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Research Article

Hyperbolic 3-Manifolds with Boundary Which are Side-Pairings of Two Tetrahedra as Exteriors of Knotted Graphs in the 3-Sphere

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Abstract: In this paper, we give a generalization of Ivanšić's method for hyperbolic 3-manifolds without boundary, which allows us to recognize if a hyperbolic 3-manifold with totally geodesic boundary, given by an isometric side-pairing of two hyperbolic truncated tetrahedra, is the exterior of a knotted graph; i.e., it is the complement of a 1-manifold with isolated singularities embedded in $\mathbb{S}^3$, in which case we get the corresponding diagram of the knotted isotopy class of its boundary. Otherwise, we obtain that the corresponding 3-manifold with boundary is the exterior of a knotted graph embedded in some lens space. Finally, we apply this method to a noncompact 3-manifold with a totally geodesic surface boundary of genus 2.

***Keywords*:** 3-manifolds with boundary, link complement, knotted graph

MSC: 57Q15, 57Q05

1. Introduction

A *hyperbolic link* (or *hyperbolic knot*) L is a topological 1-link (knot) whose complement in the 3-sphere, $\mathbb{S}^3 - L$, admits a geodesically complete hyperbolic structure; in other words, its complement is a non-compact complete Riemannian 3-manifold of constant sectional curvature equal to – 1.

In [1], Riley gave the first example of a hyperbolic knot, namely the figure-eight knot. He computes its hyperbolic structure using a discrete representation of the figure-eight knot group in the group of isometries of the hyperbolic space, $\mathbb{H}^3$. Thurston, in [2], and [3], described geometrically the hyperbolic structures of the complements of the figure-eight knot and other hyperbolic links in $\mathbb{S}^3$, by gluing faces of ideal polyhedra.

Moreover, Thurston gave a hyperbolic structure to a 3-manifold whose boundary is a compact hyperbolic surface of genus two, which can be obtained by gluing pairs of the corresponding hexagonal faces of two truncated hyperbolic tetrahedra. This 3-manifold with boundary is precisely the exterior of the most simple knotted multigraph embedded in $\mathbb{S}^3$, the theta graph, which looks like the Greek letter theta and consists of two vertices joined by three edges, or its equivalent by homotopic deformation for our purposes: the graph with loops obtained by the wedge product of two circles.

Michihiko Fujii found in [4] that there are eight mutually non-isometric compact oriented hyperbolic 3-manifolds with totally geodesic boundaries, which can decompose into two hyperbolic truncated tetrahedra.

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In this paper, we apply the Ivanšić's method to these eight compact oriented hyperbolic 3-manifolds to show that four of them are homeomorphic to the exteriors of knotted graphs in $\mathbb{S}^3$. In other words, we prove the following.

Theorem 1 Let M be a compact oriented hyperbolic 3-manifold whose boundary is a totally geodesic surface of genus 2, which can be obtained by side-pairing two truncated tetrahedra. Then, there are only four of them that are the exteriors of the knotted theta graph in the 3-sphere. Figure 1 shows their isotopy diagrams.

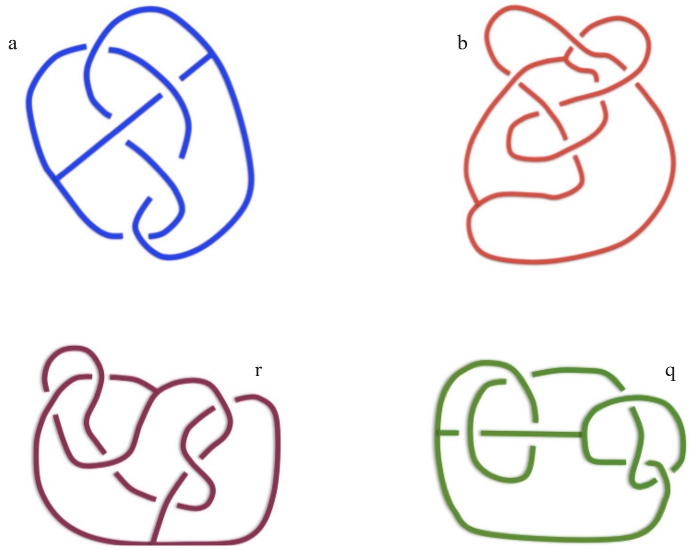

Figure 1. The four hyperbolic knotted graphs whose exteriors in the 3-sphere are hyperbolic manifolds with totally geodesic boundaries such that they can decompose into two hyperbolic truncated tetrahedra

Notice that if we consider the set $\mathcal{M}$ of all compact oriented hyperbolic 3-manifolds, each of which has a totally geodesic surface boundary of genus 2, and we compute their volume, then the eight hyperbolic 3-manifolds found by Fujii in [4], have the same volume, which is the minimal value that the volume function from $\mathcal{M}$ to $\mathbb{R}$ can take.

In Section 2, we describe Ivanšić's method [5] to recognize if a non-compact hyperbolic 3-manifold is a link complement in $\mathbb{S}^3$. If it does, we generate its link diagram.

In Section 3, we generalize Ivanšić's method to recognize if a hyperbolic 3-manifold with a totally geodesic boundary obtained by a side-pairing polyhedron is the exterior of a knotted graph embedded in the complement of a link (knot) L in $\mathbb{S}^3$, $\mathbb{S}^3 - L$.

In the last section, we apply this process to an example of a hyperbolic non-compact 3-manifold with a compact boundary described again by Fujii in [6], and we find the isotopy class of its complement.

Theorem 2 The exterior of the knotted theta graph in the complement $\mathbb{S}^3 - C$ of a circle C in the 3-sphere, whose isotopy class is shown in Figure 2, is a non-compact oriented hyperbolic 3-manifold with a totally geodesic closed surface boundary of genus 2.

