



## Research Article

# Idealization of a Semigroup by Using Bruck-Reilly Monoids

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**Abstract:** The aim of this paper is to create a new semigroup by defining a special idealization operation on semigroups. Additionally, by considering amalgamation, we will present novel distinguishing results on idealization over semigroups. Initially, we will combine the definitions of idealization and the Bruck-Reilly extension. Consequently, by integrating these two structures, we will provide a significant result on idealization in semigroups that will be crucial for future studies. Finally, we will conclude by discussing the ideal extension for the new semigroup structure, namely idealization.

**Keywords:** idealization, semigroup, Bruck-Reilly extension, idempotent

**MSC:** 16D25, 17C27, 18B40, 20E220

## 1. Introduction and preliminaries

One of the notable advantages of considering a new and more general construction is the unification of existing results under a novel structure. Moreover, it offers a concise specification, facilitates the derivation of new designs, and provides an economical approach to proving the correctness of certain properties. Additionally, when we view extensions as a means of combining known structures to create a new structure, this approach yields similar benefits in a distinct and effective manner. As an illustrative example, in [1-3], the authors introduced a new semigroup called  $\mathcal{N}$  and characterized it by extensively studying its properties.

One of the most important constructions, as indicated in the above paragraph, is idealization (*trivial extension or ringification*), which is built on a ring with two operations and has been used to produce some interesting results. For a commutative ring  $R$  with the identity and an  $R$ -module  $M$ , the idealization  $R(+M)$  of  $M$  was first introduced by Nagata in [4] via additive  $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$  and multiplicative operations,

$$(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2) \quad (1)$$

for all  $r_1, r_2 \in R$  and  $m_1, m_2 \in M$ . Idealization is the process of enclosing  $M$  in a commutative ring  $A$  so that its structure as a  $R$ -module is almost identical to its structure as an  $A$ -module, or as an ideal of  $A$ . Hence, it is helpful for generalizing results from rings to modules, reducing results about submodules to the ideal case, and creating

examples of commutative rings with zero divisors. Although there are so many important studies about idealization in the literature (see, for instance, in [5-8]), different versions of this subject are still of interest. For instance, for a ring  $R$ , an ideal  $I$ , and an  $R$ -module  $M$ , the authors (in [9]) defined the so-called *amalgamated duplication*  $R \triangleright \triangleleft I = \{(r, r+i) : r \in R, i \in I\} \subset R \times R$ , which produces a new ring as well as satisfying many properties coinciding with the idealization if  $M = I$ . Meanwhile, in [10], it has been proposed that another possible generalization of the duplication is as follows: Let  $R$  and  $U$  be commutative rings with unity; let  $J$  be an ideal of  $U$ ; and let  $f: R \rightarrow U$  be a ring homomorphism. In this setting, there exists a subring  $R \triangleright \triangleleft^f J = \{(r, f(r)+j) : r \in R, j \in J\}$  of  $R \times U$ , which is called the *amalgamation* of  $R$  with  $U$  along  $J$  with respect to  $f$ . This construction is a generalization of the amalgamated duplication, and so it contains the idealization as well.

In addition to the above studies, a survey was presented in [11] on some ring constructions and showed how one might produce some analogous semigroup constructions. In light of the idea, the main purpose of this paper is to introduce another approach to construct the idealization for semigroups by using the (single) semigroup operation. However, the biggest challenge when doing this is reducing binary operations on rings to a single operation on semigroups. To solve it, we will use the operation defined on Bruck-Reilly extensions in our construction (see Theorem 2.1 below).

Hence, let us remind ourselves of the Bruck-Reilly monoid that will definitely be needed in this paper. Assume that  $A$  is a monoid and  $\theta$  is an endomorphism such that  $A\theta$  is in the  $\mathcal{H}$ -class (this class is a binary relation defined on the elements of a semigroup, where two elements are related if and only if they generate the same principal right ideals. See [12] Chapter 2 of the identity  $1_A$  of  $A$ ). Thus, for the set of all non-negative integers  $\mathbb{N}^0$ , the set  $\mathbb{N}^0 \times A \times \mathbb{N}^0$  with the multiplication

$$(m, a, n)_B (m', a', n') = (m - n + t, (\theta^{t-n}(a))(\theta^{t-m'}(a')), n' - m' + t), \quad (2)$$

where  $t = \max(n, m')$  and  $\theta^0$  is the identity map on  $A$ , forms a monoid with the identity  $(0, 1_A, 0)$ . Then this monoid is called the *Bruck-Reilly extension* of  $A$ , determined by  $\theta$  and denoted by  $BR(A, \theta)$  (cf. [13-15]). The Bruck-Reilly extension is considered one of the fundamental constructions depending on the isomorphism and is presented as a characterization for the theory of semigroups. For example, between any bisimple regular w-semigroup and the Bruck-Reilly extension of a group or between any simple regular w-semigroup and the Bruck-Reilly extension of a finite chain of groups, there exist isomorphisms (see [16-17]). We may suggest [15, 18-19] for some other examples of characterizations via Bruck-Reilly extensions to the reader.

