile function. Employing two real-life datasets, goodness-of-fit measures such as Akaike Information Criterion (AIC), Corrected Akaike Information Criterion, Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), and chi-square goodness-of-fit tests demonstrate the adaptability and flexibility of the GOM-Kumaraswamy distribution against competing distributions, including Kumaraswamy-Kumaraswamy and Kumaraswamy-Burr III. The results reveal that the GOM-Kumaraswamy distribution exhibits the lowest AIC, CAIC, BIC, and HQIC values and the highest goodness-of-fit values compared to other models, suggesting its superiority as the preferred fit for both dataset forms. This proposed distribution contributes to practical applications, unveiling its potential significance in modeling real-world phenomena within the domains of hydrology and engineering.

Keywords: generalized odd Maxwell-generated class, generalized odd Maxwell-Kumaraswamy, kurtosis, moments, skewness

MSC: 60E05, 60E99, 62P10, 62Q05

1. Introduction

The choice of a statistical distribution to describe a given dataset significantly influences the course of statistical

DOI: https://doi.org/10.37256/cm.5120242888

analysis. In response to the increasing complexity and diversity of datasets, researchers are developing more robust statistical models that provide accurate and reliable predictions of underlying processes [1]. To enhance the versatility of the distribution in various real-world applications, researchers in distribution theory frequently introduce additional parameters to the underlying distribution for data modeling [2]. This addition of parameters is crucial as it improves the model's adaptability, enabling it to better capture the complexity of real-world phenomena across diverse applications.

Kumaraswamy [3] observed that classical distributions like Beta, Lognormal, and Gaussian do not effectively model datasets related to hydrological and other constrained random processes. These processes, characterized by limits at both lower and upper ends, pose challenges in determining key statistical features such as mean, standard deviation, skewness, and kurtosis using simulated data. In response, [4] introduced an expanded set of variables that accommodate hydrological data features, derived from simulations presented in [5]. This extension of variables, initially discussed in [6] and termed Kumaraswamy's distribution, provides a framework for handling data bounded between 0 and 1. The first four moments of this distribution were obtained through simulated data, with applications in hydrological scenarios. As defined in [6], the cumulative distribution function (cdf) and probability density function (pdf) are expressed as follows:

$$g(x; a, b) = abx^{a-1}(1-x^a)^{b-1}, \ a, b > 0; 0 < x < 1 \tag{1}$$

and

$$G(x; a, b) = 1-(1-x^a)^b, \ a, b > 0; 0 < x < 1 \tag{2}$$

where a and b are shape parameters.

This distribution has found application in various fields, with authors utilizing it for different purposes. Employing the expectation approach, [7] utilized the Kumaraswamy distribution for estimating annual reservoir storage capacity data. Jones [8] observed that the Kumaraswamy distribution demonstrates constant, increasing, decreasing, and unimodal density shapes based on parameter values, akin resembling the Beta random variable. While the distribution and quantile function of the Kumaraswamy random variable do not involve any special cases, the distribution and quantile function of the Beta random variable are presented in a closed-form expression, as elucidated by the author in discussing distinctions between Kumaraswamy and Beta random variables. Lemonte [9] scrutinized biases and maximum likelihood estimators of the Kumaraswamy distribution, varying its parameters for different sample sizes. An extension of the Kumaraswamy distribution was proposed by [10] to study the Generalized Transmuted-Kumaraswamy distribution, defining various properties in the literature. Numerous other extensions of the Kumaraswamy distribution are notable, including Kumaraswamy-Kumaraswamy by [11], Exp-Kumaraswamy by [12], Marshall-Olkin-extended inverted Kumaraswamy by [13], Generalized Modification of the Kumaraswamy by [14], Log-Kumaraswamy distributions by [15], and many more.

The Maxwell distribution, commonly referred to as the Maxwell-Boltzmann distribution, was first introduced in the fields of statistical mechanics and the kinetic theory of gases, as outlined in [16-17]. This distribution has found application in various domains, notably in reliability studies, owing to its appropriateness for modeling right-skewed datasets characterized by increasing failure rates. The detailed description of the Maxwell distribution, including its pdf and cdf, is expounded upon in the works of [16-17], are respectively given as follows:

$$m(x; \alpha) = \sqrt{\frac{2}{\pi}} \frac{x^2 e^{-\frac{x^2}{2\alpha^2}}}{\alpha^3}, \ \alpha > 0; x \geq 0 \tag{3}$$

and

$$M(x; \alpha) = \int_0^x m(t; \alpha) dt = \sqrt{\frac{2}{\pi}} \gamma\left(\frac{3}{2}, \frac{x^2}{2\alpha^2}\right), \ \alpha > 0; x \geq 0. \tag{4}$$

Many researchers have considered the Maxwell distribution as a suitable lifetime model and have applied it in various applications. For example, [18] obtained the reliability, Bayesian, and minimum variance unbiased estimators of the Maxwell distribution. The classical and Bayesian estimators of the generalized Maxwell distribution were discussed by [19]. Bayes estimators of the Maxwell model were studied in [20-21] respectively. Hossain and Huerta [22] determined the parameters of the Maxwell distribution using different estimation methods, including Bayes, maximum likelihood, survival, and hazard estimators under type II censored data. A study conducted by [23] introduced an extension of the Weibull distribution using the Maxwell-X approach, and various properties of this distribution were studied in [23], and this distribution stands out among existing ones. Furthermore, [24] generalized the Maxwell model to the Weighted Maxwell distribution by introducing an additional parameter to improve the flexibility of the traditional Maxwell model. The physical properties of this distribution were derived, showing that the proposed distribution could be considered compared to other Weighted models. Singh and Sharma [25] added an extra parameter to the Maxwell distribution and defined the Location-scale family of generalized Maxwell distributions. The failure rate of this distribution could be increasing, and bathtub shaped. Annual rainfall data were used to illustrate the performance of the new model, and it was found that the proposed model fitted this dataset.

In recent years, there has been a notable surge in interest in enhancing conventional distribution models to better accommodate real-life data through the utilization of a generalized class of distributions. This includes the extended Gumbel-Weibull distribution introduced by [26], the novel flexible exponentiated Weibull distribution proposed by [27], the generalized odd beta prime family of distributions presented by [28], the extended Topp-Leone exponential distribution innovated by [29], the log-Topp-Leone distribution introduced by [30], the Marshall-Olkin extended Gumbel type-II distribution outlined by [31], the exponentiated odd Lomax exponential distribution suggested by [32], and various others. Notably, [33] has recently introduced a novel family of distributions known as the odd beta prime-G (OBP-G) family. The OBP-G class has been instrumental in extending several baseline distributions, resulting in the creation of new compound distributions with diverse properties and applications. For instance, [34] proposed the OBP-logistic distribution, the OBP-Fréchet distribution by [35], the OBP-Burr X distribution by [36], and OBP-Inverted Kumaraswamy distribution by [37].

Recently, a study performed by [38] defined the distribution and density functions of the Maxwell-G family, studying some properties of the extended model using this family. Parameters were derived using the method of maximum likelihood, and simulations using its quantile function showed that this distribution fitted the strengths of glass fibers and Nigerian Naira to Japanese Yen exchange rates datasets. In [39], the Maxwell-Dagum distribution was introduced, with its density shape being right-skewed and symmetric. The failure rate of the proposed distribution has a monotonically decreasing, increasing, and upside-down bathtub shape. The quantile function, moments, moment generating function, and order statistics were discussed, demonstrating that the Maxwell-Dagum distribution could be chosen apart from Weibull-Dagum, Topp-Leone-Dagum, Odd Log-Logistic-Dagum, and Gamma-Dagum models by applying two datasets. Some other generalized Maxwell distributions can be mentioned such as the development of Maxwell-Dagum distribution introduced by [40], Maxwell-Exponentiated Exponential distribution discussed in [41], Maxwell-Lomax distribution studied by [42], Maxwell-Burr X distribution by [43], among others.

This study introduces a novel family of distributions termed the Generalized Odd Maxwell-Generated (GOM-G) family, employing it to propose a new extension of the Kumaraswamy distribution known as the Generalized Odd Maxwell-Kumaraswamy (GOM-Kw) distribution. The GOM-Kw distribution demonstrates enhanced flexibility in modeling datasets with prolonged right and left tails when compared to other commonly utilized distributions. Consequently, the GOM-Kw proves to be highly effective for long-term reliability estimates, providing accurate predictions of extreme values in both the right and left tails of the distribution, outperforming alternative distributions.

The introduction of the Generalized Odd Maxwell-G (GOM-G) family and GOM-Kw distribution is motivated by the limitations observed in classical Kumaraswamy distribution. Therefore, the rationale and justification behind introducing the GOM-G family and GOM-Kw distribution are outlined as follows:

i. To enhance the overall performance of the classical Kumaraswamy distribution, accommodating both right-skewed and left-skewed datasets in comparison to other competitive models;

ii. To formulate a model with diverse shapes, including right-skewed and left-skewed characteristics;

iii. To introduce a novel model featuring various hazard functions capable of capturing bathtub and upside-down shapes;

iv. To consistently provide superior fit when compared to well-established generated distributions for the same baseline distribution.

For these reasons, we proposed the GOM-G family and GOM-Kw distribution, formed by combining the GOM-G family of distributions with the Kumaraswamy distribution.

The research is organized as follows: In Section 2, the cdf anf pdf of the proposed Generalized Odd Maxwell-Generated family of distributions are defined. Section 3 presents the cdf and pdf of the extended family of distributions, considering the Kumaraswamy model as a special case, along with its survival and hazard functions, mixture representations, and the statistical table of the density function. Section 4 examines the structural properties of the Generalized Odd Maxwell-Kumaraswamy distribution, including moments, skewness, and kurtosis. The estimates of the proposed distribution, derived using the maximum likelihood method, are presented in Section 5. Section 6 includes two simulation studies utilizing the quantile function. In Section 7, the Generalized Odd Maxwell-Kumaraswamy distribution is applied alongside other existing models using two real datasets. Section 8 concludes the research, while Section 9 presents future research.

2. The generalized odd maxwell-generated family of distributions

Alzaatreh et al. [44] introduces a technique for formulating novel family of continuous probability distributions after noticing that the approach used to develop the Beta-G family of distribution by [45] and the Kumaraswamy-G family of distribution by [46] could support the distribution ranging between 0 and 1. Consequently, [44] suggested a novel family of continuous probability distributions capable of accommodating all limits of probability distributions.

Assume X represents a random variable with the cdf $G(x)$ and the corresponding pdf $g(x)$. Let be another random variable, with the cdf and pdf given as $R(t)$ and $r(t)$, respectively, defined on the interval $[a, b]$. The cdf of the novel family of distribution discussed in [44] is then provided by

$$F(x) = \int_{a}^{\Delta(G(x))} r(t) dt. \tag{5}$$

In this regards, $\Delta(G(x))$ is the function of cdf of the random variable X and $r(t)$ is the pdf of the random variable T, for any $T \in [a, b]$ and $-\infty < a < b < \infty$. Therefore, $\Delta(G(x))$ must satisfied the following conditions:

i. $\Delta(G(x)) \in [a, b]$,

ii. $\Delta(G(x))$ is differentiable and monotonically non-decreasing, and

iii. $\Delta(G(x)) \to a$ as $x \to -\infty$ and $\Delta(G(x)) \to b$ as $x \to \infty$.

Different versions of $\Delta(G(x))$ might produce a new class of continuous distributions, albeit this is dependent on the limits of a random variable T. Many researchers regarded [44] technique to studying diverse families of distributions. For instance, [47] introduced Odd F generalized class of distribution. Some properties and real-life applications of the novel Odd F-Weibull distribution are provided in [47].

Let $G(x, \xi)$ and $g(x, \xi)$ denote the cdf and pdf of the baseline model with vector parameter ψ. Consider the pdf defined in (3). The cdf of the proposed family of distribution, referred to as the Generalized Odd Maxwell-Generated (GOM-G) family of distributions, is derived by replacing the argument $\Delta(G(x))$ in (5) with the link function $\frac{G^{\beta}(x, \psi)}{1 - G^{\beta}(x, \psi)}$, defined as:

$$F(x; \alpha, \beta, \psi) = \frac{\sqrt{\frac{2}{\pi}}}{\alpha^{3}} \int_{0}^{\frac{G^{\beta}(x, \psi)}{1-G^{\beta}(x, \psi)}} t^{2} e^{-\frac{t^{2}}{2\alpha^{2}}} dt. \tag{6}$$

Let

$$y = \frac{t^{2}}{2\alpha^{2}}, \Rightarrow t = \left(2\alpha^{2} y\right)^{1/2}, dt = \frac{\alpha^{2} dy}{t}. \tag{7}$$

Substituting (7) into (6) becomes

$$F(x;\ \alpha,\ \beta,\ \psi) = \frac{\sqrt{\frac{2}{\pi}}}{\alpha^3}\int_0^{\frac{1}{2\alpha^2}\left(\frac{G^\beta(x,\ \psi)}{1-G^\beta(x,\ \psi)}\right)^2} t^2 e^{-y}\frac{\alpha^2 dy}{t}$$

$$= \frac{\sqrt{\frac{2}{\pi}}}{\alpha}\int_0^{\frac{1}{2\alpha^2}\left(\frac{G^\beta(x,\ \psi)}{1-G^\beta(x,\ \psi)}\right)^2} \left(2\alpha^2 y\right)^{1/2} e^{-y} dy$$

$$= \frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{2\alpha^2}\left(\frac{G^\beta(x,\ \psi)}{1-G^\beta(x,\ \psi)}\right)^2} y^{\left(\frac{1}{2}+1\right)-1} e^{-y} dy$$

$$= \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\ \frac{1}{2\alpha^2}\left(\frac{G^\beta(x,\ \psi)}{1-G^\beta(x,\ \psi)}\right)^2\right),\ \alpha,\ \beta > 0;\ x \in \Re \tag{8}$$

which is the cdf of the GOM-G family distributions. The corresponding pdf of this family of distributions is obtained by differentiating (8) with respect to the function of x as

$$f(x;\ \alpha,\ \beta,\ \psi) = \frac{2}{\sqrt{\pi}}\frac{d}{dx}\left(\gamma\left(\frac{3}{2},\ \frac{1}{2\alpha^2}\left(\frac{G^\beta(x,\ \psi)}{1-G^\beta(x,\ \psi)}\right)^2\right)\right)$$

$$= \frac{2}{\sqrt{\pi}}\frac{d}{dx}\left(\int_0^{\frac{1}{2\alpha^2}\left(\frac{G^\beta(x,\ \psi)}{1-G^\beta(x,\ \psi)}\right)^2} y^{\left(\frac{1}{2}+1\right)-1} e^{-y} dy\right)$$

$$= \frac{2}{\sqrt{\pi}}\left(\frac{1}{2\alpha^2}\left(\frac{G^\beta(x,\ \psi)}{1-G^\beta(x,\ \psi)}\right)^2\right)^{\frac{1}{2}} e^{-\frac{1}{2\alpha^2}\left(\frac{G^\beta(x,\ \psi)}{1-G^\beta(x,\ \psi)}\right)^2}\frac{d}{dx}\left(\frac{1}{2\alpha^2}\left(\frac{G^\beta(x,\ \psi)}{1-G^\beta(x,\ \psi)}\right)^2\right)$$

$$= \frac{2G^\beta(x,\ \psi)}{\alpha\sqrt{2\pi}\left(1-G^\beta(x,\ \psi)\right)}\exp\left(-\frac{1}{2\alpha^2}\left(\frac{G^\beta(x,\ \psi)}{1-G^\beta(x,\ \psi)}\right)^2\right)\frac{d}{dx}\left(\frac{1}{2\alpha^2}\left(\frac{G^\beta(x,\ \psi)}{1-G^\beta(x,\ \psi)}\right)^2\right). \tag{9}$$