In this context, in [7], Frigerio et al. have extended the census of hyperbolic 3-manifolds with boundary. In [8], Damian Heard's program, *Orb*, implements the reverse process: going from a diagram of a knotted graph to a triangulation of its complement, which is a generalization of the Weeks' work in [9] and in SnapPea. For more details, see [10] and [11].

We would like to remark that many papers have shown that some hyperbolic 3-manifolds are link complements in $\mathbb{S}^3$ (compare [9-13]), giving a hyperbolic structure to the link complement using an ideal triangulation and glue equations, instead of starting with a given polyhedral decomposition of a hyperbolic manifold and show that it is topologically equivalent to a link complement in $\mathbb{S}^3$. As far as we know, there is no more bibliography on finding the topological classes of hyperbolic 3-manifolds with totally geodesic boundaries except the Thurston's seminar works [3-4].

Figure 2. A hyperbolic graph whose exterior in the 3-sphere is a hyperbolic manifold whose boundary is a geodesically complete hyperbolic surface of genus 2 in the complement of a circle in the 3-sphere

2. Preliminaries

This section briefly recalls some definitions of knots and hyperbolic manifolds. For more details, see [13] and [14].

A topologically embedded 1-sphere $K \subset \mathbb{S}^3$ is a *topological knot* or *1-knot*. We say that L is a link if L is homeomorphic to a disjoint union of a finite number of 1-spheres. Given two knots or links L_1 and L_2, we say that L_1 is *equivalent* to L_2, if there is a continuous function $\varphi: \mathbb{S}^3 \to \mathbb{S}^3$ such that $\varphi(L_1) = L_2$.

There exists an essential connection between knots or links and hyperbolic 3-manifolds, hence we will start recalling some basic definitions about the hyperbolic 3-space.

The 3-dimensional hyperbolic space is a simply connected complete Riemannian 3-manifold, with a constant negative sectional curvature equal to –1, which is unique in the following sense. Any two Riemannian 3-manifolds which satisfy these properties are isometric. There are some models of this space, but the most used are:

Poincaré half-plane model: Consists of the upper-half space of $\mathbb{R}^3$, i.e., $\{(x_1, x_2, x_3)\} \in \mathbb{R}^3 | x_3 > 0\}$, endowed with the metric $\dfrac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}$.

Poincaré disk model: Consists of the unit Euclidean ball of $\mathbb{R}^3$, with the metric $4\dfrac{dx_1^2 + dx_2^2 + dx_3^2}{(1 - x_1^2 - x_2^2 - x_3^2)^2}$.

These two spaces are conformal hyperbolic models and are isometric via the stereographic projection. We will refer to either of these models with its metric as hyperbolic 3-space and denote it by $\mathbb{H}^3$.

The geodesics in $\mathbb{H}^3$, considering the half-plane model, are the Euclidean semicircles orthogonal to the *xy*-plane and vertical half-lines. We denote the group of preserving-orientation isometries of $\mathbb{H}^3$ by $\mathrm{Isom}^+(\mathbb{H}^3)$, which is isomorphic to the group PSL(2, $\mathbb{C}$) consisting of all the 2×2 matrices with complex entries and determinant 1.

A *hyperbolic 3-manifold M* is a topological manifold of dimension 3 equipped with a hyperbolic metric, that is a Riemannian metric which has all its sectional curvatures equal to – 1. Due to the uniqueness of the 3-hyperbolic space, we have that the universal covering of any hyperbolic 3-manifold is $\mathbb{H}^3$; as a consequence, M is isometric to the quotient space $\mathbb{H}^3/\Gamma$, where Γ is a torsion-free subgroup of the group of isometries of $\mathbb{H}^3$.

Hyperbolic geometry is very rich since many topological 3-manifolds are hyperbolic; for instance, Thurston proved [16] that the complement of any knot K, which is not either a satellite knot or a torus knot, is hyperbolic. A knot or link K is a *hyperbolic knot* or *hyperbolic link* if its complement, in $\mathbb{S}^3$, is a hyperbolic 3-manifold.

Hyperbolic 3-manifolds of finite volume have particular importance in 3-dimensional topology due to the Mostow rigidity theorem, which states that the homotopy type determines the unique hyperbolic structure of a hyperbolic 3-manifold of finite volume. A corollary implies that the volume is a topological invariant used to define new topological invariants.

In [5], Ivanšić gave a method to recognize if a non-compact hyperbolic 3-manifold is a link complement in $\mathbb{S}^3$. If it does, it generates its link diagram. Roughly speaking, we start with a hyperbolic 3-manifold M^3 given by a side-pairing of a polyhedron, then the Ivanšić's method gives us a handle decomposition of the Dehn filling of M^3 as follows.

We start projecting the diagram of the side-pairing of M^3 into the plane $\mathbb{R}^2$ to get a handle decomposition diagram. Next, we obtain a handle decomposition of the manifold $\tilde{M}^3$ resulting from M^3 after applying a hyperbolic Dehn filling on each of its tori boundary components; which consists of adding, for each torus on its boundary, one 2-handle to the handle decomposition diagram of M^3. If the handle diagram of $\tilde{M}^3$, after applying handle moves, consists of just one 0-handle and one 3-handle, we have shown by Alexander's lemma [17] that $\tilde{M}^3$ is homeomorphic to $\mathbb{S}^3$, and as a consequence, M^3 is a link complement in $\mathbb{S}^3$; where the longitudes in the boundary of the added solid tori are parallel to their meridians which are the components of the hyperbolic link, yielding the link diagram.

3. Hyperbolic knotted graphs

Let $K^1 \subset \mathbb{S}^3$ be a compact 1-manifold with a finite number of isolated singular points. A point $x \in K^1$ is a singular point if it has a neighborhood homeomorphic to a cone over a set of n points ($n \geq 3$), and its apex is x. A point $x \in K^1$ is regular if it has a neighborhood homeomorphic to the real line, which is homeomorphic to a cone over two points. For instance, in Figure 1, we can see four regular diagrams of knotted compact 1-manifolds in $\mathbb{S}^3$, each with two isolated singular points with a neighborhood homeomorphic to a cone over a set of 3 points.