## 2. A new semigroup via Bruck-Reilly operation

As stated in the previous section, our goal is to use the Bruck-Reilly operation to build a new structure based on the idealization of a semigroup. To do that, we will combine the operations presented in (1) and (2), and hence we will capture the infrastructure relationship between semigroups and (sub-Bruck-Reilly) monoids similarly as in the idealization between modules and rings.

For any two submonoids  $SM_1$  and  $SM_2$  of the monoid  $M$ , let us assume that the Bruck-Reilly extension  $B = BR(M, \theta)$  is defined on the set  $\mathbb{N}^0 \times (M, \cdot) \times \mathbb{N}^0$  while each of the sub-Bruck-Reilly extensions  $SB_i = BR(SM_i, \theta)$  is defined on the set  $\mathbb{N}^0 \times (SM_i, \cdot) \times \mathbb{N}^0$ , where  $i = 1, 2$ . Throughout this paper, we will assume there exists an ideal relationship among our monoids,  $SM_2 \subset SM_1 \subset M$ , and especially we will take  $SM_2 = 1_M$ , which can be thought of as the minimal submonoid. Now, for arbitrary elements  $a, c \in SB_i (i = 1, 2)$  and  $b, d \in B$ , let us consider the mapping  $(SB_i \times B) \times (SB_i \times B) \rightarrow (SB_i \times B)$  with the binary operation  $\otimes$  as

$$((a, b), (c, d)) \mapsto (a, b) \otimes (c, d) = (a_B c, (a_B d)_B (c_B b)) \quad (3)$$

such that, for arbitrary elements,  $k \in SB_i$  and  $t \in B$ , the notation  $k_B t$  in (3) denotes the Bruck-Reilly operation given in (2). With this approach, a connection will be established between idealization and this new semigroup since the operation given in (2) and used in (3) has the same meaning as (1) in the definition of idealization. This approach certainly fits

with the way defined in [11] to show how it may produce some analogies on a semigroup construction.

On the other hand, when a monoid is trivial, its Bruck-Reilly extension is isomorphic to the bicyclic monoid. This is very valuable in that the semigroup formed by the trivial monoid belongs to an important type of semigroup.

Thus, we have the following first pre-result of this paper:

**Proposition 2.1** The set  $SB_i \times B$  with the operation given in (3) defines a semigroup which will be denoted by  $SB_i(\otimes)B$ .

**Proof.** It suffices to prove the well-defined and associative properties of (3).

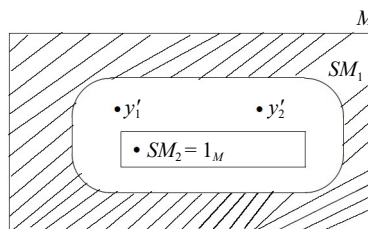
Suppose that  $a = (x_1, y_1, z_1)$ ,  $c = (x_2, y_2, z_2) \in SB_i$ , and  $b = (x'_1, y'_1, z'_1)$ ,  $d = (x'_2, y'_2, z'_2) \in B$ . Now, by taking into account (2), let us calculate  $a_B c$ ,  $a_B d$  and  $c_B b$  separately. Firstly, we have

$$a_B d = (x_1, y_1, z_1)_B(x_2, y_2, z_2) = (x_1 - z_1 + t, \theta^{t-z_1}(y_1)\theta^{t-x_2}(y_2), z_2 - x_2 + t), \quad (4)$$

where  $t = \max(z_1, x_2)$  and  $\theta$  is a monoid homomorphism (no necessary to know the rule) having one of the possibilities  $SM_1 \rightarrow SM_1$ ,  $SM_1 \rightarrow SM_2$ ,  $SM_2 \rightarrow SM_1$  or  $SM_2 \rightarrow SM_2$ . Actually, one of these homomorphisms must exist in (4) due to the definition. Secondly,