The differential part of (9) can be expressed as

$$\frac{d}{dx}\left(\frac{1}{2\alpha^2}\left(\frac{G^\beta(x,\ \psi)}{1-G^\beta(x,\ \psi)}\right)^2\right) = \frac{1}{2\alpha^2}\frac{d}{dx}\left(\left(\frac{G^\beta(x,\ \psi)}{1-G^\beta(x,\ \psi)}\right)^2\right)$$

$$= \frac{1}{2\alpha^2}\left(2\left(\frac{G^\beta(x,\ \psi)}{1-G^\beta(x,\ \psi)}\right)\frac{d}{dx}\left(\frac{G^\beta(x,\ \psi)}{1-G^\beta(x,\ \psi)}\right)\right)$$

$$= \frac{G^\beta(x,\ \psi)}{\alpha^2\left(1-G^\beta(x,\ \psi)\right)}\left(\frac{d}{dx}\left(\frac{G^\beta(x,\ \psi)}{1-G^\beta(x,\ \psi)}\right)\right)$$

$$= \frac{G^\beta(x,\ \psi)}{\alpha^2\left(1-G^\beta(x,\ \psi)\right)}\left(\frac{\beta g(x,\ \psi)G^{\beta-1}(x,\ \psi)}{\left(1-G^\beta(x,\ \psi)\right)^2}\right) = \frac{\beta g(x,\ \psi)G^{2\beta-1}(x,\ \psi)}{\alpha^2\left(1-G^\beta(x,\ \psi)\right)^3}. \tag{10}$$

Substituting (10) in (9) gives

$$f(x;\ \alpha,\ \beta,\ \psi)=\frac{2\beta g(x,\ \psi)G(x,\ \psi)^{3\beta-1}}{\alpha^3\sqrt{2\pi}\left(1-G^\beta(x,\ \psi)\right)^4}\exp\left(-\frac{1}{2\alpha^2}\left(\frac{G^\beta(x,\ \psi)}{1-G^\beta(x,\ \psi)}\right)^2\right),\ \alpha,\ \beta>0;\ x\in\Re \tag{11}$$

which is the pdf of the proposed Generalized Odd Maxwell-G family of distributions, where $g(x, \psi)$ and $G(x, \psi)$ are pdf and cdf of the baseline model, α and β are scale and shape parameters. Henceforth, a random variable X with density function defined in (11) denoted by $X \sim GOM - G(\alpha, \beta, \psi)$ reads as X follows Generalized Odd Maxwell-G family with parameter α, β and ψ. Thus, we can omit the dependence on vector parameter ψ by simply writing $F(x; \alpha, \beta, \psi) = F(x)$ and $f(x; \alpha, \beta, \psi) = f(x)$. Moreover, the pdf in (11) is a special case of the Maxwell generalized family of distributions given in [38] when parameter $\beta = 1$.

3. The generalized odd maxwell-kumaraswamy distribution

The cdf of the Generalized Odd Maxwell-Kumaraswamy distribution is derived by replacing the cdf of the baseline model in (8) with the distribution function defined in (2), as follows:

$$F(x)=\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\ \frac{1}{2\alpha^2}\left(\frac{\left(1-\left(1-x^a\right)^b\right)^\beta}{1-\left(1-\left(1-x^a\right)^b\right)^\beta}\right)^2\right),\ 0<x<1 \tag{12}$$

where α is scale parameter and a, b, β are shape parameters. Its corresponding pdf is obtained by replacing the pdf and cdf of baseline model in (11) with (1) and (2) respectively as

$$f(x)=\frac{2\beta abx^{a-1}\left(1-x^a\right)^{b-1}\left(1-\left(1-x^a\right)^b\right)^{3\beta-1}}{\alpha^3\sqrt{2\pi}\left(1-\left(1-\left(1-x^a\right)^b\right)^\beta\right)^4}e^{\left(-\frac{1}{2\alpha^2}\left(\frac{\left(1-\left(1-x^a\right)^b\right)^\beta}{1-\left(1-\left(1-x^a\right)^b\right)^\beta}\right)^2\right)}, \tag{13}$$

where $0 < x < 1$. However, for a random variable X with apdf given in (13), we denote $X \sim GOM - Kw(\alpha, \beta, a, b)$, indicating that X follows the Generalized Odd Maxwell-Kumaraswamy distribution with parameters α, β, a and b. Statistical tables for the proposed distribution with different parameter values are provided in Tables 1-4, respectively, by considering GOM-Kumaraswamy distribution as the probability distribution.

Table 1 presents statistical table for the GOM-Kumaraswamy distribution, where β, a and b are kept constant, and the values of α are interchanged from 0.5 to 1.2.

As observed from Table 1, the density values of the GOM-Kumaraswamy distribution increases with the increase of parameter α.

Table 1. Statistical table of GOM-Kumaraswamy distribution for $\beta = 0.5$, $a = 1.15$, $b = 1.2$ and $\alpha = 0.5$ to 1.2

x	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$	$\alpha = 1.1$	$\alpha = 1.2$
0.01	0.2328	0.1353	0.0854	0.0573	0.0403	0.0294	0.0221	0.0170
0.02	0.4455	0.2605	0.1651	0.1110	0.0782	0.0571	0.0430	0.0331
0.03	0.6696	0.3944	0.2511	0.1694	0.1195	0.0874	0.0659	0.0508
0.04	0.9067	0.5387	0.3447	0.2333	0.1650	0.1209	0.0912	0.0704
0.05	1.1563	0.6937	0.4464	0.3033	0.2151	0.1579	0.1192	0.0922
0.06	1.4168	0.8592	0.5566	0.3798	0.2701	0.1987	0.1503	0.1163
0.07	1.6860	1.0349	0.6752	0.4629	0.3303	0.2435	0.1845	0.1430
0.08	1.9613	1.2199	0.8023	0.5529	0.3959	0.2926	0.2221	0.1724
0.09	2.2397	1.4133	0.9377	0.6498	0.4671	0.3462	0.2634	0.2048
0.10	2.5177	1.6141	1.0811	0.7539	0.5442	0.4046	0.3084	0.2402
0.11	2.7916	1.8206	1.2322	0.8650	0.6273	0.4679	0.3576	0.2790
0.12	3.0573	2.0313	1.3903	0.9831	0.7166	0.5364	0.4110	0.3213
0.13	3.3104	2.2442	1.5548	1.1081	0.8120	0.6102	0.4689	0.3674
0.14	3.5466	2.4572	1.7247	1.2398	0.9138	0.6896	0.5315	0.4174
0.15	3.7612	2.6676	1.8991	1.3776	1.0218	0.7746	0.5990	0.4716
0.16	3.9499	2.8729	2.0765	1.5213	1.1360	0.8653	0.6716	0.5302
0.17	4.1082	3.0701	2.2556	1.6701	1.2563	0.9619	0.7494	0.5934
0.18	4.2322	3.2561	2.4347	1.8233	1.3823	1.0643	0.8326	0.6614
0.19	4.3183	3.4278	2.6118	1.9800	1.5137	1.1725	0.9213	0.7343
0.20	4.3637	3.5819	2.7848	2.1391	1.6501	1.2864	1.0156	0.8124

Table 2. Statistical table of GOM-Kumaraswamy distribution for $\alpha = 0.5$, $a = 1.15$, $b = 1.2$ and $\beta = 0.6$ to 1.3

x	$\beta = 0.6$	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$	$\beta = 1.0$	$\beta = 1.1$	$\beta = 1.2$	$\beta = 1.3$
0.01	0.0533	0.0124	0.0029	0.0007	0.0002	0.0000	0.0000	0.0000
0.02	0.1250	0.0360	0.0105	0.0031	0.0009	0.0003	0.0001	0.0000
0.03	0.2116	0.0686	0.0227	0.0076	0.0025	0.0009	0.0003	0.0001
0.04	0.3126	0.1103	0.03v97	0.0145	0.0053	0.0020	0.0007	0.0003
0.05	0.4274	0.1610	0.0618	0.0241	0.0095	0.0037	0.0015	0.0006
0.06	0.5560	0.2212	0.0896	0.0368	0.0153	0.0064	0.0027	0.0011
0.07	0.6981	0.2912	0.1233	0.0530	0.0230	0.0101	0.0045	0.0020
0.08	0.8534	0.3713	0.1636	0.0731	0.0331	0.0151	0.0069	0.0032
0.09	1.0213	0.4619	0.2109	0.0975	0.0456	0.0216	0.0102	0.0049
0.10	1.2011	0.5632	0.2656	0.1267	0.0611	0.0298	0.0146	0.0072
0.11	1.3920	0.6755	0.3283	0.1611	0.0799	0.0400	0.0202	0.0102
0.12	1.5926	0.7989	0.3995	0.2013	0.1024	0.0526	0.0272	0.0142
0.13	1.8015	0.9334	0.4795	0.2476	0.1290	0.0678	0.0359	0.0191
0.14	2.0170	1.0788	0.5689	0.3007	0.1601	0.0860	0.0465	0.0254
0.15	2.2369	1.2349	0.6680	0.3611	0.1963	0.1076	0.0594	0.0330
0.16	2.4589	1.4011	0.7771	0.4292	0.2381	0.1330	0.0748	0.0423
0.17	2.6801	1.5768	0.8964	0.5056	0.2858	0.1625	0.0931	0.0536
0.18	2.8976	1.7609	1.0260	0.5907	0.3402	0.1968	0.1146	0.0671
0.19	3.1080	1.9522	1.1660	0.6851	0.4017	0.2363	0.1398	0.0832
0.20	3.3077	2.1493	1.3161	0.7891	0.4708	0.2815	0.1691	0.1021

Observations from Table 2 indicate that the density values of the GOM-Kumaraswamy distribution also increase with an increase in the parameter β.

Table 2 displays results from the statistical table of the proposed distribution, with β ranging from 0.6 to 1.3 while keeping α, a and β constant.

Similarly, Table 3 presents results for the GOM-Kumaraswamy distribution with varying values of a from 1.20 to 1.55, while keeping α, β and b as constant parameters.

Table 3. Statistical table of GOM-Kumaraswamy distribution for $\alpha = 0.5$, $\beta = 0.5$, $b = 1.2$ and $a = 1.20$ to 1.55

x	$a = 1.20$	$a = 1.25$	$a = 1.30$	$a = 1.35$	$a = 1.40$	$a = 1.45$	$a = 1.50$	$a = 1.55$
0.01	0.1664	0.1191	0.0854	0.0613	0.0441	0.0317	0.0228	0.0165
0.02	0.3327	0.2489	0.1865	0.1400	0.1052	0.0792	0.0597	0.0450
0.03	0.5133	0.3941	0.3030	0.2334	0.1800	0.1390	0.1074	0.0832
0.04	0.7087	0.5545	0.4344	0.3408	0.2676	0.2105	0.1657	0.1306
0.05	0.9183	0.7297	0.5803	0.4619	0.3681	0.2937	0.2345	0.1875
0.06	1.1410	0.9189	0.7402	0.5966	0.4813	0.3886	0.3140	0.2540
0.07	1.3753	1.1210	0.9135	0.7445	0.6070	0.4953	0.4044	0.3304
0.08	1.6193	1.3348	1.0994	0.9051	0.7452	0.6138	0.5057	0.4170
0.09	1.8707	1.5587	1.2968	1.0779	0.8956	0.7441	0.6183	0.5140
0.10	2.1269	1.7908	1.5045	1.2620	1.0577	0.8860	0.7421	0.6216
0.11	2.3851	2.0290	1.7209	1.4564	1.2308	1.0392	0.8770	0.7400
0.12	2.6418	2.2708	1.9442	1.6599	1.4142	1.2033	1.0229	0.8691
0.13	2.8936	2.5131	2.1721	1.8707	1.6068	1.3775	1.1794	1.0088
0.14	3.1366	2.7530	2.4023	2.0871	1.8072	1.5610	1.3459	1.1589
0.15	3.3667	2.9869	2.6318	2.3068	2.0138	1.7525	1.5217	1.3190
0.16	3.5796	3.2111	2.8577	2.5274	2.2246	1.9508	1.7058	1.4883
0.17	3.7712	3.4217	3.0764	2.7461	2.4374	2.1540	1.8969	1.6661
0.18	3.9371	3.6147	3.2844	2.9597	2.6498	2.3601	2.0936	1.8513
0.19	4.0735	3.7861	3.4778	3.1650	2.8588	2.5670	2.2940	2.0424
0.20	4.1764	3.9317	3.6529	3.3583	3.0614	2.7718	2.4960	2.2378

Table 3 displays the results of the density values of the GOM-Kumaraswamy distribution while keeping α, β and c as constant parameters.

Finally, Table 4 presents the results of the GOM-Kumaraswamy distribution for different parameter values of α, β and a when b is set to 1.21 and 1.28.

The pdf values of the GOM-Kumaraswamy distribution increase as the parameter c increases, corresponding to the increase in the value of x, as presented in Table 4.

The plots of density function of the GOM-Kumaraswamy distribution are provided in Figure 1.

As presented in Figure 1, the proposed density function of the GOM-Kumaraswamy distribution is (a) right-skewed, and (b) left and symmetric.