We can associate to K^1 a combinatorial finite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ such that, its set of vertices $\mathcal{V}$ is equal to the singular set of K^1, which is a finite set of points, and its set of edges $\mathcal{E}$ corresponds to the set of connected components of $K^1 - \mathcal{V}$, each homeomorphic to $\mathbb{R}$. Observe that the end-points of any edge are the boundary points of the closure of the corresponding connected component of the regular set. In this way, we can think of the 1-manifold K^1 as a knotted graph $\mathcal{G}$ embedded in either a link or knot complement in $\mathbb{S}^3$.

Let N_K^3 be a sufficiently small open tubular neighborhood of K^1 considered as a knotted graph embedded in $\mathbb{S}^3$ (or the complement of either a link or knot L in $\mathbb{S}^3$). So, the boundary of N_K^3 is a closed surface with a genus greater than one. We say that a 3-hyperbolic manifold M^3 with boundary is the *exterior* of K^1 in $\mathbb{S}^3 - L$, if M^3 is the complement of N_K^3 in $\mathbb{S}^3 - L$. In this case, we say that K^1 is a "*hyperbolic knotted graph*".

4. From polyhedra to a handle decomposition

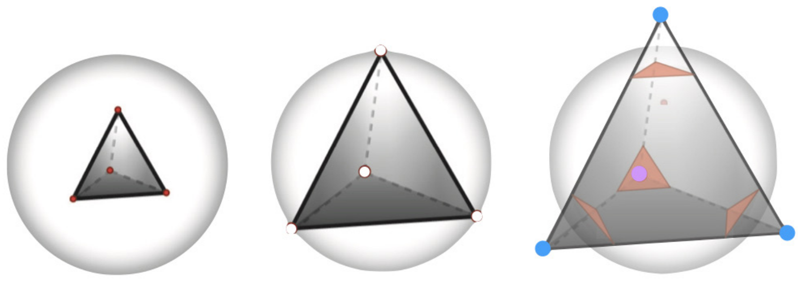

Figure 3. Three types of hyperbolic tetrahedra: a tetrahedron, an ideal tetrahedron and a hyper-ideal tetrahedron in the projective model of the hyperbolic 3-space

Let $\hat{P}^3$ be a hyperbolic polyhedron in the hyperbolic 3-space $\mathbb{H}^3$ with a side-pairing defined on it, then the quotient space $\hat{M}^3$ is an open hyperbolic 3-manifold. We are interested in determining if a hyperbolic polyhedron (a triangular bipyramid) with a given side-pairing is a hyperbolic 3-manifold $\hat{M}^3$, which is the complement of a singular 1-manifold (link or knot) in $\mathbb{S}^3$.

In Figure 3, using the Klein projective ball model of $\mathbb{H}^3$, we show the three types of vertices a hyperbolic polyhedron can have in tetrahedra: a compact tetrahedron, an ideal tetrahedron, and a hyper ideal tetrahedron. The action given by the side-pairing applies on ideal vertices yields the parabolic ends of the manifold $\hat{M}^3$ (links and knots in $\mathbb{S}^3$).

This action on hyper ideal vertices results in the hyperbolic ends (the boundary of $\hat{M}^3$).

Let P^3 be the compact hyperbolic polyhedron obtained from $\hat{P}^3$ truncating its ideal vertices by horospheres and its hyper ideal vertices by geodesic planes in $\mathbb{H}^3$. Then, P^3 has a side-pairing defined on it that gives a compact 3-manifold M^3 such that its boundary ∂M^3 is a geodesically complete submanifold obtained from $\hat{M}^3$ compactifying all its ends; for more details, see [18].

Notice that any compact 3-manifold M^3 can be described as a handle decomposition, i.e., by gluing a finite number of disjoint pieces such that each is homeomorphic to a 3-ball, and their boundaries glue these according to some rules. We factorize each 3-ball as a k-cell, $\partial B^k \times B^{3-k} = \mathbb{S}^{k-1} \times B^{3-k}$, k = 0, 1, 2, 3, and gluing it to a 3-manifold through its boundary $\partial B^k \times B^{3-k} = \mathbb{S}^{k-1} \times B^{3-k}$ (see Figure 4). Summarizing, we can decompose a compact 3-manifold in a finite set of disjoint 3-balls and reassemble it. The first stage to recover M^3 is to take a 3-ball called 0-handle, and from it, we will stick new 3-balls to get M^3 (see [17]).

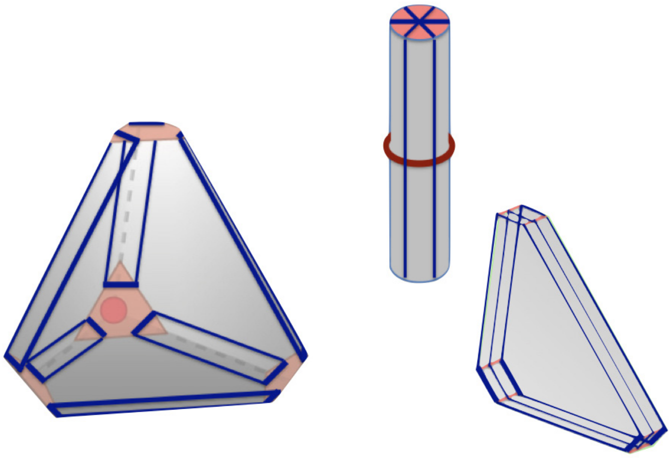

Figure 4. Three types of k-handles for k = 0, 2, 1

Next, we choose closed neighborhoods of the k-cells in M^3 in ascendent order such that these neighborhoods should match via the side-pairing to get 3-balls as follows.