$$a_B d = (x_1, y_1, z_1)_B(x'_2, y'_2, z'_2) = (x_1 - z_1 + t', \beta^{t-z_1}(y_1)\beta^{t-x'_2}(y'_2), z'_2 - x'_2 + t'),$$

where  $t' = \max(z_1, x'_2)$  and  $\beta$  must be a homomorphism with the form either  $\beta : SM_1 \rightarrow M$  or  $\beta : SM_2 \rightarrow M$ . However, for an element  $y'_2 \in M$ , we need to observe that  $y'_2 \notin M \setminus SM_1$ . Thus, by the fact that  $y_2$  is an element of both  $M$  and  $SM_1$ , we obtain the required homomorphism as  $\beta : SM_1 \rightarrow M$ . Observe that the homomorphism  $\beta$  might have been taken as  $\beta : SM_2 \rightarrow M$  (which certainly exists), but since it provides the trivial case, the other option has been chosen (see Figure 1).



**Figure 1.** While the scanned area represents  $M \setminus SM_1$ , bold characters represent the monoid and its submonoids so as not to cause confusion

Finally, again by (2),

$$c_B b = (x_2, y_2, z_2)_B(x'_1, y'_1, z'_1) = (x_2 - z_2 + t'', \chi^{t''-z_2}(y_2)\chi^{t''-x'_1}(y'_1), z'_1 - x'_1 + t''),$$

where  $t'' = \max(z_2, x'_1)$  and  $\chi$  must be a homomorphism either in the form  $SM_1 \rightarrow M$  or  $SM_2 \rightarrow M$ . For an element  $y'_1 \in M$ , such that  $y'_1 \notin M \setminus SM_1$ , by the same idea as in the previous paragraph, we certainly have a homomorphism  $\beta : SM_1 \rightarrow M$ . Here, the existence of the homomorphism  $\chi$  from  $SM_2$  to  $M$  is obvious (see Figure 1).

Now, the next step is to combine the above elements under the Bruck-Reilly operation  $\mathcal{B}$  as presented in (3) (and so by (2)). Thus,

$$(a_B d)_B(c_B b) = (x_1 - z_1 + t', \beta^{t-z_1}(y_1)\beta^{t-x'_2}(y'_2), z'_2 - x'_2 + t')_B(x_2 - z_2 + t'', \chi^{t''-z_2}(y_2)\chi^{t''-x'_1}(y'_1), z'_1 - x'_1 + t'')$$

which is equal to the element

$$\begin{aligned} & (x_1 - z_1 - z'_2 + x'_2 + t^*, \alpha^{t^* - z'_2 + x'_2 - t'}(\beta^{t' - z_1}(y_1)\beta^{t' - x'_2}(y'_2))) \\ & \alpha^{t^* - x_2 + z_2 - t''}(\chi^{t'' - z_2}(y_2)\chi^{t'' - x'_1}(y'_1)), z'_1 - x'_1 - x_2 + z_2 + t^*), \end{aligned} \tag{5}$$

where  $t^* = \max(z'_2 - x'_2 + t', x_2 - z_2 + t'')$  and  $\alpha$  must be a homomorphism from  $M$  to  $M$ . Since  $\beta^{t' - z_1}(y_1)\beta^{t' - x'_2}(y'_2)$  and  $\chi^{t'' - z_2}(y_2)\chi^{t'' - x'_1}(y'_1)$  are the elements of  $M$ , there exists at least one homomorphism with the form  $\alpha : M \rightarrow M$ .

At this stage, by considering the elements calculated in (4) and (5), respectively, we must check that the element  $(a_B c, (a_B d)_B(c_B b))$  belongs to the set  $SB_i \times B$ . In fact, once we consider  $(a_B c)$  given in (4), it is seen that  $\theta^{t' - z_1}(y_1)\theta^{t' - x'_2}(y'_2)$  is an element of  $SM_1$  or  $SM_2 = 1_M$ . On the other hand, for the element  $(a_B d)_B(c_B b)$  in (5), it can be observed that the element

$$\alpha^{t^* - z'_2 + x'_2 - t'}(\beta^{t' - z_1}(y_1)\beta^{t' - x'_2}(y'_2))\alpha^{t^* - x_2 + z_2 - t''}\chi^{t'' - z_2}(y_2)\chi^{t'' - x'_1}(y'_1)$$

belongs to the  $M$ . These two imply that the ordered pair  $(a_B c, (a_B d)_B(c_B b))$  is an element of the set  $SB_i \times B$ , which gives the closure property.

Additionally, by taking elements  $a, b, c \in SB_i(\otimes)B$  and applying the operation in (2) as in the above calculations, it will be seen that the associative property  $(a \otimes b) \otimes c = a \otimes (b \otimes c)$  is satisfied.