Table 4. Statistical table of GOM-Kumaraswamy distribution for $\alpha = 0.5$, $\beta = 0.5$, $a = 1.15$ and $b = 1.21$ to 1.28

x	$b = 1.21$	$b = 1.22$	$b = 1.23$	$b = 1.24$	$b = 1.25$	$b = 1.26$	$b = 1.27$	$b = 1.28$
0.01	0.2360	0.2392	0.2425	0.2457	0.2490	0.2523	0.2556	0.2589
0.02	0.4518	0.4582	0.4646	0.4711	0.4776	0.4842	0.4907	0.4974
0.03	0.6793	0.6891	0.6990	0.7090	0.7190	0.7291	0.7392	0.7494
0.04	0.9201	0.9336	0.9472	0.9609	0.9747	0.9885	1.0025	1.0165
0.05	1.1736	1.1909	1.2084	1.2260	1.2437	1.2615	1.2795	1.2975
0.06	1.4380	1.4593	1.4808	1.5024	1.5242	1.5460	1.5681	1.5902
0.07	1.7112	1.7365	1.7620	1.7876	1.8134	1.8394	1.8655	1.8917
0.08	1.9904	2.0196	2.0490	2.0786	2.1083	2.1383	2.1683	2.1986
0.09	2.2725	2.3054	2.3386	2.3719	2.4053	2.4390	2.4728	2.5067
0.10	2.5539	2.5903	2.6268	2.6635	2.7003	2.7373	2.7745	2.8118
0.11	2.8308	2.8701	2.9096	2.9492	2.9889	3.0288	3.0687	3.1088
0.12	3.0989	3.1406	3.1825	3.2244	3.2664	3.3084	3.3506	3.3928
0.13	3.3538	3.3972	3.4407	3.4842	3.5277	3.5712	3.6147	3.6582
0.14	3.5909	3.6352	3.6795	3.7237	3.7678	3.8119	3.8558	3.8997
0.15	3.8056	3.8499	3.8940	3.9379	3.9817	4.0252	4.0686	4.1118
0.16	3.9933	4.0364	4.0793	4.1219	4.1643	4.2063	4.2480	4.2894
0.17	4.1496	4.1905	4.2311	4.2713	4.3110	4.3503	4.3892	4.4275
0.18	4.2704	4.3081	4.3452	4.3818	4.4177	4.4531	4.4879	4.5221
0.19	4.3523	4.3855	4.4181	4.4499	4.4810	4.5113	4.5409	4.5698
0.20	4.3923	4.4200	4.4469	4.4730	4.4981	4.5223	4.5457	4.5681

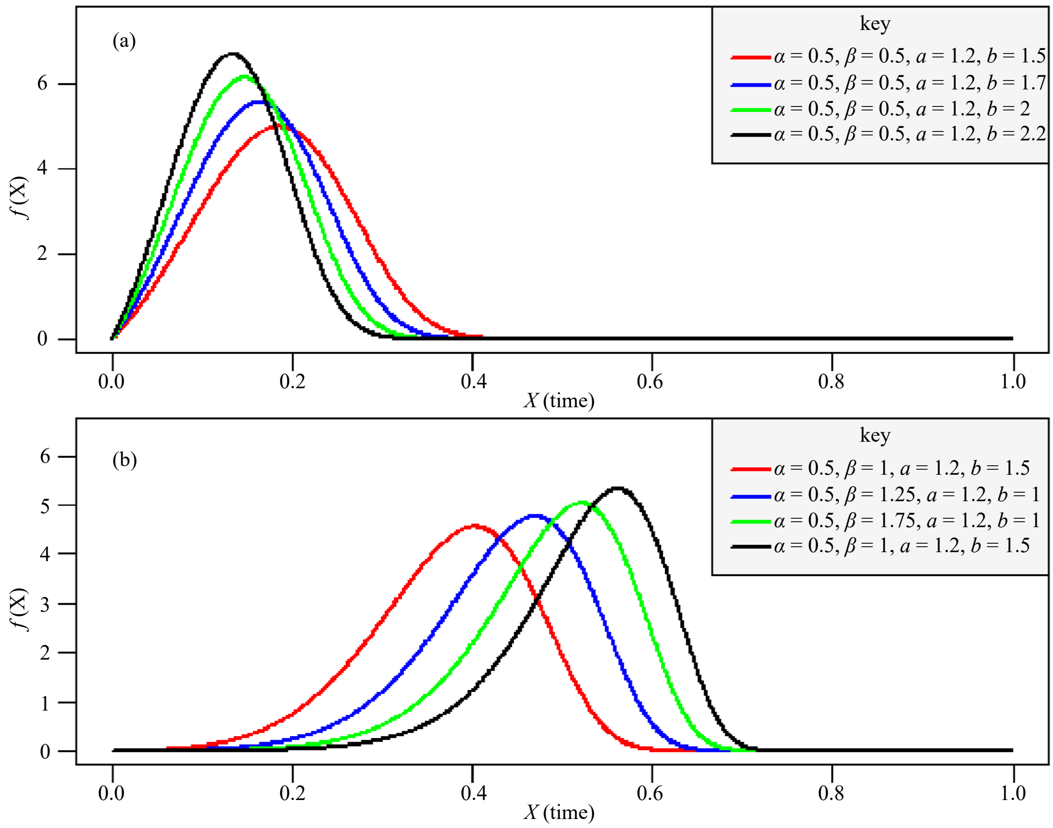


Figure 1. Density plots of GOM-Kumaraswamy at various parameter values

3.1 *Survival and Hazard functions of the GOM-Kumaraswamy distribution*

In this section, the survival and hazard functions of the GOM-Kumaraswamy distribution are provided.

3.1.1 *Survival function*

The survival function of the GOM-Kumaraswamy distribution is given by

$$S(x) = 1 - \frac{2}{\sqrt{\pi}} \gamma \left(\frac{3}{2}, \ \frac{1}{2\alpha^2} \left(\frac{\left(1-\left(1-x^a\right)^b\right)^\beta}{1-\left(1-\left(1-x^a\right)^b\right)} \right)^2 \right), \ 0 < x < 1 \tag{14}$$

where α is scale parameter and a, b, β are shape parameters.

3.1.2 *Hazard function*

The hazard function of the proposed distribution is given by

$$h(x) = \frac{2\beta ab x^{a-1}\left(1-x^a\right)^{b-1}\left(1-\left(1-x^a\right)^b\right)^{3\beta-1} e^{\left(-\frac{1}{2\alpha^2}\left(\frac{\left(1-\left(1-x^a\right)^b\right)^\beta}{1-\left(1-\left(1-x^a\right)^b\right)^\beta}\right)^2\right)}}{\alpha^3\sqrt{2\pi}\left(1-\left(1-\left(1-x^a\right)^b\right)^\beta\right)^4\left(1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \ \frac{1}{2\alpha^2}\left(\frac{\left(1-\left(1-x^a\right)^b\right)^\beta}{1-\left(1-\left(1-x^a\right)^b\right)}\right)^2\right)\right)}. \tag{15}$$

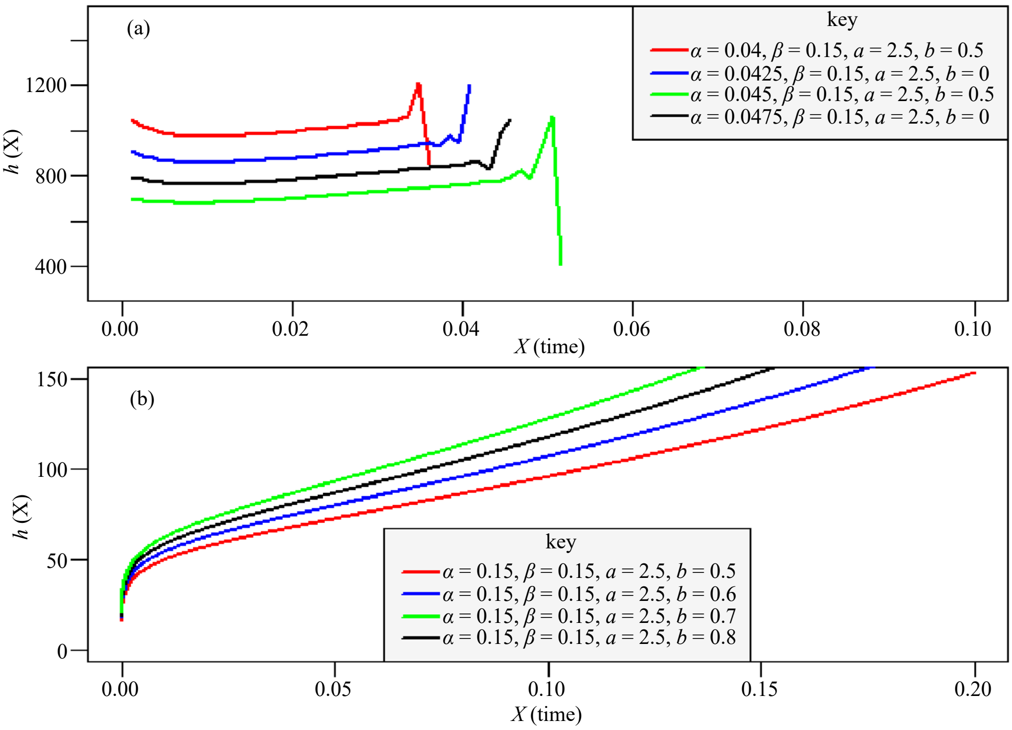


Figure 2. Hazard plots of GOM-Kumaraswamy at various parameter values

Suitable plots of hazard function of the proposed are provided in Figure 2.

The GOM-Kumaraswamy distribution has bath-tub, upside down bath-tub as displayed in Figure 2a and increasing failure rates (see, Figure 2b).

3.1.3 *Mixture representations for the cdf and pdf of GOM-Kumaraswamy distribution*

As defined in [48] Section 8.354, the expansion of incomplete gamma function can be given as

$$\gamma\left(a,\ x\right)=\sum_{t=0}^{\infty}\frac{(-1)^{t}x^{a+t}}{t!(a+t)} \tag{16}$$

Applying this expansion for the distribution function defined in (12) we get

$$F\left(x\right)=\frac{2}{\sqrt{\pi}}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!\left(\frac{3}{2}+i\right)}\left(\frac{1}{2\alpha^{2}}\left(\frac{\left(1-\left(1-x^{a}\right)^{b}\right)^{\beta}}{1-\left(1-\left(1-x^{a}\right)^{b}\right)^{\beta}}\right)^{2}\right)^{\left(\frac{3}{2}+i\right)}$$

$$=\frac{\sqrt{2}}{\sqrt{\pi}}\sum_{i=0}^{\infty}\frac{(-1)^{i}\left(1-\left(1-x^{a}\right)^{b}\right)^{\beta(3+2i)}}{i!2^{i}\left(3+2i\right)\alpha^{3+2i}\left(1-\left(1-\left(1-x^{a}\right)^{b}\right)^{\beta}\right)^{3+2i}}. \tag{17}$$

It is well known that for real non integer $s > 0$ and $|x| < 1$, the generalized binomial expansion of this can be given by

$$\left(1-x\right)^{-s}=\sum_{m=0}^{\infty}\frac{\Gamma(s+m)}{m!\Gamma(s)}x^{m}. \tag{18}$$

Applying (18) to the last term of the denominator in (17) gives

$$F\left(x\right)=\frac{\sqrt{2}}{\sqrt{\pi}}\sum_{i,j=0}^{\infty}\frac{(-1)^{i}\Gamma(3+2i+j)\left(1-\left(1-x^{a}\right)^{b}\right)^{\beta(3+2i)+\beta j}}{i!\,j!2^{i}\left(3+2i\right)\alpha^{3+2i}\Gamma(3+2i)}. \tag{19}$$

For real number $u > 0$ and $|x| < 1$, the binomial expansion of this becomes

$$\left(1-x\right)^{u}=\sum_{q=0}^{\infty}\frac{(-1)^{q}\Gamma(u+1)}{q!\Gamma(u-q+1)}x^{q}. \tag{20}$$

Thus, the cdf in (19) can be presented in (21) by applying (20)

$$F\left(x\right)=\sum_{i,\,j,\,k=0}^{\infty}\varpi_{i,\,j,\,k}\left(1-x^{a}\right)^{bk}, \tag{21}$$

which is the cdf of GOM-Kumaraswamy distribution expressed as a mixture of exponentiated-G density by [49], where

$$\varpi_{i,\,j,\,k}=\frac{\sqrt{2}}{\sqrt{\pi}}\frac{(-1)^{i+k}\Gamma(3+2i+j)\Gamma(\beta(3+2i+j)+1)}{i!\,j!k!2^{i}\left(3+2i\right)\alpha^{3+2i}\Gamma(3+2i)\Gamma(\beta(3+2i+j)-k+1)}.$$

The expansion for the density function of the proposed distribution can be obtained as follows:
Let us consider the exponential series expansion defined as

$$e^{-x}=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}x^s. \tag{22}$$

By applying this expansion to the pdf defined in (13) we can have

$$f(x)=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!(2\alpha^2)^i}\frac{2\beta abx^{a-1}\left(1-x^a\right)^{b-1}\left(1-\left(1-x^a\right)^b\right)^{3\beta+2\beta i-1}}{\alpha^3\sqrt{2\pi}\left(1-\left(1-\left(1-x^a\right)^b\right)^{\beta}\right)^{4+2i}}. \tag{23}$$

Using (18), (23) can be written as

$$f(x)=\sum_{i,j=0}^{\infty}\frac{(-1)^i\Gamma(4+2i+j)\beta ab}{i!j!2^i\alpha^{2i+3}\Gamma(4+2i)}\frac{\sqrt{2}}{\sqrt{\pi}}x^{a-1}\left(1-x^a\right)^{b-1}\left(1-\left(1-x^a\right)^b\right)^{\beta(3+2i+j)-1}. \tag{24}$$

Applying (20), the pdf in (24) can further be written as

$$f(x)=\sum_{i,j,k=0}^{\infty}\Omega_{i,j,k}abx^{a-1}\left(1-x^a\right)^{b(1+k)-1}, \tag{25}$$

where

$$\Omega_{i,j,k}=\frac{(-1)^{i+k}\Gamma(4+2i+j)\Gamma(\beta(3+2i+j))\beta}{i!j!k!2^i\alpha^{2i+3}\Gamma(4+2i)\Gamma(\beta(3+2i+j)-k)}\frac{\sqrt{2}}{\sqrt{\pi}}.$$

4. Structural properties of the GOM-Kumaraswamy distribution

The structural properties of the GOM-Kumaraswamy distribution, as provided in this section, include moments, incomplete moments, probability weighted moments, entropies, stress-strength, and order statistics.