For each vertex in the cell complex decomposition of M^3 via P^3, we consider a solid 3-ball of sufficiently small radius ϵ. Let $V_1^3, \ldots, V_m^3$ be neighborhoods of a cycle of vertices $v_1, \ldots, v_k$ of P^3, respectively (a cycle of k-faces consists of all the k-faces of P^3 that are identified by the side-pairing). Then, each V_i^3 is a solid cone in P^3 with apex v_i, and $V_1^3 \cup \ldots \cup V_m^3$ is a 3-ball V^3 in M^3 such that; if v_i is an interior vertex of M^3, then V^3 can be thought as a 3-handle glued by its boundary, but if v_i is a vertex in ∂M^3, then V^3 is a 3-ball in M^3 glued by a disk. Removing these 3-balls around all vertices in the CW decomposition of M^3, we have a 3-manifold with spherical holes.

Now, we consider disjoint closed neighborhoods of truncated edges as cylinders of radii sufficiently small, contained in M^3 minus the union of all the previous 3-balls around vertices. Let $E_1^3, \ldots, E_n^3$ be the corresponding neighborhoods of a cycle of truncated edges $e_1, \ldots, e_n$, which are identified in M^3. Then, $E_1^3 \cup \ldots \cup E_n^3$ assembles into a solid cylinder around a truncated interior edge, which becomes a 2-handle $E^3 = B^2 \times B^1$ glued by an annulus $\mathbb{S}^1 \times B^1$. If the truncated edge lies on ∂M^3, then $E_1^3 \cup \ldots \cup E_n^3$ assembles into a 3-ball glued by a disk.

Let H_1^3 be the solid simplex obtained by removing neighborhoods of vertices and truncated neighborhoods of edges from P^3. The boundary of H_1^3 is the truncated 2-faces of P^3 that identify, so H_1^3 projects to a handlebody H^3 in M^3. The feet of the 1-handles of H^3 correspond to the truncated 2-faces of P^3 on H_1^3 (for more details, see [5]).

Finally, we consider disjoint 3-balls as prisms of sufficiently small height, which are neighborhoods around truncated 2-faces of P^3 in H^3. These become into 1-handles as follows. Each one can decompose as $F^3 = B^1 \times B^2$. Thus, they are glued to a 3-ball via the two disks $\mathbb{S}^0 \times B^2$. This 3-ball remains of P^3 after removing all the 3-balls around vertices, truncated edges, and faces and becomes a 0-handle. Then, this 3-ball is our start key to glue handles and recover our given 3-manifold M^3.

5. Handle decomposition diagram

A handle decomposition diagram of a 3-manifold is a picture in $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ (the boundary of the unique 0-handle of the 3-manifold), where we place a pair of disks representing the corresponding feet of 1-handles and curves outside of the disks representing the corresponding attaching circles of 2-handles glued to the 3-handle.

In this section, we will describe a way to go from a 3-manifold M^3 given as a polyhedron P^3 with a side-pairing to its handle decomposition diagram (see Figure 5). In order to do that, we start with a convex polyhedron P^3 with a side-pairing, and we truncate all its k-faces to obtain the handles as in the previous section. We remove all the k-handles from P^3 (k = 1, 2, 3) to get the 0-handle P_0 homeomorphic to a 3-ball whose boundary is a 2-sphere $\mathbb{S}^2$. Now, we apply the following steps (for more details, see [5]).

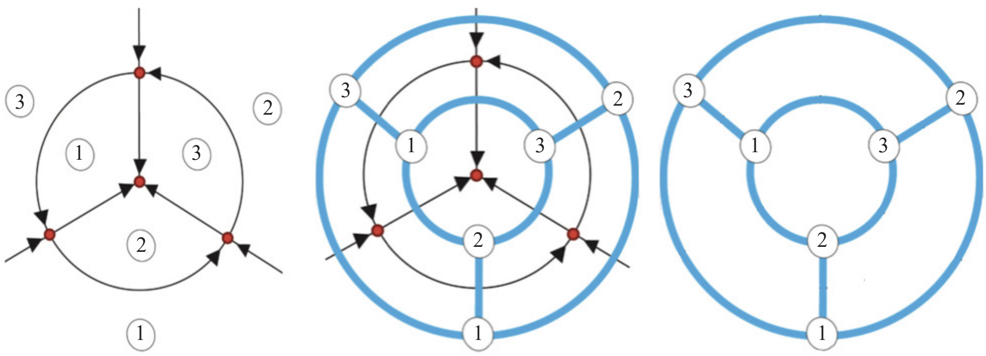

Figure 5. From the polyhedral diagram to the handle decomposition diagram

1. We project the boundary of P^3 to $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ via the stereographic projection, obtaining D_0 which is called *polyhedral diagram* of M^3. Notice that each k-face of D_0 is labeled according to the side-pairing of the polyhedron P^3.

2. Draw a disk inside every side of P^3 in D_0 that represents one of the feet of a 1-handle (paired sides correspond to feet of 1-handles). Notice that one of the disks may be outside the diagram since a sphere (the surface of P^3) is projected on $\mathbb{R}^2$ (see Figure 5).

3. For each pair of adjacent faces of P^3 along the edge e, draw an arc in D_0, crossing the corresponding projected edge e transversally and joining the two disks corresponding to the adjacent faces. The attaching circle for a 2-handle consists of the union of arcs crossing edges in the same cycle.

Observe that by Alexander's lemma [17], there is only one way to attach 3-handles. Furthermore, if P^3 has only ideal and hyper-ideal vertices, there are no 3-handles. The resulting diagram is called the *handle decomposition diagram* for P^3.