Hence, the result.

The following result shows in which cases this new semigroup  $SB_i(\otimes)B$  defined in Proposition 2.1 will have the form of a monoid.

**Theorem 2.2** Suppose all elements  $n_0 \in SB_i(\otimes)B$  are formed as

$$((0, SM_2, 0), (d, e, f))$$

such that  $d, f \in \mathbb{N}^0$  and  $e \in M$ . Then,  $SB_i(\otimes)B$  is a monoid with the identity  $((0, 1_M, 0), (0, 1_M, 0))$ .

**Proof.** By the assumption, let  $n_0 = ((0, 1_M, 0), (d, e, f)) \in SB_i(\otimes)B$ . Also, let us consider the element  $\mathcal{I}_d = ((0, 1_M, 0), (0, 1_M, 0))$ . Taking into account the identity of the Bruck-Reilly monoid as  $(0, 1_M, 0)$  if we apply the operation in (3), then we obtain that  $n_0 \otimes \mathcal{I}_d$  is actually equal to

$$\begin{aligned} & = (((0, 1_M, 0)_B(0, 1_M, 0)), ((0, 1_M, 0)_B(0, 1_M, 0))_B((0, 1_M, 0)_B(d, e, f))) \\ & = ((0, 1_M, 0), ((0, 1_M, 0)_B(0 - 0 + t, \alpha^{t-0}(1M)\alpha^{t-d}(e), f - d + e))) \end{aligned}$$

where  $t = \max(0, d) = d$ . Replacing  $t$  by  $d$ , the last equality is equal to

$$\begin{aligned} & = ((0, 1_M, 0), ((0, 1_M, 0)_B(0 - 0 + d, \alpha^d(1M)\alpha^0(e), f - d + d))) \\ & = ((0, 1_M, 0), ((0, 1_M, 0)_B(d, e, f))) \\ & = ((0, 1_M, 0), (d, e, f)) = n_0 \end{aligned}$$

Applying a similar calculation, we get  $\mathcal{I}_d \otimes n_0 = n_0$ , and so, as a result of Proposition 2.1, we obtain  $SB_i(\otimes)B$  is a monoid with the identity element  $\mathcal{I}_d$ , as required.

Recall that an *idempotent element* of a semigroup  $S$  is defined as  $a^2 = a$  for any  $a \in S$ , and if every  $a \in S$  is an idempotent element,  $S$  is called the *idempotent semigroup* or *band*. Recall that the notation  $E(S)$  denotes the set of idempotents. The idempotent elements and bands play an important role in semigroup theory. Since our new semigroup  $SB_i(\otimes)B$  is not a band (see Lemma 2.3 below), we give our attention to showing that the set  $E(SB_i(\otimes)B)$  is a band (and a semilattice) under the condition that  $M$  is a commutative monoid (see Theorem 2.5 below).

The following lemma gives an explicit description of the idempotent elements of  $SB_i(\otimes)B$ .

**Lemma 2.3** The element  $((x, y, z), (d, e, f)) \in SB_i(\otimes)B$  is an idempotent if and only if  $y \in SM_2$ ,  $x = z = d = f$  and

$e \in E(M)$ .

**Proof.** The necessity part: Let us assume an idempotent element  $n = (x, y, z), (d, e, f)$  of  $SB_i(\otimes)B$  exists. Then, it is easy to see that  $n = n \otimes n$  holds. Now, once we look at the settlement of this algebraic property, it is seen that

$$\begin{aligned} n &= ((x, y, z)_B(x, y, z), ((x, y, z)_B(d, e, f))_B((x, y, z)_B(d, e, f))) \\ &= ((x - z + t, \theta^{t-z}(y)\theta^{t-x}(y), z - x + t), (x - z + t', \beta^{t'-z}(y)\beta^{t'-d}(e), \\ &\quad f - d + t')_B(x - z + t', \beta^{t'-z}(y)\beta^{t'-d}(e), f - d + t')) \end{aligned}$$

where  $t = \max(z, x)$  and  $t' = \max(z, d)$ . If we keep going one more step, then we have

$$\begin{aligned} n &= ((x - z + t, \theta^{t-z}(y)\theta^{t-x}(y), z - x + t), (x - z - f + d + t'', \\ &\quad \lambda^{t''-f+d-t'}(\beta^{t'-z}(y)\beta^{t'-d}(e))\lambda^{t''-x+z-t'}(\beta^{t'-z}(y)\beta^{t'-d}(e)), \\ &\quad f - d - x + z + t'')) \end{aligned}$$

such that  $t'' = \max(z - x + t, x - z + t')$ . By the truthfulness of  $n = n \otimes n$ , the equalities in the following two cases are definitely satisfied:

Case I:  $(x - z + t, \theta^{t-z}(y)\theta^{t-x}(y), z - x + t) = (x, y, z)$ ,

Case II:  $(x - z - f + d + t'', \lambda^{t''-f+d-t'}(\beta^{t'-z}(y)\beta^{t'-d}(e))\lambda^{t''-x+z-t'}(\beta^{t'-z}(y)\beta^{t'-d}(e)), f - d - x + z + t'') = (d, e, f)$ .

For Case I, there further exist the following three subcases:

- (i) For the first component, we already have  $x - z + t = x$ , which implies the equality  $t = x = z$ .
- (ii) For the second component, we have  $\theta^{t-z}(y)\theta^{t-x}(y) = y$ . But, by (i), since  $t = x = z$ , and since  $\theta^0$  is the identity homomorphism, we get

$$\theta^0(y)\theta^0(y) = y \Rightarrow yy = y.$$

We know that  $y \in SM_i$  for  $i = 1, 2$ . However, if  $y \in SM_1$ , then  $yy = y$  implies  $y \in E(SM_1)$ .

The trivial case  $y \in SM_2 = 1_M$  is clear.

- (iii) For the third component, we have  $z - x + t = z$ , which yields  $x = z$ , as in (i).

Similarly, for Case II, we have the following three subcases for Case II:

- (i)' For the first components, we have  $x - z - f + d + t''$  and this implies the equality  $x = z = d = f$ .
- (ii)' For the second components, we have

$$\lambda^{t''-f+d-t'}(\beta^{t'-z}(y)\beta^{t'-d}(e))\lambda^{t''-x+z-t'}(\beta^{t'-z}(y)\beta^{t'-d}(e)) = e.$$

By (i)', since  $x = z = d = f$ , we have  $t' = z = d$ , and so

$$\lambda^0(\beta^0(y)\beta^0(e))\lambda^0(\beta^0(y)\beta^0(e)) = e \Rightarrow (ye)(ye) = e.$$

The only option to achieve the final equality is  $y \in SM_2 = 1_M$ , which clearly implies  $e \in E(M)$ .

- (iii)' For the third components, we have  $f - d - x + z + t'' = f$ , which gives the equality  $x = z = d = f$  as in (i)'.

*The sufficiency part:* Conversely, let us assume that conditions  $y \in SM_2, x = z = d = f$  and  $e \in E(M)$  are all held. Thus, we observe all elements of  $SB_i(\otimes)B$  are formed as  $s = ((x, y = 1_M, x), (x, e, x))$ . By a simple calculation, we clearly obtain  $s \otimes s = s$ , as required.

By Lemma 2.3, we basically say that the set of idempotent elements of the semigroup  $SB_i(\otimes)B$  is defined by

$$E(SB_i(\otimes)B) = \left\{ ((x, 1_M, x), (x, e, x)) : e \in E(M) \right\}. \quad (6)$$

In the same way as in the proof of Proposition 2.1, we can show that  $E(SB_i(\otimes)B)$  is a semigroup under the operation defined in (3). Furthermore, as a consequence of Theorem 2.2, the elements in the set (6) cannot define a monoid unless they are formed as  $((0, SM_2, 0), b)$  such that  $b$  is an element of  $B$ . Hence, we obtain the following proposition.

**Proposition 2.4** The set  $E(SB_i(\otimes)B)$  in (6) defines a semigroup. Moreover, it defines a monoid if we choose  $x = 0$ .

Nevertheless, since a commutative idempotent semigroup is called a *semilattice* (cf. [12]), we then have the following theorem as a next step to Proposition 2.4.

**Theorem 2.5** Let  $M$  be a commutative monoid. Then,  $E(SB_i(\otimes)B)$  is not only a band but also a semilattice.