4.1 *Moments*

Suppose $X \sim GOM-Kw(\alpha, \beta, a, b)$ with the density function $f(x)$ as defined in (24), then the r^{th} non-central moments of the random variable X are given by:

$$E(X^r)=\int_{-\infty}^{\infty}x^r f(x)dx=\sum_{i,j,k=0}^{\infty}\Omega_{i,j,k}ab\int_0^1 x^{r+a-1}\left(1-x^a\right)^{b(1+k)-1}dx, \tag{26}$$

where

$$\Omega_{i,j,k}=\frac{(-1)^{i+k}\Gamma(4+2i+j)\Gamma(\beta(3+2i+j))\beta}{i!j!k!2^i\alpha^{2i+3}\Gamma(4+2i)\Gamma(\beta(3+2i+j)-k)}\frac{\sqrt{2}}{\sqrt{\pi}}.$$

Letting

$$A=x^a \Rightarrow x=A^{1/a}, \text{ and } dx=\frac{dA}{ax^{a-1}}. \tag{27}$$

Substituting (27) in the integral part (26) we can obtain

$$\int_0^1 x^{r+a-1}\left(1-x^a\right)^{b(1+k)-1} dx = \frac{1}{a}\int_0^1 x^r\left(1-A\right)^{b(1+k)-1} dA$$

$$= \frac{1}{a}\int_0^1 A^{\frac{r}{a}}\left(1-A\right)^{b(1+k)-1} dA$$

$$= \frac{1}{a}\int_0^1 A^{\left(\frac{r}{a}+1\right)-1}\left(1-A\right)^{b(1+k)-1} dA$$

$$= \frac{1}{a}\beta\left(1+\frac{r}{a},\ b(1+k)\right). \tag{28}$$

Inserting (28) into (26) gives the r^{th} non-central moments of the GOM-Kumaraswamy distribution given in (29)

$$E(X^r) = \sum_{i,\,j,\,k=0}^{\infty} \Omega_{i,\,j,\,k} b\beta\left(1+\frac{r}{a},\ b(1+k)\right)$$

$$= K\sum_{k=0}^{\infty}\beta\left(1+\frac{r}{a},\ b(1+k)\right), \tag{29}$$

where $K = b\sum_{i,\,j}^{\infty}\Omega_{i,\,j,\,k}$, and $\beta\left(c,\ d\right) = \int_0^1 t^{c-1}\left(1-t\right)^{d-1} dt$ is a beta function of first kind. Given r = 1, 2, 3, and 4, (29) becomes the first, second, third and fourth moments of the GOM-Kumaraswamy distribution given in (30)-(33) respectively as

$$E(X) = K\sum_{h=0}^{\infty}\beta\left(1+\frac{1}{a},\ b(1+k)\right), \tag{30}$$

$$E(X^2) = K\sum_{k=0}^{\infty}\beta\left(1+\frac{2}{a},\ b(1+k)\right), \tag{31}$$

$$E(X^3) = K\sum_{k=0}^{\infty}\beta\left(1+\frac{3}{a},\ b(1+k)\right), \tag{32}$$

and

$$E(X^4) = K\sum_{k=0}^{\infty}\beta\left(1+\frac{4}{a},\ b(1+k)\right), \tag{33}$$

From the relations in (30) and (31), the variance of the GOM-Kumaraswamy distribution is given by

$$\sigma^2 = E(X^2) - \left(E(X)\right)^2 = K\sum_{k=0}^{\infty}\beta\left(1+\frac{2}{a},\ b(1+k)\right) - \left(K\sum_{k=0}^{\infty}\beta\left(1+\frac{1}{a},\ b(1+k)\right)\right)^2, \tag{34}$$

and the standard deviation (σ) of this is the square-root of (34).

4.1.1 *Skewness and Kurtosis of the GOM-Kumaraswamy distribution*

The skewness and kurtosis of the GOM-Kumaraswamy distribution are presented in (35) and (36) respectively as

$$S_k = \sigma^{-3}(E(X^3) - 3E(X)E(X^2) + 2(E(X))^3), \tag{35}$$

and

$$K_t = \sigma^{-4}(E(X^4) - 4E(X)E(X^3) + 6(E(X))^2 E(X^2) - 3(E(X))^4), \tag{36}$$

where σ is standard deviation, $E(X)$, $E(X^2)$, $E(X^3)$ and $E(X^4)$ are defined in (30)-(31).

4.2 *Incomplete moments*

Let $X \sim GOM - Kw(\alpha, \beta, a, b)$ with pdf defined in (25), then the r^{th} incomplete moments of X is given by

$$\psi_r(t) = \int_{-\infty}^{t} x^r f(x)dx. \tag{37}$$

Substituting (25) into (37) we have

$$\psi_r(t) = \sum_{i,j,k=0}^{\infty} \Omega_{i,j,k} ab \int_0^t x^{r+a-1}\left(1-x^a\right)^{b(1+k)-1} dx, \tag{38}$$

where $\Omega_{i,j,k} = \dfrac{(-1)^{i+k}\Gamma(4+2i+j)\Gamma(\beta(3+2i+j))\beta}{i!j!k!2^i\alpha^{2i+3}\Gamma(4+2i)\Gamma(\beta(3+2i+j)-k)}\dfrac{\sqrt{2}}{\sqrt{\pi}}$. Thus, by applying (27) into (38) gives

$$\begin{aligned}\psi_r(t) &= \sum_{i,j,k=0}^{\infty} \Omega_{i,j,k} b \int_0^{t^a} x^r \left(1-A\right)^{b(1+k)-1} dA \\ &= \sum_{i,j,k=0}^{\infty} \Omega_{i,j,k} b \int_0^{t^a} A^{\frac{r}{a}} \left(1-A\right)^{b(1+k)-1} dA \\ &= \sum_{i,j,k=0}^{\infty} \Omega_{i,j,k} b \int_0^{t^a} A^{\left(\frac{r}{a}+1\right)-1} \left(1-A\right)^{b(1+k)-1} dA \\ &= \sum_{i,j,k=0}^{\infty} \Omega_{i,j,k} b\beta_{t^a}\left(1+\frac{r}{a},\ b(1+k)\right),\end{aligned} \tag{39}$$

which is the proposed r^{th} incomplete moments of GOM-Kumaraswamy distribution, where $\beta_x(c, d) = \int_0^x t^{c-1}(1-t)^{d-1}$ is an incomplete beta function of first kind.

4.3 *Probability weighted moments*

For real numbers p, q and $t = 0$ [50], defined probability weighted moments of random variable X as

$$\varrho_{p,q,t=0} = \int_{-\infty}^{\infty} x^p f(x)(F(X))^q dx, \tag{40}$$

where $F(x)$ and $f(x)$ are distribution and density functions defined in (12) and (13) respectively. The term $f(x)(F(x))^q$ in (40) can be expanded as

$$f(x)(F(X))^q = \frac{2\beta ab x^{a-1}\left(1-x^a\right)^{b-1}\left(1-\left(1-x^a\right)^b\right)^{3\beta-1}}{\alpha^3\sqrt{2\pi}\left(1-\left(1-\left(1-x^a\right)^b\right)^\beta\right)^4}\exp\left(-\frac{1}{2\alpha^2}\left(\frac{\left(1-\left(1-x^a\right)^b\right)^\beta}{1-\left(1-\left(1-x^a\right)^b\right)^\beta}\right)^2\right)\gamma_1\left(c, z\right)^q, \tag{41}$$

where $\gamma_1(c, z) = \gamma(c, z)/\Gamma(c)$ is the ratio of incomplete gamma function with $c = 3/2$ and

$$z = \frac{1}{2\alpha^2}\left(\frac{\left(1-\left(1-x^a\right)^b\right)^\beta}{1-\left(1-\left(1-x^a\right)^b\right)^\beta}\right)^2.$$

Applying exponential expansion defined in (22), (41) becomes

$$f(x)(F(X))^q = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!\left(2\alpha^2\right)^l}\frac{2\beta abx^{a-1}\left(1-x^a\right)^{b-1}\left(1-\left(1-x^a\right)^b\right)^{3\beta+2\beta l-1}}{\alpha^3\sqrt{2\pi}\left(1-\left(1-\left(1-x^a\right)^b\right)^\beta\right)^{4+2l}}\gamma_1\left(c,\, z\right)^q. \tag{42}$$

The incomplete gamma function in (42) can be simplified as

$$\left[\gamma_1\left(c,\, z\right)\right]^q = \left[1-\left(1-\gamma_1\left(c,\, z\right)\right)\right]^q. \tag{43}$$

Applying (20) into (43) we can have

$$\left[\gamma_1\left(c,\, z\right)\right]^q = \sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(q+1)}{i!\Gamma(q-i+1)}\left(1-\gamma_1\left(c,\, z\right)\right)^i$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^{i+j}\Gamma(q+1)\Gamma(i+1)}{i!\,j!\Gamma(q-i+1)\Gamma(i-j+1)}\gamma_1\left(c,\, z\right)^j. \tag{44}$$

Substitute for $\sum_{i=0}^{\infty}\sum_{j=0}^{i}$ to $\sum_{j=0}^{\infty}\sum_{i=j}^{\infty}$ in (44) gives

$$\left[\gamma_1\left(c,\, z\right)\right]^q = \sum_{j=0}^{\infty}\sum_{i=j}^{\infty}\frac{(-1)^{i+j}\Gamma(q+1)\Gamma(i+1)}{i!\,j!\Gamma(q-i+1)\Gamma(i-j+1)}\gamma_1\left(c,\, z\right)^j$$

$$= \sum_{j=0}^{\infty}C_i(q)\gamma_1\left(c,\, z\right)^j, \tag{45}$$

where $C_i(q) = \sum_{i=j}^{\infty}\frac{(-1)^{i+j}\Gamma(q+1)\Gamma(i+1)}{i!\,j!\Gamma(q-i+1)\Gamma(i-j+1)}$. As defined by [48] in Section 0.314, the power series expansion for the ratio of incomplete gamma function in (45) can be expressed as

$$\gamma_1\left(c,\, z\right)^j = \frac{z^{cj}}{\left(\Gamma(c)\right)^j}\sum_{k=0}^{\infty}\Delta_{j,\,k}z^k, \tag{46}$$

where $\Delta_{j,\,k} = \left(kz_0\right)^{-1}\sum_{h=1}^{k}\left(jh-k+h\right)z_h\Delta_{j,\,k-h},\ k \geq 1$ with $z_h = (-1)^h/h!(c+h)$. Substituting this into (45) gives

$$\left[\gamma_1(c,\ z)\right]^q = \sum_{j=0}^{\infty}\frac{C_i(q)z^{cj}}{\left(\Gamma(c)\right)^j}\sum_{k=0}^{\infty}\Delta_{j,\,k}z^k$$

$$= \sum_{j,\,k=0}^{\infty}\frac{C_i(q)\,\Delta_{j,\,k}}{\left(\Gamma\left(\frac{3}{2}\right)\right)^j\left(2\alpha^2\right)^{\frac{3j+2k}{2}}}\left(\frac{\left(1-\left(1-x^a\right)^b\right)^\beta}{1-\left(1-\left(1-x^a\right)^b\right)^\beta}\right)^{3j+2k}. \tag{47}$$

Inserting (47) into (42) we get

$$f(x)(F(X))^q = \sum_{l=0}^{\infty}\frac{(-1)^l\,2\beta abx^{a-1}\left(1-x^a\right)^{b-1}\left(1-\left(1-x^a\right)^b\right)^{3\beta+2\beta l-1}}{l!\left(2\alpha^2\right)^l\alpha^3\sqrt{2\pi}\left(1-\left(1-\left(1-x^a\right)^b\right)^\beta\right)^{4+2l}}\sum_{j,\,k=0}^{\infty}\frac{C_i(q)\,\Delta_{j,\,k}}{\left(\Gamma\left(\frac{3}{2}\right)\right)^j\left(2\alpha^2\right)^{\frac{3j+2k}{2}}}\left(\frac{\left(1-\left(1-x^a\right)^b\right)^\beta}{1-\left(1-\left(1-x^a\right)^b\right)^\beta}\right)^{3j+2k}$$

$$= \sum_{l,\,j,\,k=0}^{\infty}\frac{\Psi_{l,\,j,\,k}abx^{a-1}\left(1-x^a\right)^{b-1}\left(1-\left(1-x^a\right)^b\right)^{3\beta+2\beta l+\beta(3j+2k)+\beta m-1}}{\left(1-\left(1-\left(1-x^a\right)^b\right)^\beta\right)^{4+2l+3j+2k}}, \tag{48}$$

where $\Psi_{l,\,j,\,k} = \dfrac{2(-1)^l C_i(q)\,\Delta_{j,\,k}\beta}{l!\left(2\alpha^2\right)^l\left(\Gamma\left(\frac{3}{2}\right)\right)^j\left(2\alpha^2\right)^{\frac{3j+2k}{2}}\alpha^3\sqrt{2\pi}}$. Using (18), (48) becomes

$$f(x)(F(X))^q = \sum_{l,\,j,\,k,\,m=0}^{\infty}\Psi_{l,\,j,\,k}\frac{\Gamma(4+2l+3j+2k+m)}{m!\Gamma(4+2l+3j+2k)}abx^{a-1}\left(1-x^a\right)^{b-1}\left(1-\left(1-x^a\right)^b\right)^{3\beta+2\beta l+\beta(3j+2k)+\beta m-1}. \tag{49}$$

Applying binomial expansion defined in (20), (49) can be written as

$$f(x)(F(X))^q = \sum_{l,\,j,\,k,\,m=0}^{\infty}\Psi_{l,\,j,\,k}\frac{\Gamma(4+2l+3j+2k+m)}{m!\Gamma(4+2l+3j+2k)}abx^{a-1}\left(1-x^a\right)^{b(1+n)-1}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma\left(\beta\left(+l+(3j+2k)+m\right)\right)}{n!\Gamma\left(\beta\left(+l+(3j+2k)+m\right)-n\right)} \tag{50}$$

$$= ab\sum_{l,\,j,\,k,\,m,\,n=0}^{\infty}T_{l,\,j,\,k,\,m,\,n}x^{a-1}\left(1-x^a\right)^{b(1+n)-1}, \tag{51}$$

where

$$T_{l,\,j,\,k,\,m,\,n} = \Psi_{l,\,j,\,k}\frac{(-1)^n\Gamma(4+2l+3j+2k+m)\Gamma\left(\beta\left(+l+(3j+2k)+m\right)\right)}{m!n!\Gamma(4+2l+3j+2k)\Gamma\left(\beta\left(+l+(3j+2k)+m\right)-n\right)}. \tag{52}$$

Inserting (50) into (40) we can easily get

$$\varrho_{p,\,q,\,t=0} = ab\sum_{l,\,j,\,k,\,m,\,n=0}^{\infty}T_{l,\,j,\,k,\,m,\,n}\int_0^1 x^{p+a-1}\left(1-x^a\right)^{b(1+n)-1}dx. \tag{53}$$

From the definition of integral part of (53) in (28), we have

$$\varrho_{p,\,q,\,t=0} = b \sum_{l,\,j,\,k,\,m,\,n=0}^{\infty} T_{l,\,j,\,k,\,m,\,n} \int_0^1 x^p \left(1-A\right)^{b(1+n)-1} dA$$

$$= b \sum_{l,\,j,\,k,\,m,\,n=0}^{\infty} T_{l,\,j,\,k,\,m,\,n} \int_0^1 A^{\frac{p}{a}} \left(1-A\right)^{b(1+n)-1} dA$$

$$= b \sum_{l,\,j,\,k,\,m,\,n=0}^{\infty} T_{l,\,j,\,k,\,m,\,n} \int_0^1 A^{\left(\frac{p}{a}+1\right)-1} \left(1-A\right)^{b(1+n)-1} dA$$

$$= b \sum_{l,\,j,\,k,\,m,\,n=0}^{\infty} T_{l,\,j,\,k,\,m,\,n} \beta\left(1+\frac{p}{a},\, b(1+n)\right), \tag{54}$$

which is the probability weighted moments of GOM-Kumaraswamy distribution, where $\beta(c,\, d) = \int_0^1 t^{c-1}(1-t)^{d-1}\, dt$ is a beta function of first kind.