6. Exterior of knotted graphs in the 3-sphere

In this section, we will apply the previous method to the eight compact oriented hyperbolic 3-manifolds whose boundaries are totally geodesic closed surface of genus two, which can decompose as two hyperbolic truncated tetrahedra, to determine if they are the exterior of a knotted graph in the 3-sphere. In other words, if these 3-manifolds are homeomorphic to the complement of an open tubular neighborhood of a knotted graph in $\mathbb{S}^3$, and if it does, then we will also get the knotted graph isotopy diagram.

Fujii in [4] completely classifies all hyperbolic 3-manifolds with totally geodesic boundaries, decomposed into two hyperbolic truncated tetrahedra. He constructs such 3-manifolds by identifying, via isometries, the hexagonal faces of two hyperbolic truncated tetrahedra. Then, Fujii showed that the boundary surface of these hyperbolic 3-manifolds is a totally geodesic closed surface of genus two and that there are precisely eight mutually non-isometric compact oriented hyperbolic 3-manifolds with this property. Kojima et al. showed in [19] that these eight hyperbolic 3-manifolds have the same volume of 6.452, and this is the minimal one among all compact oriented hyperbolic 3-manifolds with totally geodesic closed boundaries.

Fujii uses a purely combinatorial method to describe all these 3-manifolds in [4]. First, he considers two truncated tetrahedra and glues two faces between these to obtain a triangular bipyramid. Then, he labels all their faces and edges as shown in its projection in Figure 6. Fujii shows that there are only two ways of identifying a pair of faces of two tetrahedra (or better, a triangular bipyramid). The two cases are: $A-E$, $B-G$, $C-F$ and $A-E$, $B-C$, $F-G$. In each case, he considers the number of possible gluing diagrams such that all edges identify. Then, he reduces this number by considering the symmetries of the diagrams and using the fact that for any complete hyperbolic 3-manifold satisfying his hypothesis, a unique minimizing geodesic exists that intersects perpendicularly at both ends of its boundary. This minimal geodesic is the embedded cycle of edges in the manifold, obtained by identifying all edges of the two hyperbolic truncated tetrahedra (for more details, see [4]).

In Table 1, we show the labels of the orientations of the edges in the tetrahedra using as reference Figure 6 of each of the eight Fujii's 3-manifolds M^3 which he calls a, b, j for the case of side-pairing $A-E$, $B-G$, $C-F$, and q, r, u, v, x for the other case $A-E$, $B-C$, $F-G$. If in an edge there is a positive sign the orientation is preserved; if there is a negative sign, the orientation is reversed.

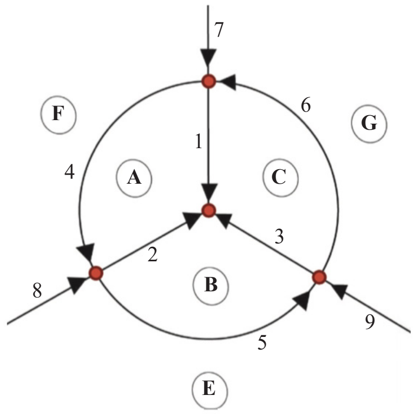

Figure 6. The stereographic projection of a triangular bipyramid with labeled sides and edges to describe the eight Fujii's 3-manifolds obtained by gluing the sides of two tetrahedra

Table 1. Orientations of the edges of the eight Fujii manifolds that are obtained by identifying the sides of two hyperbolic truncated tetrahedrons

Edges M^3	a	b	j	q	r	u	v	x
1	+	+	+	+	+	+	+	+
2	+	+	–	–	–	–	–	–
3	+	–	+	+	+	+	+	–
4	+	–	–	–	–	–	–	–
5	+	+	–	–	–	–	–	–
6	+	–	+	–	–	–	–	–
7	+	+	+	+	–	+	–	–
8	+	–	–	–	–	–	–	–
9	+	+	+	+	+	+	+	+

In Table 2, we show the side-pairing of two tetrahedra using as reference diagram in Figure 7. Moreover, we can read on Table 2 the side-pairing of the triangular links of the truncated hyper-ideal vertices of the tetrahedra. In Figure 8, there are the eight hyperbolic surfaces of genus two associated to the boundaries of the Fujii's manifolds.

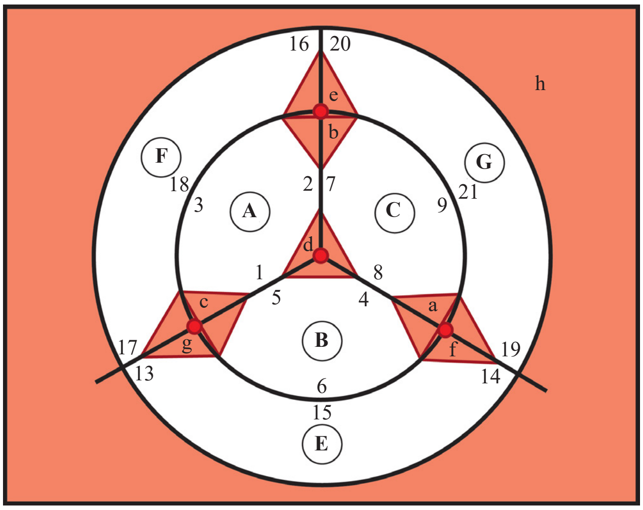

Figure 7. The stereographic projection of a truncated triangular bipyramid with labeled sides and edges to describe the eight Fujii's 3-manifolds obtained by gluing the sides of two tetrahedra

We will prove our main theorem using the method described above.

Theorem 1 Let M be a compact oriented hyperbolic 3-manifold whose boundary is a totally geodesic surface of genus 2, which can be obtained by side-pairing two truncated tetrahedra. Then, there are only four of them that are the exteriors of the knotted theta graph in the 3-sphere. Figure 1 shows their isotopy diagrams.