**Proof.** We know that  $E(SB_i(\otimes)B)$  is a semigroup (by Proposition 2.4) and a band (by Lemma 2.3). Now, assume that  $M$  is a commutative monoid. Therefore, for arbitrary elements

$$a_1 = ((x, 1_M, x), (x, e, x)), a_2 = ((x', 1_M, x'), (x', d, x')) \in E(SB_i(\otimes)B),$$

we will show that  $a_1 \otimes a_2 = a_2 \otimes a_1$ , and so, it will imply  $E(SB_i(\otimes)B)$  is commutative for all, such as these elements. We have

$$\begin{aligned} a_1 \otimes a_2 &= ((x, 1_M, x), (x, e, x)) \otimes ((x', 1_M, x'), (x', d, x')) \\ &= \left( \left( (x, 1_M, x)_{\beta} (x', 1_M, x') \right), \left( (x, 1_M, x)_{\beta} (x', d, x') \right)_{\beta} \left( (x', 1_M, x'), (x, e, x) \right) \right) \\ &= \left( \left( t, \theta^{t-x}(1_M) \theta^{t-x'}(1_M), t \right), \left( t, \theta^{t-x}(1_M) \theta^{t-x'}(d), t \right)_{\beta} \left( t, \theta^{t-x}(1_M) \theta^{t-x}(e), t \right) \right) \\ &= \left( \left( t, \theta^{t-x}(1_M) \theta^{t-x'}(1_M), t \right), \left( t, \beta^0 \left( \theta^{t-x}(1_M) \theta^{t-x'}(d) \right) \beta^0 \left( \theta^{t-x'}(1_M) \theta^{t-x}(e) \right), t \right) \right) \\ &= \left( \left( t, \theta^{t-x}(1_M) \theta^{t-x'}(1_M), t \right), \left( t, \theta^{t-x'}(d) \theta^{t-x}(e), t \right) \right). \end{aligned} \quad (7)$$

On the other hand, we have

$$\begin{aligned} a_1 \otimes a_2 &= ((x', 1_M, x'), (x', d, x')) \otimes ((x, 1_M, x), (x, e, x)) \\ &= \left( \left( (x', 1_M, x')_{\beta} (x, 1_M, x) \right), \left( (x', 1_M, x')_{\beta} (x, e, x) \right)_{\beta} \left( (x, 1_M, x), (x', d, x') \right) \right) \\ &= \left( \left( t, \theta^{t-x'}(1_M) \theta^{t-x}(1_M), t \right), \left( t, \theta^{t-x'}(1_M) \theta^{t-x}(e), t \right)_{\beta} \left( t, \theta^{t-x}(1_M) \theta^{t-x}(d), t \right) \right) \\ &= \left( \left( t, \theta^{t-x'}(1_M) \theta^{t-x}(1_M), t \right), \left( t, \beta^0 \left( \theta^{t-x'}(1_M) \theta^{t-x}(e) \right) \beta^0 \left( \theta^{t-x}(1_M) \theta^{t-x'}(d) \right), t \right) \right) \\ &= \left( \left( t, \theta^{t-x'}(1_M) \theta^{t-x}(1_M), t \right), \left( t, \theta^{t-x}(e) \theta^{t-x'}(d), t \right) \right), \end{aligned} \quad (8)$$

where  $t = \max(x, x')$  and both  $\theta$  and  $\beta$  are any two monoid homomorphisms (such that no need to know the rules of these homomorphisms for our calculations).

Since  $M$  is commutative, the expressions in (7) and (8) are equal to each other, which yields  $E(SB_i(\otimes)B)$  is a commutative semigroup and so a semilattice, as required.

### 3. An ideal extension for $SB_i(\otimes)B$

In this section, we will focus on a special extension of the semigroup  $SB_i(\otimes)B$  defined in Proposition 2.1. Extensions were first systematically studied by Clifford [20], who gave the first general structure theorem in the case when  $S$  is weakly reductive (see [21] Theorem 4.21). Later on, this result was extended to arbitrary semigroups by Yoshida [22].

Let  $K$  and  $T$  be disjoint semigroups such that  $K$  has an identity and  $T$  has a zero element. A semigroup  $\mathcal{S}$  is called an (ideal) extension of  $K$  by  $T$  if it contains  $K$  as an ideal and if  $\mathcal{S}/K \cong T$ . Special types of ideal extensions, namely *strict* and *pure*, have also been introduced in the literature. We may refer, for instance, to [23-24] for a detailed introduction to ideal extensions and examples illustrating the strict or pure types.

Let us suppose that  $1_M \in SM_1$ . Then, we certainly have an ideal  $\mathcal{K}$  of  $SB_i(\otimes)B$  since the closure of the every element  $((0, 1_M, 0), (d, e, f))$  with  $e \in SM_1$  holds. Thus, we can consider the quotient monoid  $[SB_i(\otimes)B]/\mathcal{K}$ . In fact, the coming theorem will use this monoid and will be a fundamental structure to obtain an ideal extension for our new semigroup.