4.4 *Entropies*

The two entropies such as Rényi and q-entropies of the GOM-Kumaraswamy distribution are discussed in this section.

4.4.1 *Rényi entropy*

By definition, the Rényi entropy of a random variable X with density function $f(x)$ defined in (13) is given by

$$R_\rho(x) = \frac{1}{1-\rho}\left[\int_{-\infty}^{\infty} f(x)^\rho\, dx\right],\ \rho > 0.\ \rho \neq 1;\ x \in \Re. \tag{55}$$

The term $f(x)^\rho$ in (55) can be expressed as

$$f(x)^\rho = \left(\frac{2\beta ab}{\alpha^3\sqrt{2\pi}}\right)^\rho \frac{x^{\rho(a-1)}\left(1-x^a\right)^{\rho(b-1)}\left(1-\left(1-x^a\right)^b\right)^{\rho(3\beta-1)}}{\left(1-\left(1-\left(1-x^a\right)^b\right)^\beta\right)^{4\rho}} e^{\left(-\frac{\rho}{2\alpha^2}\left(\frac{\left(1-\left(1-x^a\right)^b\right)^\beta}{1-\left(1-\left(1-x^a\right)^b\right)^\beta}\right)^2\right)}. \tag{56}$$

Using (23), then (56) becomes

$$f(x)^\rho = \sum_{i=0}^{\infty} \frac{(-1)^i(\rho)^i}{i!(2\alpha^2)^i}\left(\frac{2\beta ab}{\alpha^3\sqrt{2\pi}}\right)^\rho x^{\rho(a-1)}\left(1-x^a\right)^{\rho(b-1)} \frac{\left(1-\left(1-x^a\right)^b\right)^{\rho(3\beta-1)+2\beta i}}{\left(1-\left(1-\left(1-x^a\right)^b\right)^\beta\right)^{4\rho+2i}}. \tag{57}$$

Applying (18) to the last term of the denominator in (57) becomes

$$f(x)^\rho = \sum_{i,\,j=0}^{\infty} \frac{(-1)^i(\rho)^i\Gamma(4\rho+2i+j)}{i!\,j!(2\alpha^2)^i\Gamma(4\rho+2i)}\left(\frac{2\beta ab}{\alpha^3\sqrt{2\pi}}\right)^\rho x^{\rho(a-1)}\left(1-x^a\right)^{\rho(b-1)}\left(1-\left(1-x^a\right)^b\right)^{\rho(3\beta-1)+2\beta i+\beta j}, \tag{58}$$

and this can be written as

$$f(x)^{\rho} = \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+k}(\rho)^i \Gamma(4\rho+2i+j)\Gamma(\rho(3\beta-1)+2\beta i+\beta j+1)}{i!j!k!(2\alpha^2)^i \Gamma(4\rho+2i)\Gamma(\rho(3\beta-1)+2\beta i+\beta j-k+1)} \left(\frac{2\beta ab}{\alpha^3\sqrt{2\pi}}\right)^{\rho} x^{\rho(a-1)} \left(1-x^a\right)^{\rho(b-1)+bk}$$

$$= \sum_{i,j,k=0}^{\infty} \Phi_{i,j,k} x^{\rho(a-1)} \left(1-x^a\right)^{\rho(b-1)+bk}, \tag{59}$$

where $\Phi_{i,j,k} = \frac{(-1)^{i+k}(\rho)^i \Gamma(4\rho+2i+j)\Gamma(\rho(3\beta-1)+2\beta i+\beta j+1)}{i!j!k!(2\alpha^2)^i \Gamma(4\rho+2i)\Gamma(\rho(3\beta-1)+2\beta i+\beta j-k+1)} \left(\frac{2\beta ab}{\alpha^3\sqrt{2\pi.}}\right)^{\rho}.$

By substituting (59) into (55), the integral part of (55) can be simplified as

$$\int_0^1 f(x)^{\rho} dx = \sum_{i,j,k=0}^{\infty} \Phi_{i,j,k} \int_0^1 x^{\rho(a-1)} \left(1-x^a\right)^{\rho(b-1)+bk} dx, \tag{60}$$

which can be expressed as

$$\int_0^1 f(x)^{\rho} dx = \sum_{i,j,k=0}^{\infty} \frac{\Phi_{i,j,k}}{a} \int_0^1 A^{\frac{\rho(a-1)-a+1}{a}} (1-A)^{\rho(b-1)+bk} dA$$

$$= \sum_{i,j,k=0}^{\infty} \frac{\Phi_{i,j,k}}{a} \int_0^1 A^{\frac{\rho(a-1)+1}{a}-1} (1-A)^{\rho(b-1)+bk} dA$$

$$= \sum_{i,j,k=0}^{\infty} \frac{\Phi_{i,j,k}}{a} \beta\left(\frac{\rho(a-1)+1}{a},\ \rho(b-1)+bk\right). \tag{61}$$

The Rényi entropy of the GOM-Kumaraswamy distribution is thus obtained by inserting (61) into (55) as

$$R_{\rho}(x) = \frac{1}{1-\rho}\left[\sum_{i,j,k=0}^{\infty} \frac{\Phi_{i,j,k}}{a} \beta\left(\frac{\rho(a-1)+1}{a},\ \rho(b-1)+bk\right)\right],\ \rho>0.\ \rho\neq 1;\ 0<x<1. \tag{62}$$

4.4.2 *Q-entropy*

The q-entropy of a random variable X with density function $f(x)$ is given by

$$Q_{\eta}(x) = \frac{1}{\eta-1}\left[1-\int_{-\infty}^{\infty} f(x)^{\eta} dx\right],\ \eta\neq 1;\ x\in\Re \tag{63}$$

where $f(x)$ is as defined in (13). The integral part of (63) has been defined in (61) for $\rho = \eta$. Therefore, the proposed q-entropy of the GOM-Kumaraswamy distribution is obtained by inserting (61) into (63) as

$$Q_{\eta}(x) = \frac{1}{\eta-1}\left[1-\sum_{i,j,k=0}^{\infty} \frac{\Phi_{i,j,k}}{a} \beta\left(\frac{\eta(a-1)+1}{a},\ \eta(b-1)+bk\right)\right],\ \eta\neq 1;\ 0<x<1 \tag{64}$$

where $\Phi_{i,j,k} = \frac{(-1)^{i+k}(\rho)^i \Gamma(4\rho+2i+j)\Gamma(\rho(3\beta-1)+2\beta i+\beta j+1)}{i!j!k!(2\alpha^2)^i \Gamma(4\rho+2i)\Gamma(\rho(3\beta-1)+2\beta i+\beta j-k+1)} \left(\frac{2\beta ab}{\alpha^3\sqrt{2\pi}}\right)^{\rho}.$

4.5 *Stress-strength model*

The stress-strength model is usually called Reliability denoted by **R**, where **R** = $P(X_2 < X_1)$. Suppose X_1 and X_2 are two independent random variables having density functions $f(x_1; \alpha, \beta, a, b = b_1)$ and $f(x_2; \alpha, \beta, a, b = b_2)$ defined in (13).

The reliability **R** can be presented by

$$\mathbf{R} = \int_{-\infty}^{\infty} f(x_1;\ \alpha,\ \beta,\ a,\ b = b_1) F(x_2;\ \alpha,\ \beta,\ a,\ b = b_2) dx_1.\ x \in \Re. \tag{65}$$

Since $f(x_1; \alpha, \beta, a, b = b_1)$ and $F(x_2; \alpha, \beta, a, b = b_2)$ are presented as a mixture of exponentiated-G density defined respectively in (21) and (25). By taking the products of these functions we can have

$$\mathbf{R} = \sum_{i, j, k, l, m, n=0}^{\infty} \Omega_{i, j, k} \varpi_{l, m, n} ab_1 \int_0^1 x^{a-1} \left(1 - x^a\right)^{b_1(1+k)+b_2k-1} dx_1, \tag{66}$$

Where $\Omega_{i, j, k} = \dfrac{(-1)^{i+k} \Gamma(4+2i+j)\Gamma(\beta(3+2i+j))\beta}{i!\,j!\,k!\,2^i \alpha^{2i+3} \Gamma(4+2i)\Gamma(\beta(3+2i+j)-k)} \dfrac{\sqrt{2}}{\sqrt{\pi}}$, and

$$\varpi_{l, m, n} = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{(-1)^{l+n} \Gamma(3+2l+m)\Gamma(\beta(3+2l+m)+1)}{l!\,m!\,n!\,2^l \left(3+2l\right) \alpha^{3+2l} \Gamma(3+2l)\Gamma(\beta(3+2l+m)-n+1)}.$$

Inserting (27) into (66) gives

$$\mathbf{R} = \sum_{i, j, k, l, m, n=0}^{\infty} \Omega_{i, j, k} \varpi_{l, m, n} b_1 \int_0^1 \left(1 - A\right)^{b_1(1+k)+b_2k-1} dA. \tag{67}$$

If we let

$$Z = 1 - A,\ \Rightarrow dA = -dZ, \tag{68}$$

then substituting this into (67) gives the stress-strength of the GOM-Kumaraswamy distribution given in (69)

$$\mathbf{R} = \sum_{i, j, k, l, m, n=0}^{\infty} \Omega_{i, j, k} \varpi_{l, m, n} b_1 \int_0^1 Z^{b_1(1+k)+b_2k-1} dZ = \sum_{i, j, k, l, m, n=0}^{\infty} \frac{\Omega_{i, j, k} \varpi_{l, m, n} b_1}{b_1(1+k)+b_2k}. \tag{69}$$

4.6 *Order statistics*

Suppose $X_1, X_2, X_3, X_4, \ldots, X_n$ are independent random samples of size n with density and distribution functions given by $f(x)$ and $F(x)$ respectively. Let $X_{1:n}, X_{2:n}, X_{3:n}, \ldots, X_{n:n}$ denoted the order statistics of the corresponding $X_1, X_2, X_3, \ldots, X_n$ derived from this samples. By definition, the p^{th} order statistics is defined as

$$f_{p:n}(x) = \frac{\Gamma(n+1) f(x)}{\Gamma(p)\Gamma(n-p+1)} F(x)^{p-1} \left[1 - F(x)\right]^{n-p}. \tag{70}$$

Using binomial expansion defined in (20), the p^{th} order statistics is

$$f_{p:n}(x) = \frac{\Gamma(n+1)}{\Gamma(p)\Gamma(n-p+1)} \sum_{s=0}^{\infty} \frac{(-1)^s \Gamma(n-p+1)}{s!\,\Gamma(n-p-s+1)} f(x) F(x)^{p+s-1}. \tag{71}$$

To obtain p^{th} order statistics of the proposed GOM-Kumaraswamy distribution, the term $f(x)F(x)^{p+s-1}$ in (71) can be replaced by (50) for $q = p + s - 1$. By substituting this we get

$$f_{p:n}(x) = ab \sum_{n=0}^{\infty} \Pi_n x^{a-1} \left(1 - x^a\right)^{b(1+n)-1}, \tag{72}$$

where

$$\Pi_n = \sum_{l,\, j,\, k,\, m,\, s=0}^{\infty} \frac{(-1)^s \Gamma(n-p+1)\Gamma(n+1)}{s!\Gamma(n-p-s+1)\Gamma(p)\Gamma(n-p+1)} T_{l,\, j,\, k,\, m,\, n}, \tag{73}$$

$$T_{l,\, j,\, k,\, m,\, n} = \Psi_{l,\, j,\, k} \frac{(-1)^n \Gamma(4+2l+3j+2k+m)\Gamma\left(\beta\left(+l+(3j+2k)+m\right)\right)}{m!n!\Gamma(4+2l+3j+2k)\Gamma\left(\beta\left(+l+(3j+2k)+m\right)-n\right)},$$

$$\Psi_{l,\, j,\, k} = \frac{2(-1)^l C_i(q)\, \Delta_{j,\, k} \beta}{l!\left(2\alpha^2\right)^l \left(\Gamma\left(\frac{3}{2}\right)\right)^j \left(2\alpha^2\right)^{\frac{3j+2k}{2}} \alpha^3 \sqrt{2\pi}},$$

$$C_i(q) = \sum_{i=j}^{\infty} \frac{(-1)^{i+j} \Gamma(q+1)\Gamma(i+1)}{i!\, j!\Gamma(q-i+1)\Gamma(i-j+1)},$$

and $\Delta_{j,\, k} = \left(kz_0\right)^{-1} \sum_{h=1}^{k} \left(jh-k+h\right) z_h \Delta_{j,\, k-h}$, for $k \geq 1$and $z_h = (-1)^h / h!(a+h)$.

4.6.1 *Moments of order statistics*

Suppose X is a random variable with density function defined (27). The moments of p^{th} order statistics is defined as

$$E(X_{p:n}^r) = \int_{-\infty}^{\infty} x^r f_{p,\, n}(x) dx. \tag{74}$$

Inserting (72) into (74) we can have

$$E(X_{p:n}^r) = ab \sum_{n=0}^{\infty} \Pi_n \int_0^1 x^{r+a-1} \left(1-x^a\right)^{b(1+n)-1} dx, \tag{75}$$

where Π_n is defined in (73). Applying (27) into (75) gives

$$\begin{aligned} E(X_{p:n}^r) &= b \sum_{n=0}^{\infty} \Pi_n \int_0^1 A^{\frac{r}{a}} \left(1-A\right)^{b(1+n)-1} dA \\ &= b \sum_{n=0}^{\infty} \Pi_n \int_0^1 A^{\left(\frac{r}{a}+1\right)-1} \left(1-A\right)^{b(1+n)-1} dA \\ &= b \sum_{n=0}^{\infty} \Pi_n \beta\left(1+\frac{r}{a},\ b(1+n)\right). \end{aligned} \tag{76}$$

5. Parameter estimation

The estimates of the parameters of the GOM-Kumaraswamy distribution are derived in this section.