Proof. We will describe the method using the compact hyperbolic 3-manifolds M^3 with a totally geodesic boundary surface of genus two, decomposed into two hyperbolic truncated tetrahedra pairing by isometries [4]. Notice that if we glue two faces of both tetrahedra, it becomes a triangular bipyramid. In the left part of Figure 4, we draw the polyhedral diagram of a hyperbolic 3-manifold M^3, which comes from pairing the sides of a triangular bipyramid. Notice that it has one vertex at the point at infinity.

Table 2. Side-pairing of the eight Fujii's manifolds

	a	*b*	*j*		*q*	*r*	*u*	*v*	*x*
1	15	15	14	1	15	15	14	14	14
2	14	14	13	2	14	14	13	13	13
3	13	13	15	3	13	13	15	15	15
4	21	21	19	4	9	9	9	9	7
5	20	20	21	5	8	8	8	8	9
6	19	19	20	6	7	7	7	7	8
7	18	17	16	16	21	19	21	19	19
8	17	16	18	17	20	21	20	21	21
9	15	18	17	18	19	20	19	20	20

Since only one cycle of hyper-ideal vertices exists, then the eight Fujii's 3-manifolds have only one boundary component. There is also, for each 3-manifold M^3, only one cycle of edges and three cycles of faces, so we have in the handle decomposition of M^3 one 2-handle, three 1-handles, and one 0-handle (see the right side of Figure 5). Hence, the handle decomposition diagram has three pairs of disks: the feet of the 1-handles and nine arcs on the attaching circle of the 2-handle.

Next, we truncate the tetrahedra and set a geodesically complete hyperbolic triangle around each hyper-ideal vertex. These polygons are called *vertex links* and have an edge-pairing given by the corresponding restriction of the side-pairing of the polyhedra associated with their 3-manifolds. The left side of Figure 7 shows the polyhedral diagram with the truncation of hyper-ideal vertices of the two tetrahedra.

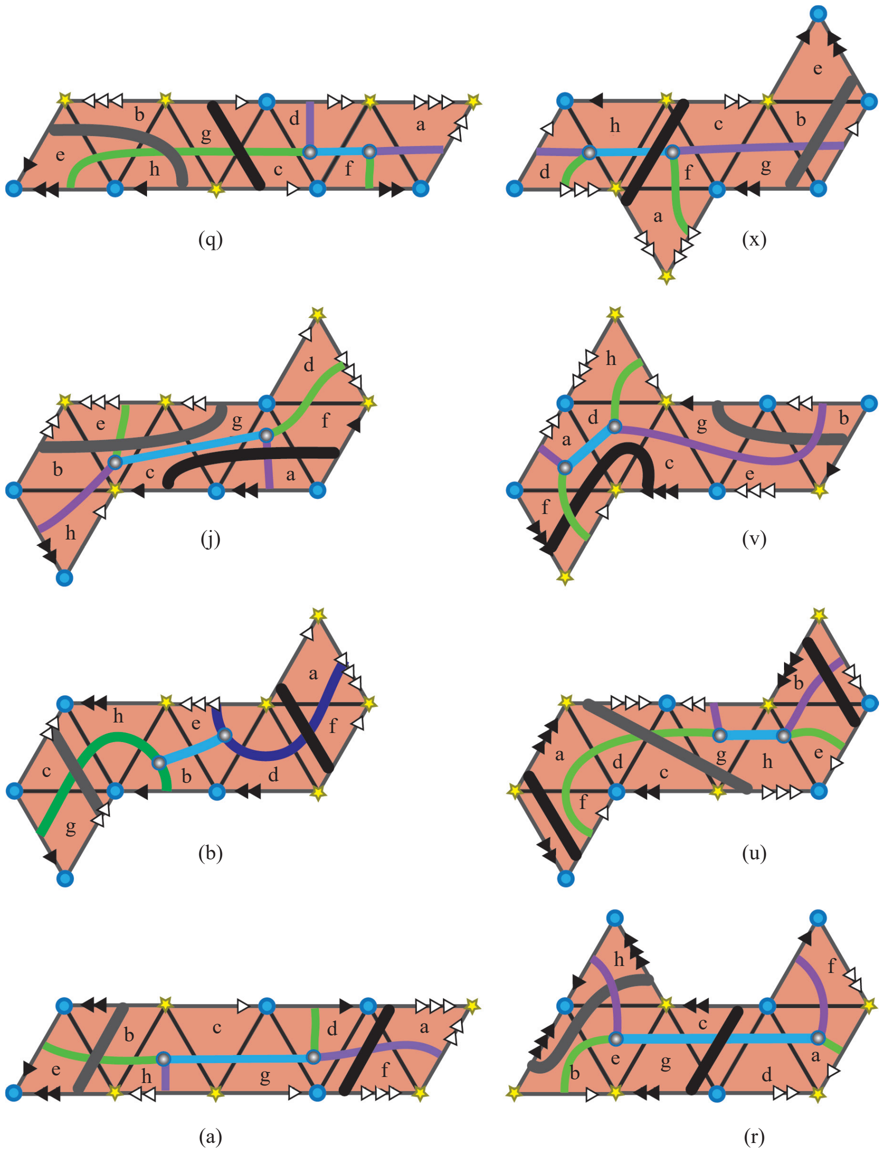

Figure 8. Theta graphs and attaching circles of the 2-handles embedded in the boundaries surfaces of the eight Fujii's manifolds