Before presenting the following result, it should be noted that the index sets  $I$  and  $J$  in these Rees matrix semigroups are considered the non-negative integer set  $\mathbb{N}^0$  in this theorem.

**Theorem 3.1** Suppose that  $RM_1$  and  $R$  are two Rees matrix semigroups, which are defined on the sets  $\mathbb{N}^0 \times SM_1 \times \mathbb{N}^0$  and  $\mathbb{N}^0 \times M \times \mathbb{N}^0$ , respectively. Then, we have

$$[SB_i(\otimes)B]/\mathcal{K} \cong (RM_1, R) \otimes (SB_1 / SB_2, B).$$

**Proof.** For simplicity, let us denote  $(RM_1, R) \otimes (SB_1 / SB_2, B)$  by  $\mathcal{F}$ . Also, for a fixed  $a \in SB_1 \subset B$ , suppose that the element  $a$  satisfies the property

$$a_B((n_i)_B(n_j)) = (a_B(n_i))_B(a_B(n_j)).$$

By Theorem 2.2, since  $SB_i(\otimes)B$  is a monoid, all elements of  $SB_i(\otimes)B$  will be the form of  $((0, 1_M, 0), b)$  where  $b \in B$ . Now, let us consider the map  $f : SB_i(\otimes)B \rightarrow \mathcal{F}$  defined by

$$((0, 1_M, 0), b) \mapsto ((0, 1_M, 0), b) \otimes ((0, 1_M, 0), a) = ((0, 1_M, 0), a_B b).$$

For arbitrary elements, the equality  $((0, 1_M, 0), b_i) = ((0, 1_M, 0), b_j)$  certainly implies

$$((0, 1_M, 0), b_i) \otimes ((0, 1_M, 0), a) = ((0, 1_M, 0), b_j) \otimes ((0, 1_M, 0), a),$$

and so, the map  $f$  is well defined. By the assumption on the element  $a$ , we have

$$\begin{aligned} f(((0, 1_M, 0), b_i) \otimes ((0, 1_M, 0), b_j)) &= (((0, 1_M, 0), b_i) \otimes ((0, 1_M, 0), b_j)) \otimes ((0, 1_M, 0), a) \\ &= ((0, 1_M, 0), b_{j_B} b_i) \otimes ((0, 1_M, 0), a) \\ &= ((0, 1_M, 0), a_B (b_{j_B} b_i)). \end{aligned}$$

On the other hand,

$$f((0, 1_M, 0), b_i) \otimes f((0, 1_M, 0), b_j) = ((0, 1_M, 0), (a_B b_j)_B (a_B b_i)),$$

and hence, by the associativity of  $\mathcal{B}$ , we obtain  $f$  is a homomorphism. Furthermore, since for all  $((0, 1_M, 0), a_B b) \in \mathcal{F}$ , there always exists an element of the form  $((0, 1_M, 0), b) \in SB_i(\otimes)B$ , in other words  $\text{Im} f = \mathcal{F}$ , which implies that  $f$  is an epimorphism. Finally, let us consider the monoid  $SB_i(\otimes)B$  as a semigroup. Then, for the elements as the form

$$X = \{((0, 1_M, 0), b_i), ((0, 1_M, 0), b_j)\} \in SB_i(\otimes)B \times SB_i(\otimes)B,$$

since

$$\begin{aligned} \text{Kerf} &= \{X : f((0, 1_M, 0), b_i) = f((0, 1_M, 0), b_j)\} \\ &= \{X : ((0, 1_M, 0), b_i) \otimes ((0, 1_M, 0), a) = ((0, 1_M, 0), b_j) \otimes ((0, 1_M, 0), a)\} \\ &= \{X : ((0, 1_M, 0), a b_i) = ((0, 1_M, 0), a b_j)\} \\ &= \{X : b_i = b_j\}, \end{aligned}$$

we get the equality  $\text{Kerf} = \{((0, 1_M, 0), b) : b \in SB_i(\otimes)B\} = \mathcal{K}$ . Thus, the required isomorphism is obtained.

Since an extension  $V$  of  $S$  is *strict* if and only if, for every  $a \in V$ , there exists  $c \in S$  such that  $ax = cx$  and  $xa = xc$  for all  $x \in S$ . Strict extensions are closely related to extensions determined by partial homomorphisms. The following lemma (see [23], Proposition 2.4) offers a practical way to show the existence of strict extensions, which will be used to classify the type of extension over  $SB_i(\otimes)B$ .