Suppose $X_1, X_2, X_3, \ldots, X_n$ are random samples of size n drawn from the GOM-Kumaraswamy distribution with parameters expressed in vector form given by $\xi = (a, b, \alpha, \beta)^T$. To derive the estimates of ξ parameter, the likelihood function (13) is given by

$$\ell\left(a,\ b,\ \alpha,\ \beta / x_i\right)=\left(\frac{2\beta ab}{\alpha^3\sqrt{2\pi}}\right)^n \prod_{i=1}^{n} \frac{x_i^{a-1}\left(1-x_i^a\right)^{b-1}\left(1-\left(1-x_i^a\right)^b\right)^{3\beta-1}}{\left(1-\left(1-\left(1-x_i^a\right)^b\right)^\beta\right)^4} \prod_{i=1}^{n} \exp\left(-\frac{1}{2\alpha^2}\left(\frac{\left(1-\left(1-x_i^a\right)^b\right)^\beta}{1-\left(1-\left(1-x_i^a\right)^b\right)^\beta}\right)^2\right). \quad (77)$$

The natural logarithm of (77), i.e. $L=\log\ell\left(a,\ b,\ \alpha,\ \beta / x_i\right)$ is given by

$$L=n\log(2)+n\log(\beta)+n\log(a)+n\log(b)-3n\log(\alpha)-\frac{n}{2}\log(2\pi)+(a-1)\sum_{i=1}^{n}\log(x_i)$$

$$+(b-1)\sum_{i=1}^{n}\log(1-x_i^a)+(3\beta-1)\sum_{i=1}^{n}\log\left(A_i\right)-4\sum_{i=1}^{n}\log\left(1-A_i^\beta\right)-\frac{1}{2\alpha^2}\sum_{i=1}^{n}\left(\frac{A_i^\beta}{1-A_i^\beta}\right)^2, \quad (78)$$

where $A_i(x) = A_i = 1 - (1 - x_i^a)^b$. The maximum likelihood estimates $\hat{\xi}$ of the parameters ξ will be derived by differentiating (78) partially with respect to parameter α, β, a and b respectively, and setting all results to zero as

$$\frac{\partial L}{\partial\alpha}=-\frac{3n}{\alpha}+\frac{1}{\alpha^3}\sum_{i=1}^{n}V_i^2=0, \quad (79)$$

$$\frac{\partial L}{\partial\beta}=\frac{n}{\beta}+3\sum_{i=1}^{n}\log\left(A_i\right)+4\sum_{i=1}^{n}\left(V_i\log\left(A_i\right)\right)-\frac{1}{2\alpha^2}\sum_{i=1}^{n}\left(V_i^2 Z_i\right)=0, \quad (80)$$

$$\frac{\partial L}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}\log(x_i)-(b-1)\sum_{i=1}^{n}\left(\frac{x_i^a\log\left(x_i\right)}{1-x_i^a}\right)+b\left(3\beta-1\right)\sum_{i=1}^{n}\left(\frac{C_i}{A_i}\right)+4b\beta\sum_{i=1}^{n}\left(\frac{C_iV_i}{A_i}\right)-\frac{b\beta}{\alpha^2}\sum_{i=1}^{n}\left(\frac{C_iV_i^2}{A_i\left(1-A_i^\beta\right)}\right)=0 \quad (81)$$

and

$$\frac{\partial L}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\log\left(1-x_i^a\right)-\left(3\beta-1\right)\sum_{i=1}^{n}\left(\frac{W_i}{A_i}\right)-4\beta\sum_{i=1}^{n}\left(\frac{V_iW_i}{A_i}\right)+\frac{\beta}{\alpha^2}\sum_{i=1}^{n}\left(\frac{V_i^2W_i}{A_i\left(1-A_i^\beta\right)}\right)=0, \quad (82)$$

where $C_i=x_i^a\log\left(x_i\right)\left(1-x_i^a\right)^{b-1}$, $V_i=A_i^\beta/1-A_i^\beta$, $W_i=\left(1-x_i^a\right)^b\log\left(1-x_i^a\right)$, and $Z_i=\log\left(A_i\right)/1-A_i^\beta$.

Consequently, (79)-(82) are nonlinear functions and cannot be derived analytically. Consequently, one might use statistical software like Matlab, R-package, and many others to get an estimate of its parameters. Therefore, R-package can be utilized in this context to estimate the model parameters.

6. Simulation study

To assess the consistency of maximum likelihood estimators (MLEs) within the context of the GOM-Kumaraswamy distribution, two comprehensive simulation studies are conducted in this section using the R package, based on the quantile function.

6.1 *Quantile function*

The Quantile function ($Q(u)$) of the GOM-Kumaraswamy distribution can be obtained by inverting the cdf in (12) as applied by [51] given (83) as

$$1-\left(1-x^{a}\right)^{b}=\left(\frac{\left[2\alpha^{2}\gamma^{-1}\left(\frac{3}{2},\,u\Gamma\left(\frac{3}{2}\right)\right)\right]^{\frac{1}{2}}}{1+\left[2\alpha^{2}\gamma^{-1}\left(\frac{3}{2},\,u\Gamma\left(\frac{3}{2}\right)\right)\right]^{\frac{1}{2}}}\right)^{\beta^{-1}}, \tag{83}$$

this implies that

$$\left(1-x^{a}\right)^{b}=1-\left(\frac{\left[2\alpha^{2}\gamma^{-1}\left(\frac{3}{2},\,u\Gamma\left(\frac{3}{2}\right)\right)\right]^{\frac{1}{2}}}{1+\left[2\alpha^{2}\gamma^{-1}\left(\frac{3}{2},\,u\Gamma\left(\frac{3}{2}\right)\right)\right]^{\frac{1}{2}}}\right)^{\beta^{-1}}, \tag{84}$$

and can further be written as

$$1-x^{a}=\left(1-\left(\frac{\left[2\alpha^{2}\gamma^{-1}\left(\frac{3}{2},\,u\Gamma\left(\frac{3}{2}\right)\right)\right]^{\frac{1}{2}}}{1+\left[2\alpha^{2}\gamma^{-1}\left(\frac{3}{2},\,u\Gamma\left(\frac{3}{2}\right)\right)\right]^{\frac{1}{2}}}\right)^{\beta^{-1}}\right)^{\frac{1}{b}} \tag{85}$$

which on simplification becomes

$$x^{a}=1-\left(1-\left(\frac{\left[2\alpha^{2}\gamma^{-1}\left(\frac{3}{2},\,u\Gamma\left(\frac{3}{2}\right)\right)\right]^{\frac{1}{2}}}{1+\left[2\alpha^{2}\gamma^{-1}\left(\frac{3}{2},\,u\Gamma\left(\frac{3}{2}\right)\right)\right]^{\frac{1}{2}}}\right)^{\beta^{-1}}\right)^{\frac{1}{b}}. \tag{86}$$

Therefore, the quantile function of GOM-Kw distribution is obtained from (86) as

$$x=Q(u)=\left(1-\left(1-\left(\frac{\left[2\alpha^{2}\gamma^{-1}\left(\frac{3}{2},\,u\Gamma\left(\frac{3}{2}\right)\right)\right]^{\frac{1}{2}}}{1+\left[2\alpha^{2}\gamma^{-1}\left(\frac{3}{2},\,u\Gamma\left(\frac{3}{2}\right)\right)\right]^{\frac{1}{2}}}\right)^{\beta^{-1}}\right)^{\frac{1}{b}}\right)^{\frac{1}{a}}, \tag{87}$$

where u is uniform random variable defined on interval (0, 1). Setting $u = 0.5$, (87) becomes the median (second quartile) of the Generalized Odd Maxwell-Kumaraswamy distribution. Similarly, for $u = 0.25$ and 0.75, we can obtain the first and third quartiles of $Q(u)$ respectively. Hereafter, the skewness, and kurtosis of the proposed distribution can be obtained using the quantile function defined in (87). Table 5 presents the coefficients in $Q(u)$, the skewness, and kurtosis of the proposed distribution using the first and second data sets.

Table 5. Coefficients of quantile function of the GOM-Kumaraswamy distribution for data sets

$Q(u)$	First Data	Second Data
$Q(0.25)$	0.3157	0.0929
$Q(0.5)$	0.4332	0.1478
$Q(0.75)$	0.5418	0.2043
$Q(0.125)$	0.2344	0.0590
$Q(0.375)$	0.3781	0.1212
$Q(0.625)$	0.4862	0.1746
$Q(0.875)$	0.6092	0.2427
skewness	-0.0395	0.0146
kurtosis	1.1795	1.1692

Based on (87), two simulations were conducted for various sample sizes (n) at different parameter values, defined as follows:

1. For the first simulation study, data were generated for sample sizes starting from 500, 1,000, 1,500, 2,000, and 2,500, with true parameter values $\alpha = 0.5$, $\beta = 1$, $a = 1.5$, and $b = 2.5$.

2. The second simulation data were generated for sample sizes of 250, 500, 750, 1,000, and 1,250, with true parameter values $\alpha = 1$, $\beta = 2$, $a = 3$, and $b = 5$.

The simulation was repeated 1,000 times, and the mean, bias, variance, and mean squared errors (MSE) were computed. The numerical results of these simulations are presented in Tables 6 and 7.

Table 6. Simulation results of GOM-Kumaraswamy distribution for $\alpha = 0.5$, $\beta = 1$, $a = 1.5$, and $b = 2.5$

Parameter	Sizes (n)	mean	bias	variance	MSE
α	500	0.6844	0.1844	0.9115	0.9455
	1,000	0.6059	0.1059	0.1470	0.1582
	1,500	0.5642	0.0642	0.0712	0.0753
	2,000	0.5569	0.0569	0.0476	0.0508
	2,500	0.5396	0.0396	0.0293	0.0308
β	500	1.8767	0.8767	3.1529	3.9215
	1,000	1.6332	0.6332	1.9465	2.3474
	1,500	1.4863	0.4863	1.3381	1.5746
	2,000	1.4005	0.4005	1.0220	1.1824
	2,500	1.3182	0.3182	0.8006	0.9018
a	500	1.7018	0.2018	2.1610	2.2017
	1,000	1.5177	0.0177	1.0404	1.0407
	1,500	1.4479	-0.0521	0.5424	0.5451
	2,000	1.4412	-0.0588	0.4461	0.4495
	2,500	1.4553	-0.0447	0.3441	0.3461
b	500	3.0586	0.5586	6.1695	6.4815
	1,000	2.6821	0.1821	1.7147	1.7478
	1,500	2.5395	0.0395	0.6604	0.6620
	2,000	2.5326	0.0326	0.5789	0.5800
	2,500	2.5106	0.0106	0.3154	0.3156

It is observed from Table 6 that the mean of the estimates decreases and approaches the true parameter values at $\alpha = 0.5$, $\beta = 1$, $a = 1.5$, and $b = 2.5$ as the sample size increases from 500 to 2,500. Additionally, the bias, variance, and MSE of each estimate decrease as the sample size (n) increases.

Table 7. Simulation results of GOM-Kumaraswamy distribution for $\alpha = 0.5$, $\beta = 2$, $a = 3$, and $b = 5$

Parameter	Sizes (n)	mean	bias	variance	MSE
	250	1.3085	0.3085	3.7579	3.8530
	500	1.1861	0.1861	1.4225	1.4571
α	750	1.1964	0.1964	1.3364	1.3749
	1,000	1.1260	0.1260	0.5380	0.5538
	1,250	1.0817	0.0817	0.2832	0.2898
	250	2.4867	0.4867	3.9616	4.1985
	500	2.3098	0.3098	2.1324	2.2284
β	750	2.2203	0.2203	1.2948	1.3433
	1,000	2.1703	0.1703	1.0593	1.0883
	1,250	2.1290	0.1290	0.6247	0.6413
	250	3.8668	0.8668	7.6580	8.4094
	500	3.3990	0.3990	3.4099	3.5691
a	750	3.1990	0.1990	1.8843	1.9239
	1,000	3.1230	0.1230	1.0021	1.0173
	1,250	3.0638	0.0638	0.5580	0.5621
	250	7.5140	2.5140	80.5697	86.8897
	500	5.8919	0.8919	22.2885	23.0839
b	750	5.4614	0.4614	11.8116	12.0245
	1,000	5.2128	0.2128	3.3963	3.4416
	1,250	5.0927	0.0927	1.2863	1.2949

Similarly, the mean of the estimates in Table 7 approaches the true parameter values for $\alpha = 1$, $\beta = 2$, $a = 3$, and $b = 5$ as the sample size (n) increases from 250 to 1,250. The bias, variance, and MSE of each estimate also decrease as the sample size (n) increases gradually.

7. Application

This section illustrates thorough applications of the GOM-Kumaraswamy distribution using two real-world datasets, which may be accomplished using statistical software (R-Package).

7.1 *Model comparison*

The proposed GOM-Kumaraswamy distribution is compared with the Kumaraswamy-Kumaraswamy (Kw-Kw) model by [11] and the Kumaraswamy-BurrIII (Kw-BIII) model by [52], respectively.

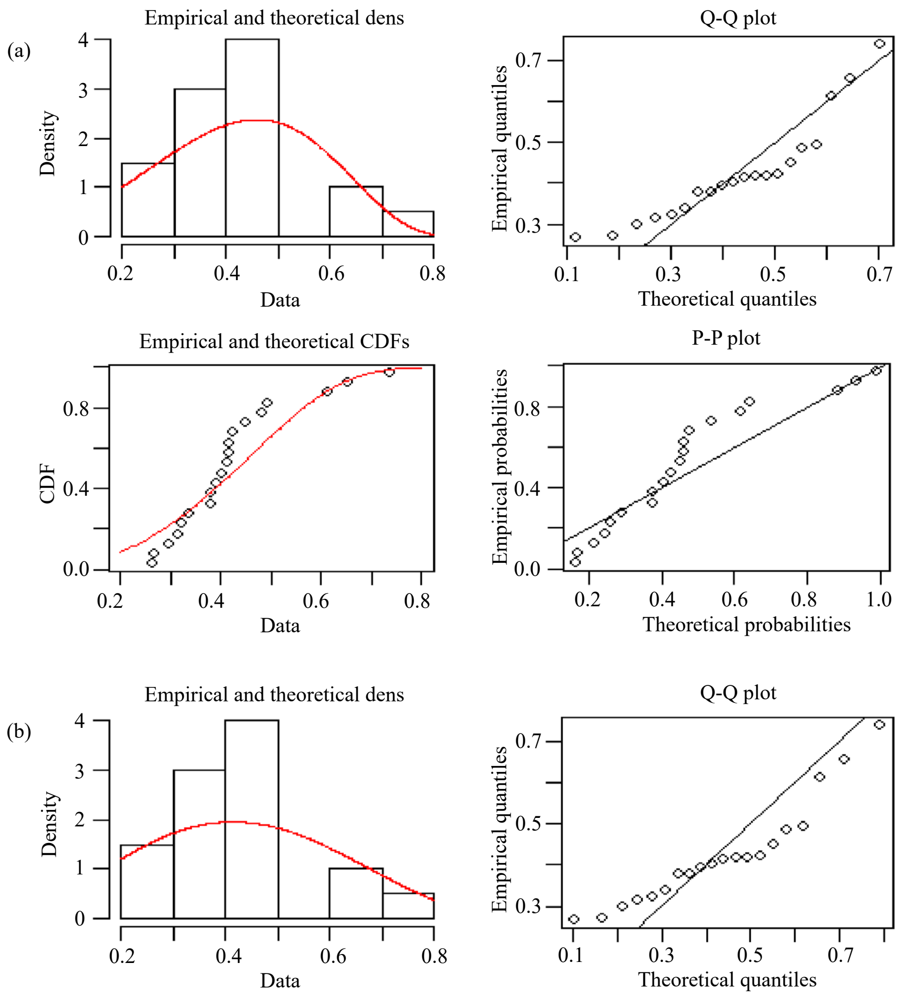
7.2 *Data sets*

We analyze two lifetime datasets to compare the flexibility of the GOM-Kumaraswamy model against its competing models. The first data set pertains to a flood data set consisting of 20 observations, as applied in [53] and [54], respectively. The second data set comprises 50 observations for the length of hole diameter and its sheet thickness of 9 mm and 2 mm, studied in [55]. The data sets are presented in Table 8.