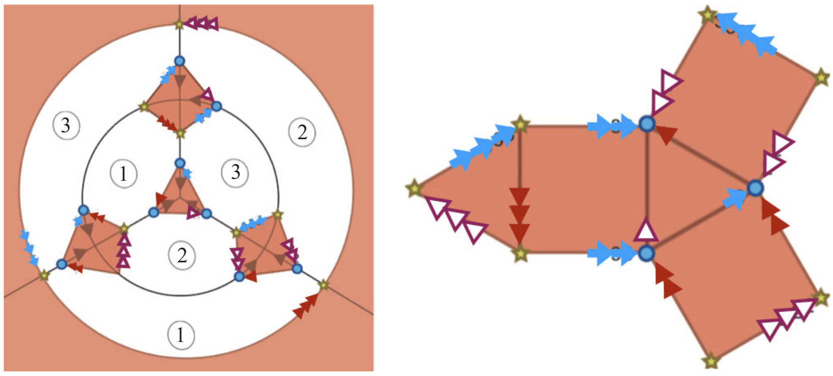

Figure 9. Left: The truncated polyhedral diagram. Right: the corresponding hyperbolic surface, which is the boundary of the 3-manifold

The pairing of edges of these polygons' vertex links are the geodesically complete hyperbolic surfaces, which are the components of the boundaries of their 3-manifolds. The right side of Figure 7 shows this hyperbolic surface of genus two as the gluing of the eight triangles with an edge-pairing inherited by the restriction of the side-pairing of the two tetrahedra, or bipyramid P^3, of Fujii's example denoted by **a**.

In Fujii's examples, there is only one cycle of hyper-ideal vertices. All the vertex links of the tetrahedra are equilateral triangles. In Figure 8, the vertex links from the cycle how they assemble into hyperbolic connected polygons that give rise to the boundary component of the 3-manifold.

In the general case for hyperbolic knotted graphs, we have compact orientable hyperbolic 3-manifolds M^3 with hyperbolic surfaces boundary components of genus higher than 1. If we glue to each boundary component of M^3 of genus g a handlebody with g handles, then the result is a closed 3-manifold $\bar{M}^3$ in such a way that M^3 is diffeomorphic to $\bar{M}^3 - G$, where G is the interior of the handlebody that we have attached. The classic knot theory studies the case where the closed 3-manifold is exactly the 3-sphere.

We can see a handlebody in $\mathbb{S}^3$ as a knotted solid graph, for instance, an ϵ-neighborhood in $\mathbb{S}^3$ of an embedded graph.

At this point, we have glued to every component of genus g of ∂M^3 a g-handlebody by its boundary. This filling process is equivalent to attaching g 2-handles and one 3-handle to the handle decomposition of M^3. The attaching circles of the 2-handles can be any set of two-by-two disjoint simple closed curves on the surface, which are the corresponding boundaries of g two by two disjoint disks of the attached handlebody, so we may assume that they are meridians (see Figure 10). The complement of these g 2-handles in the handlebody is a 3-ball which we consider as a 3-handle.

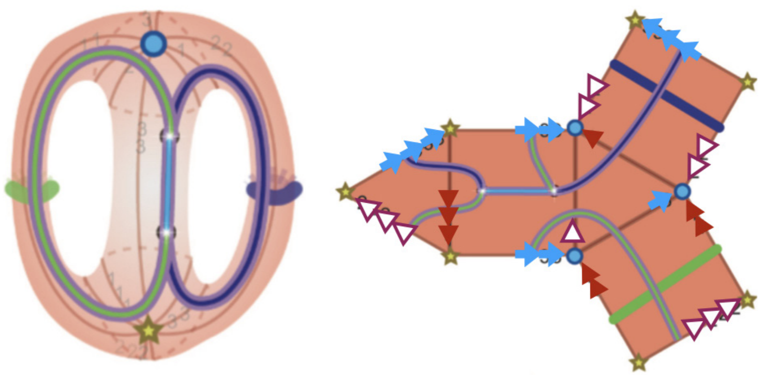

Figure 10. A theta graph and attaching circles of the 2-handles embedded in the surface such that the boundary of the exterior of this graph is a surface isotopic to the original one

On the right side of Figure 10 lies the surface of genus two of Fujii's example **a**, and on the left side lies its corresponding handle body. Now, we draw in this hyperbolic surface two meridians as attaching circles of 2-handles and a graph along the surface whose neighborhood, in the corresponding 1-manifold in $\mathbb{S}^3$, is isotopic to the handlebody. On the right side of Figure 10 is a picture of the theta graph, including arcs joining opposite sides of two rhombuses of the polygon corresponding to the attaching circles of the 2-handles. The diagrams of the eight surfaces of genus two of the Fujii's manifolds with theta graphs and attaching circles of the 2-handles are in Figure 8.

The right side of Figure 5 shows the handle decomposition diagram of M^3 obtained using the method described in Section 3. In contrast, Figure 10 exhibits the handle decomposition of M^3 for Fujii's example **a**, the arcs of attaching circles of the attached 2-handles, and the arcs of the graph whose neighborhood is isotopic to the attached handlebody in the closed 3-manifold $\bar{M}^3$. Notice that we have a correspondence between the polygonal faces of the polyhedral diagram and the sections of the handle-decomposition diagram on both sides of Figure 5.

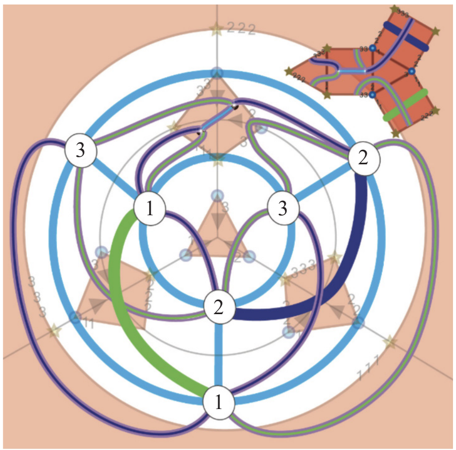

Figure 11. From the polyhedral diagram to a handle decomposition diagram

We want to remark that the handle decomposition of Fujii's examples of compact hyperbolic 3-manifolds, constructed with two truncated tetrahedra [4], have no 3-handles since we have cut the vertices, and their boundaries are always hyperbolic surfaces of genus 2. However, if we close off the boundary of a Fujii's 3-manifold with a handlebody of genus two as a Dehn filling, we must add one 3-handle (and two 2-handle) to its decomposition handle diagram, but there is no need to track.