**Lemma 3.2** ([23]). Every extension of a semigroup  $S$  is strict if and only if it has an identity.

The remaining goal of this section is to classify the ideal extension obtained in Theorem 3.1. Thus, we present the next main theorem of this paper.

**Theorem 3.3**  $SB_i(\otimes)B$  is a strict ideal extension of  $\mathcal{F}$  by  $\mathcal{K}$ .

**Proof.** The idea of the proof will be constructed over the definition of an ideal extension given at the beginning of this section. Hence, we need to prove the truthfulness of the following conditions:

(i)  $\mathcal{K}$  and  $\mathcal{F}$  must be disjoint semigroups:

We know that  $\mathcal{K}$  is an ideal of  $SB_i(\otimes)B$  and each  $SM_i$  is an ideal of  $M$ . So, we can consider the quotient  $M/SM_i$ . We need to examine the structure of  $\mathcal{F}$  or in other words, the structure of  $(RM_1, R) \otimes (SB_1 / SB_2, B)$ . Now, by Theorem 3.1, it is easy to observe that  $RM_1$  and  $R$  have the same forms as Rees matrix semigroups, and also  $SB_1$ ,  $SB_2$ , and  $B$  have the same production as Bruck-Reilly monoids. Therefore, these structures are totally disjoint under the operation in (3).

(ii)  $\mathcal{K}$  must have the identity:

For simplicity, let  $\mathcal{I}$  denotes the identity element

$$[((0, 1_M, 0), (0, 1_M, 0)), ((0, 1_M, 0), (0, 1_M, 0))]$$

of the monoid  $SB_i(\otimes)B \times SB_i(\otimes)B$ . By the proof of Theorem 3.1, we know that  $\mathcal{K} = \text{Kerf}$ . Then, the set of elements of  $\text{Kerf}$  is defined by

$$\begin{aligned} &\{\mathcal{I}, [((0, 1_M, 0), n_1), ((0, 1_M, 0), n_1)], [((0, 1_M, 0), n_2), ((0, 1_M, 0), n_2)], \\ &[((0, 1_M, 0), n_3), ((0, 1_M, 0), n_3)], \dots, [((0, 1_M, 0), n_k), ((0, 1_M, 0), n_k)]\} \end{aligned}$$

where  $n_1, n_2, \dots, n_1, n_2, \dots, n_k \in SB_i(\otimes)B$ . So,  $\mathcal{K}$  consists of the identity element  $\mathcal{I}$  of  $SB_i(\otimes)B$ .

(iii)  $\mathcal{F}$  must have the zero element, since if the semigroup  $\mathcal{F}$  consists of Bruck-Reilly monoids and the Rees matrix semigroups inside of it have zero elements, then the zero element is clearly in  $\mathcal{F}$ .

Since these above conditions are satisfied,  $SB_i(\otimes)B$  is an ideal extension. Additionally, by considering Lemma 3.2, it is a strict ideal extension.

Hence, the result.



## 4. Future problems

In this paper, by using an operation given in (3), we mainly introduce and then classify a new semigroup (or, in a special case, a monoid). However, it still needs to be investigated whether this new structure satisfies the homological conditions for semigroup extensions.

On the other hand, as indicated in [11], there exists a close relationship between rings and semigroups in some cases, and so people transfer special results from rings to semigroups (see, for instance, [25-26]). In ring theory, we also know that the idealization can be generalized to what is called a *semi-trivial extension*. We can briefly remind you of it as follows: For a commutative ring  $R$ , an  $R$ -module  $M$ , and an  $R$ -module homomorphism  $\mathcal{G} : M \otimes_R M \rightarrow R$  satisfying

$$\mathcal{G}(m \otimes m') = \mathcal{G}(m' \otimes m) \text{ and } \mathcal{G}(m \otimes m')m'' = m\mathcal{G}(m' \otimes m''),$$

if we obtain a commutative ring according to the operation  $(r, m)(r', m') = (rr', \mathcal{G}(m \otimes m'), rm' + r'm)$ , then this ring is called a semi-trivial extension of  $R$  by  $M$ . Here, the case  $\mathcal{G} = 0$  coincides with the idealization mentioned in the beginning. (We may refer to [27-28] for the details of semi-trivial extensions). As a result, one may study the semi-trivial extensions of semigroups via the operation defined in (3).

One can also transform the studies in Lemma 3.2 and Theorem 3.3 to arbitrary semigroups without assuming  $SB_i(\otimes)B$  as a monoid.

## Data availability

The article contains the data that supported the study's findings.

## Conflict of interest

The authors declare that they have no conflicts of interest.

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