Table 8. Data sets

	0.265	0.269	0.297	0.315	0.3235	0.338	0.379	0.379
First Data	0.392	0.402	0.412	0.416	0.418	0.423	0.449	0.484
	0.494	0.613	0.654	0.740	-	-	-	-
	0.06	0.12	0.14	0.04	0.14	0.16	0.08	0.26
	0.32	0.22	0.16	0.12	0.24	0.06	0.02	0.18
	0.22	0.14	0.22	0.16	0.12	0.24	0.06	0.02
Second Data	0.18	0.22	0.14	0.02	0.18	0.22	0.14	0.06
	0.04	0.14	0.22	0.14	0.06	0.04	0.16	0.24
	0.16	0.32	0.18	0.24	0.22	0.04	0.14	0.26
	0.18	0.16	-	-	-	-	-	-

The density (empirical and theoretical pdf), quantile (Q-Q plot), distribution (empirical and theoretical cdf), and probability (P-P) plots of the proposed distribution and its competing models are presented in Figures 3 and 4.



(c)

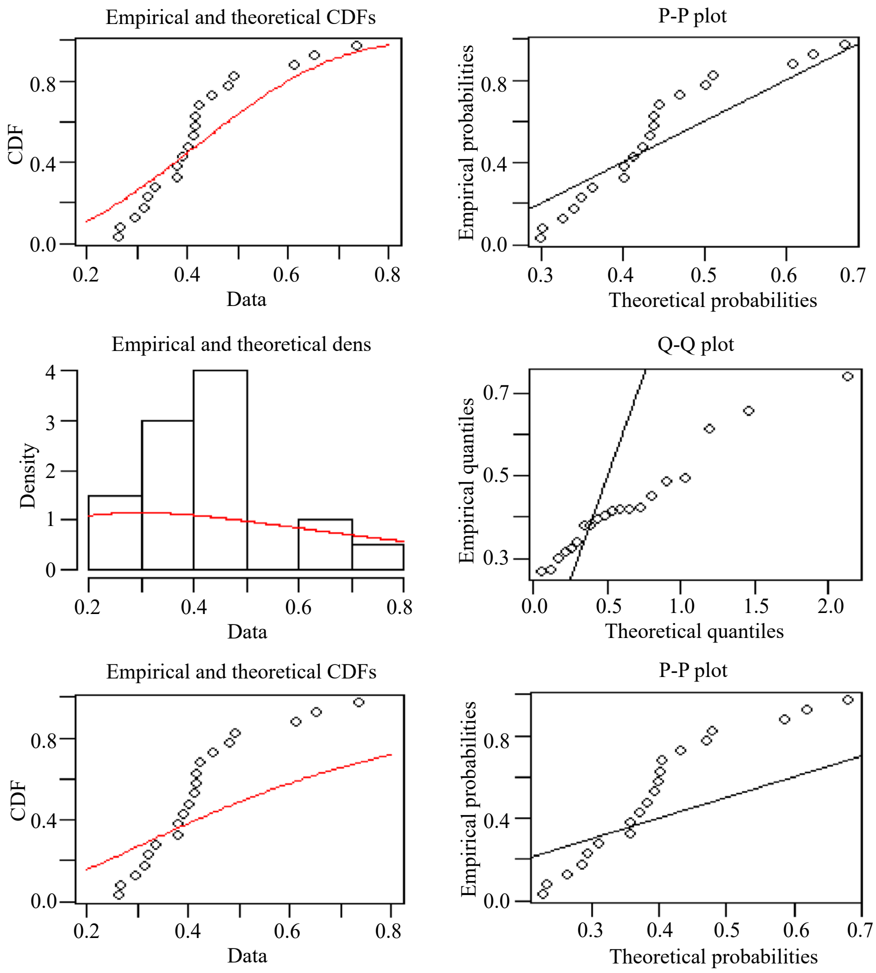
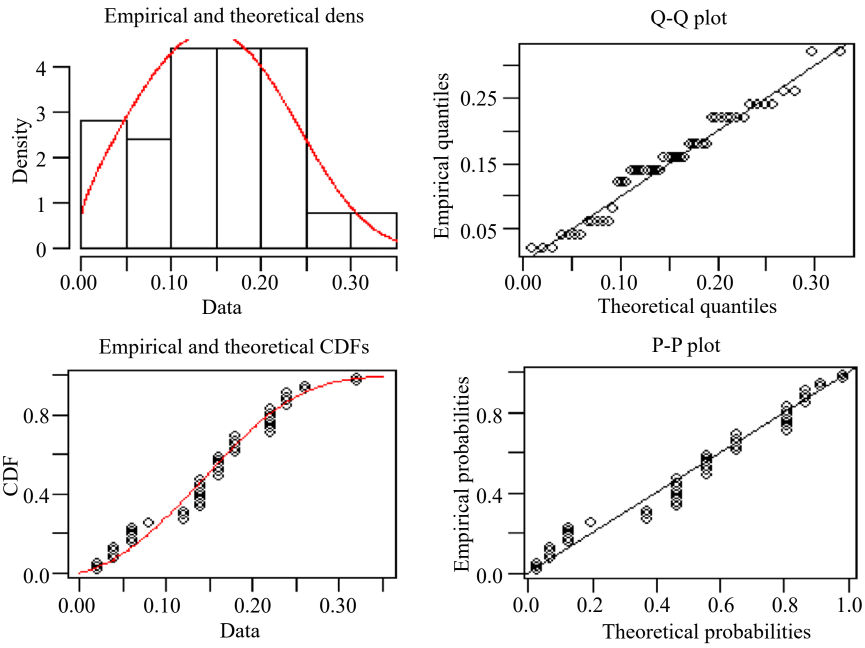


Figure 3. Density and Q-Q plots of (a) GOM-Kumaraswamy, (b) Kw-Kumaraswamy, (c) Kw-BurrIII for flood data

(a)



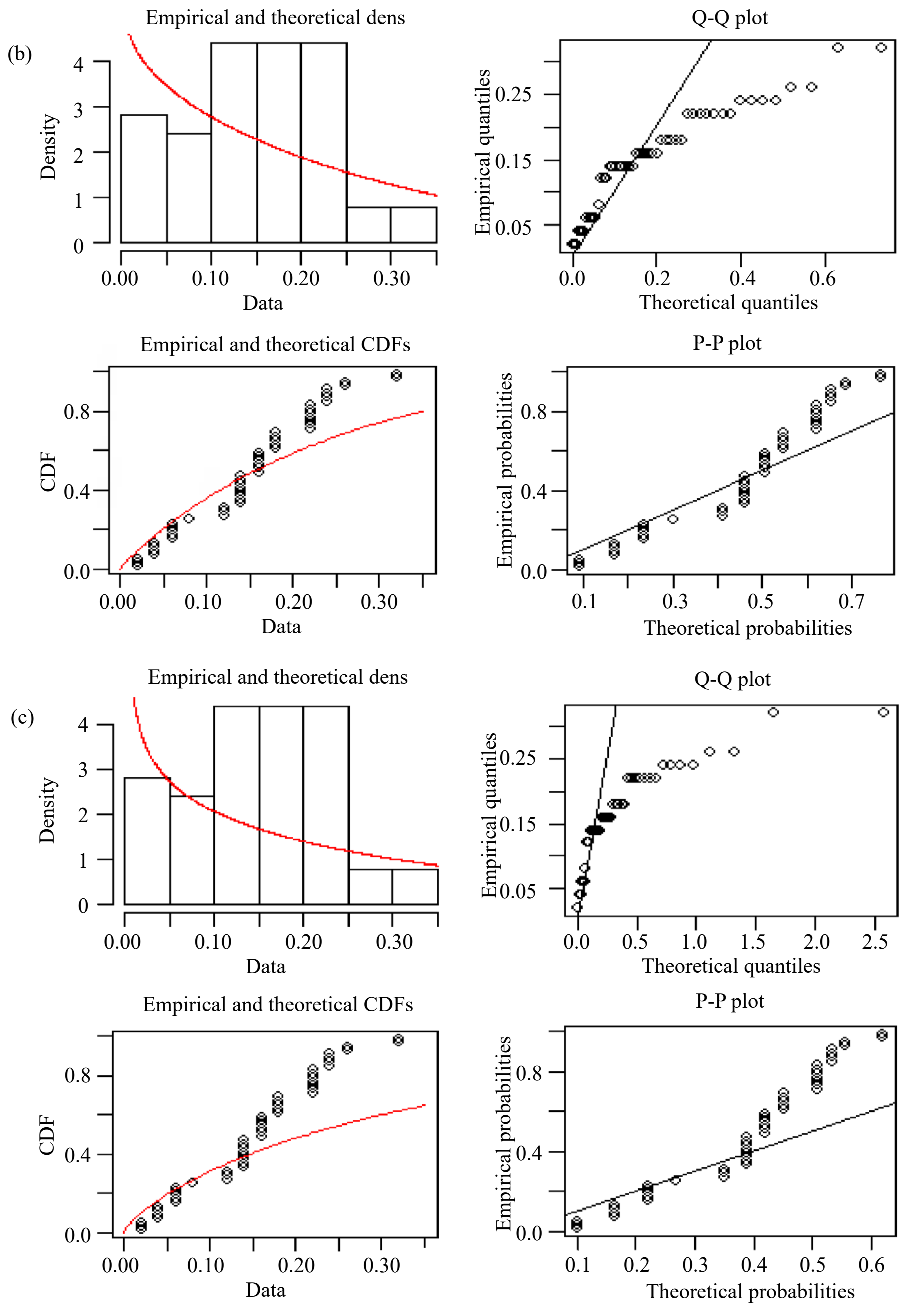


Figure 4. Density and Q-Q plots of (a) GOM-Kumaraswamy, (b) Kw-Kumaraswamy, (c) Kw-BurrIII for length of hole diameter data

It is observed that the proposed distribution appears to be suitable for all the datasets, as its density shape exhibits symmetry compared to other competing models. Additionally, the dot points in Q-Q plots, empirical cumulative distribution function, and P-P plots of the proposed distribution lie almost surrounding the lines compared to existing models. This indicates that the proposed distribution fits all the datasets.

Information criteria and goodness-of-fit measures will be used to assess the flexibility of the proposed GOM-Kumaraswamy against its competing models. The information criteria used in this research include the Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), and Hannan-Quinn Information Criterion (HQIC). The model with the least value of information criteria will be selected as the best-fitting model for the datasets. The estimates and the AIC, CAIC, BIC, and HQIC values of the proposed GOM-Kumaraswamy against its competing models, including Kw-Kumaraswamy and Kw-BurrIII distribution for the first and second datasets, are presented in Tables 9 and 10.

Table 9. Estimated values and Information criteria for the proposed and competing models using first data

Model	Estimates	AIC	CAIC	BIC	HQIC
GOM-Kw	$\alpha = 0.2567$	-15.0609	-12.3492	-11.0780	-14.2834
	$\beta = 0.5212$	-	-	-	-
	$a = 1.3439$	-	-	-	-
	$b = 0.2367$	-	-	-	-
Kw-Kw	$\mu = 0.5199$	-12.8612	-10.1945	-8.8783	-12.0837
	$\gamma = 1.9019$	-	-	-	-
	$a = 1.5344$	-	-	-	-
	$b = 1.9291$	-	-	-	-
KwBIII	$\theta = 1.9075$	6.5469	9.2134	10.5299	7.3244
	$\lambda = 0.9524$	-	-	-	-
	$a = 0.8452$	-	-	-	-
	$b = 1.9813$	-	-	-	-

As shown in Table 9, the estimates of each parameter for the proposed distribution and other competing models are presented in column 2, while the information criteria are also provided in columns 3-6. The GOM-Kumaraswamy distribution has the least value of AIC, CAIC, BIC, and HQIC compared to Kw-Kumaraswamy and Kw-BurrIII models.

Table 10. Estimated values and Information criteria for the proposed and competing models using first data

Model	Estimates	AIC	CAIC	BIC	HQIC
GOM-Kw	$\alpha = 1.2878$	-110.0699	-109.1810	-102.4218	-107.1574
	$\beta = 0.6968$	-	-	-	-
	$a = 0.4334$	-	-	-	-
	$b = 1.4160$	-	-	-	-
Kw-Kw	$\mu = 1.3848$	-74.5393	-73.6504	-66.8912	-71.6268
	$\gamma = 1.7021$	-	-	-	-
	$a = 0.6773$	-	-	-	-
	$b = 1.7937$	-	-	-	-
KwBIII	$\theta = 1.5861$	-47.8141	-46.9252	-40.1660	-44.9017
	$\lambda = 0.5149$	-	-	-	-
	$a = 0.9113$	-	-	-	-
	$b = 1.9070$	-	-	-	-

Similarly, it is observed from Table 10 that the GOM-Kumaraswamy distribution has the least value of these criteria against its competing models. This indicates that the GOM-Kumaraswamy could be considered as the best model that fits the flood and length of a hole diameter data sets against its Kw-Kumaraswamy and Kw-BurrIII models.

However, Tables 11 and 12 display the goodness-of-fits of the GOM-Kumaraswamy distribution and its competing models using the first and second data sets, respectively. In these tables, the intervals of the data and corresponding observed values are presented in columns 1 and 2. Columns 3 and 4 present the expected values of the competing models, and finally, column 5 is for the proposed model. The log-likelihood and goodness-of-fit values for all models are provided beneath the intervals and expected values. The model with the highest log-likelihood and least chi-square values could be considered as the best fit for the data sets.

Table 11. Goodness of fits for the proposed and competing models using first data

Intervals	Observed	Kw-BIII	Kw-Kw	GOM-Kw
0.0000-0.3493	6	6.6052	7.1468	6.2919
0.3494-0.4387	8	1.9606	3.4475	3.9625
0.4388-0.5281	3	1.7852	3.2918	4.1465
0.5282-0.6175	1	1.5689	2.7193	3.3381
0.6176-0.7069	1	1.3448	1.8953	1.7755
0.7070-1.0000	1	3.0103	1.4840	0.4683
Log-likelihood	-	0.7265	10.4306	11.5305
Goodness-of-fit	-	21.1230	7.8895	7.0245

Table 12. Goodness of fits for the proposed and competing models using first data

Intervals	Observed	Kw-BIII	Kw-Kw	GOM-Kw
0.00-0.04	7	8.3031	8.4385	3.5203
0.05-0.09	6	4.8245	6.2866	7.0307
0.10-0.14	11	3.7882	5.1051	9.1519
0.15-0.19	11	3.1225	4.2112	9.0810
0.20-0.24	11	2.6279	3.4883	6.7371
0.25-0.29	2	2.2366	2.8861	3.4590
0.30-1.00	2	15.5953	12.8479	1.3678
Log-likelihood	-	27.9071	41.2696	59.0349
Goodness-of-fit	-	72.6427	43.6164	7.9743

It is observed in Tables 11 and 12 that the proposed GOM-Kumaraswamy distribution has the highest values of log-likelihood and the least values of chi-square, respectively, compared to its competing models. This indicates that the proposed distribution could be considered as the best model.

8. Conclusion

We studied the Generalized Odd Maxwell-Kumaraswamy distribution within the class of Generalized Odd Maxwell generated distributions defined in Section 2. Its density shape exhibits right-skewed, left-skewed, and symmetric natures. The proposed distribution has increasing, bathtub, and upside-down bathtub failure rates. The mixture representations and structural properties of the GOM-Kumaraswamy distribution were derived. The skewness and kurtosis values of the proposed distribution were obtained using moments and the quantile function, respectively.

The parameters were estimated using simulation and the maximum likelihood method. Two data sets were used to assess the potential of the new model. As presented in Section 7, the new model fitted both data sets using Information criteria and goodness-of-fit measures compared to the Kumaraswamy-Kumaraswamy and Kumaraswamy-BurrIII models.

9. Further ctudy

The further study focuses mainly on the following:

i. Estimating the parameters of the GOM-Kumaraswamy distribution using various alternative approaches, such as maximum product of spacings, least squares, weighted least squares, Bayesian, etc.

ii. Introducing bivariate probability distributions by considering the GOM-Kumaraswamy distribution as the univariate probability distribution.

iii. Developing a new Maxwell generated class of distributions by employing some other method of generating probability distribution and evaluating how well they can work.

Contribution to knowledge

The novel GOM-G family of probability distribution developed in this study would be served as an additional family to extend any traditional continuous probability distributions. The Kumaraswamy distribution is the conventional distribution that has been modified in accordance with the GOM-G family. Some properties of the suggested distribution that did not previously exist are studied.

Data availability statement

The datasets considered in this study are freely available and they are discussed in [54-55] respectively, for the first and second data.

Conflicts of interest

Authors declare there is no conflict of interest.

References

[1] Umar AA, Usman A, Abubakar SS, Ieren TG, Falgore JY, Ishaq AI, et al. Transmuted New Weighted Weibull distribution: Some statistical properties and application to survival time data. *AIP Conference Proceedings*. 2021; 2423(1): 070008.

[2] Ishaq AI, Suleiman AA, Usman A, Daud H, Sokkalingam R. Transformed log burr III distribution: structural features and ap-plication to milk production. *Engineerimg Proceedings*. 2023; 52: 1-7.

[3] Naadimuthu G, Lee ES. Stochastic modelling and optimization of water resources systems. *Mathematical Modelling*. 1982; 3(2): 117-136.

[4] Kumaraswamy P. Extended sinepower probability density function. *Journal of Hydrology*. 1978; 37(1-2): 81-89.

[5] Kumaraswamy P. New super-kurtic probability density function for use in computer simulation. *Proceedings of the Indian Academy of Sciences Section C: Engineering Sciences*. 1979; 2: 133-140.

[6] Kumaraswamy P. A generalized probability density function for double-bounded random processes. *Journal of Hydrology*. 1980; 46(1-2): 79-88.

[7] Fletcher S, Ponnambalam K. Estimation of reservoir yield and storage distribution using moments analysis. *Journal of Hydrology*. 1996; 182(1-4): 259-275.

[8] Jones M. Kumaraswamy's distribution: A beta-type distribution with some tractability advantages. *Statistical*

Methodology. 2009; 6(1): 70-81.

[9] Lemonte AJ. Improved point estimation for the Kumaraswamy distribution. *Journal of Statistical Computation and Simulation*. 2011; 81(12): 1971-1982.

[10] Ishaq AI, Usman A, Musa T, Agboola S. On some properties of Generalized Transmuted Kumaraswamy distribution. *Pakistan Journal of Statistics and Operation Research*. 2019; 15(3): 577-586.

[11] EI-Sherpieny EI-SA, Ahmed MA. On the kumaraswamy Kumaraswamy distribution. *International Journal of Basic and Applied Sciences*. 2014; 3(4): 372.

[12] Javanshiri Z, Habibi Rad A, Arghami NR. Exp-Kumaraswamy distributions: Some properties and applications. *Journal of Sciences, Islamic Republic of Iran*. 2015; 26(1): 57-69.

[13] Usman RM, ul Haq MA. The Marshall-Olkin extended inverted Kumaraswamy distribution: Theory and applications. *Journal of King Saud University-Science*. 2020; 32(1): 356-365.

[14] Alshkaki R. A generalized modification of the Kumaraswamy distribution for modeling and analyzing real-life data. *Statistics, Optimization & Information Computing*. 2020; 8(2): 521-548.

[15] Ishaq AI, Suleiman AA, Daud H, Singh NSS, Othman M, Sokkalingam R, et al. Log-Kumaraswamy distribution: Its features and applications. *Frontiers in Applied Mathematics and Statistics*. 2023; 9(1258961): 1-13.

[16] Maxwell JC. Illustrations of the dynamical theory of gases. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*. 1860; 20(130): 21-37.

[17] Boltzmann L. Weitere studien uber das warmegleichgewicht unter gasmolekulen-sitzungsberichte der kaiserlichen akademie der wissenschaften in wien. *Mathematisch-Naturwissenschaftliche Classe*. 1872; 66: 275-370.

[18] Tyagi R, Bhattacharya S. Bayes estimation of the Maxwell's velocity distribution function. *Statistica*. 1989; 29(4): 563-567.

[19] Chaturvedi A, Rani U. Classical and Bayesian reliability estimation of the generalized Maxwell failure distribution. *Journal of Statistical Research*. 1998; 32(1): 113-120.

[20] Bekker A, Roux J. Reliability characteristics of the Maxwell distribution: A Bayes estimation study. *Communications in Statistics-Theory and Methods*. 2005; 34(11): 2169-2178.

[21] Dey S, Maiti SS. Bayesian estimation of the parameter of Maxwell distribution under different loss functions. *Journal of Statistical Theory and Practice*. 2010; 4: 279-287.

[22] Hossain AM, Huerta G. Bayesian estimation and prediction for the Maxwell failure distribution based on Type II censored data. *Open Journal of Statistics*. 2016; 6(1): 49-60.

[23] Sharma VK, Bakouch HS, Suthar K. An extended Maxwell distribution: Properties and applications. *Communications in Statistics-Simulation and Computation*. 2017; 46(9): 6982-7007.

[24] Sharma VK, Dey S, Singh SK, Manzoor U. On length and area-biased Maxwell distributions. *Communications in Statistics-Simulation and Computation*. 2018; 47(5): 1506-1528.

[25] Singh SV, Sharma VK. The location-scale family of generalized Maxwell distributions: theory and applications to annual rainfall records. *International Journal of System Assurance Engineering and Management*. 2019; 10: 1505-1515.

[26] Fayomi A, Khan S, Tahir MH, Algarni A, Jamal F, Abu-Shanab R. A new extended gumbel distribution: Properties and application. *Plos One*. 2022; 17(5): e0267142.

[27] Arif M, Khan DM, Aamir M, El-Morshedy M, Ahmad Z, Khan Z. A new flexible exponentiated-X family of distributions: characterizations and applications to lifetime data. *IETE Journal of Research*. 2022; 1-13. Available from: https://doi.org/10.1080/03772063.2022.2034537.

[28] Suleiman AA, Othman M, Ishaq AI, Abdullah ML, Indawati R, Daud H, et al. A new statistical model based on the novel generalized odd beta prime family of continuous probability distributions with applications to cancer disease data sets. *Preprints*. 2022; 1-33. Available from: https://doi.org/10.20944/preprints202212.0072.v1.

[29] Muhammad M, Liu L, Abba B, Muhammad I, Bouchane M, Zhang H, et al. A new extension of the top-Leone-family of models with applications to real data. *Annals of Data Science*. 2023; 10(1): 225-250.

[30] Usman A, Ishaq AI, Suleiman AA, Othman M, Daud H, Aliyu Y. Univariate and bivariate log-topp-leone distribution using censored and uncensored datasets. *Computer Sciences & Mathematics Forum*. 2023; 7(1): 32.

[31] Willayat F, Saud N, Ijaz M, Silvianita A, El-Morshedy M. Marshall-Olkin Extended Gumbel type-II distribution: properties and applications. *Complexity*. 2022; 2022: 1-23. Available from: https://doi.org/10.1155/2022/2219570.

[32] Dhungana GP, Kumar V. Exponentiated Odd Lomax Exponential distribution with application to COVID-19 death cases of Nepal. *PloS One*. 2022; 17(6): e0269450.

[33] Suleiman AA, Daud H, Othman M, Ishaq AI, Indawati R, Abdullah ML, et al. The odd beta prime-g family of probability distributions: properties and applications to engineering and environmental data. *Computer Sciences &*

Mathematics Forum. 2023; 7(1): 20.
[34] Suleiman AA, Daud H, Singh NSS, Othman M, Ishaq AI, Sokkalingam R. A novel odd beta prime-logistic distribution: desirable mathematical properties and applications to engineering and environmental data. *Sustainability*. 2023; 15(13): 10239.
[35] Suleiman AA, Daud H, Othman M, Singh NSS, Ishaq AI, Sokkalingam R, et al. A novel extension of the fréchet distribution: statistical properties and application to groundwater pollutant concentrations. *Journal of Data Science Insights*. 2023; 1(1): 8-24.
[36] Suleiman AA, Daud H, Singh NSS, Ishaq AI, Othman M. A new odd beta prime-burr X distribution with applications to petroleum rock sample data and COVID-19 mortality rate. *Data*. 2023; 8(9): 143.
[37] Suleiman AA, Daud H, Ishaq AI, Othman M, Sokkalingam R, Usman A, et al. The odd beta prime inverted kumaraswamy distribution with application to COVID-19 mortality rate in Italy. *Engineering Proceedings*. 2023; 56(1): 218.
[38] Ishaq AI, Abiodun AA. The Maxwell-Weibull distribution in modeling lifetime datasets. *Annals of Data Science*. 2020; 7(4): 639-662.
[39] Ishaq A, Abiodun A. A new generalization of Dagum distribution with application to financial data sets. *2020 International Conference on Data Analytics for Business and Industry: Way Towards a Sustainable Economy (ICDABI)*. Sakheer, Bahrain: IEEE; 2020.
[40] Ishaq AI, Abiodun AA. On the developments of Maxwell-Dagum distribution. *Journal of Statistical Modelling: Theory and Applications*. 2021; 2(2): 1-23.
[41] Ishaq A, Abiodun A, Panitanarak U. Modelling nigerian naira to CFA franc exchange rates with maxwell-exponentiated exponential distribution. *2021 International Conference on Data Analytics for Business and Industry (ICDABI)*. Sakheer, Bahrain: IEEE; 2021.
[42] Abiodun AA, Ishaq AI. On Maxwell-Lomax distribution: properties and applications. *Arab Journal of Basic and Applied Sciences*. 2022; 29(1): 221-232.
[43] Koobubpha K, Panitanarak T, Domthong P, Panitanarak U. The maxwell-burr X distribution: its properties and applications to the COVID-19 mortality rate in Thailand. *Thailand Statistician*. 2023; 21(2): 421-434.
[44] Alzaatreh A, Lee C, Famoye F. A new method for generating families of continuous distributions. *Metron*. 2013; 71(1): 63-79.
[45] Eugene N, Lee C, Famoye F. Beta-normal distribution and its applications. *Communications in Statistics-Theory and Methods*. 2002; 31(4): 497-512.
[46] Cordeiro GM, de Castro M. A new family of generalized distributions. *Journal of Statistical Computation and Simulation*. 2011; 81(7): 883-898.
[47] Ishaq A, Usman A, Tasi'u M, Suleiman A, Ahmad A. A new odd f-weibull distribution: properties and application of the monthly nigerian naira to british pound exchange rate data. *2022 International Conference on Data Analytics for Business and Industry (ICDABI)*. Sakhir, Bahrain: IEEE; 2022.
[48] Gradshteyn IS, Ryzhik IM. *Table of Integrals, Series, and Products*. London: Academic Press; 2014.
[49] Gupta RC, Gupta PL, Gupta RD. Modeling failure time data by Lehman alternatives. *Communications in Statistics-Theory and Methods*. 1998; 27(4): 887-904.
[50] Greenwood JA, Landwehr JM, Matalas NC, Wallis JR. Probability weighted moments: definition and relation to parameters of several distributions expressable in inverse form. *Water Resources Research*. 1979; 15(5): 1049-1054.
[51] Oluyede B, Pu S, Makubate B, Qiu Y. The gamma-Weibull-G family of distributions with applications. *Austrian Journal of Statistics*. 2018; 47(1): 45-76.
[52] Behairy S, Al-Dayian G, El-Helbawy A. The Kumaraswamy-Burr type III distribution: properties and estimation. *British Journal of Mathematics & Computer Science*. 2016; 14(2): 1-21.
[53] Ishaq A, Usman A, Tasiu M, Aliyu Y, Idris FA. A new Weibull-Kumaraswamy distribution: Theory and applications. *Nigerian Journal of Scientific Research*. 2017; 16(2): 158-166.
[54] Ishaq AI, Usman A, Usman AU. On the generalized transmuted-Rayleigh distribution. *Benin Journal of Statistics*. 2018; 1: 75-86.
[55] Dey S, Mazucheli J, Anis M. Estimation of reliability of multicomponent stress-strength for a Kumaraswamy distribution. *Communications in Statistics-Theory and Methods*. 2017; 46(4): 1560-1572.