Three types of handle moves preserve the topology of a handle decomposition. They are isotopy, handle pair creations or cancellations, and handle slides. In Figure 12, we exhibit a sequence of equivalent handle decompositions of $\bar{M}^3$ via these handle moves such that it is simplified until to get the canonical handle decomposition of the 3-sphere $\mathbb{S}^3$ consisting of one 3-handle and one 0-handle.

We start at the right top of Figure 12 with a handle decomposition diagram of a closed 3-manifold $\bar{M}^3$, which consists of one 0-handle, three 1-handles, three 2-handles, and one 3-handle. We remember that the original handle decomposition diagram of the compact 3-manifold M^3 in Figure 5 has one 0-handle, three 1-handles, and one 2-handle. We attach a handlebody of genus 2, adding two 2-handles and one 3-handle. In the handle diagrams of Figure 12, we draw the arcs of the hyperbolic graph whose isotopy class we will recover at the end of this process. This graphic lies in the boundary of the 0-handle and may cross the 1-handles; this graph's arcs can cross the pairing feet of 1-handles but do not intersect the interior of any handle.

At the left top of Figure 12, we have a handle decomposition diagram with attaching circles going across the 1-handles labeled 1 and 2 only once, respectively, so their corresponding 2-handles cancel the 1-handles. The right top of Figure 12 shows the handle decomposition after applying an isotopy, which moves the feet of the 1-handle, the arcs of the corresponding 2-handles, and the related subgraph.

At the middle right top of Figure 12, we can see the handle decomposition after canceling the 1-handle labeled 1 and joining the respective arcs. Notice that according to the handle moves, if a 2-handle cancels a 1-handle that carries one of its longitudes, it generates an undercrossing in the isotopy diagram, where the arcs fall upon the feet of the 1-handle pass over the arc joining the attaching circle of the 2-handle.

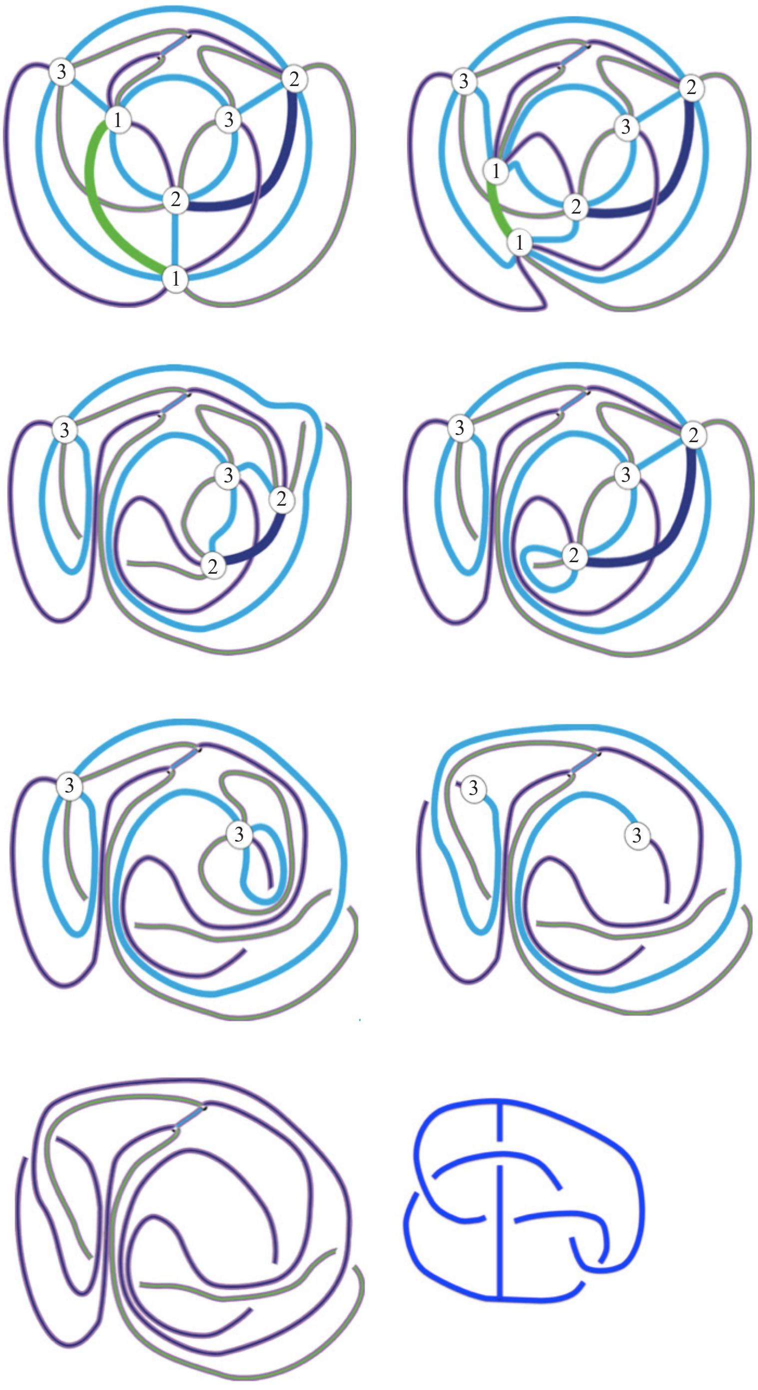

Figure 12. Handle moves and diagrams of handle decompositions to the 3-sphere which keeping track of the isotopy of the hyperbolic knotted graph

The middle-left top of Figure 12 shows the handle decomposition after sliding some part of the attaching circle of the only 2-handle in M^3, which turns around the foot of the 1-handle labeled 2, which can be pushed out of this 1-handle.

The middle-left bottom of Figure 12 shows the handle decomposition after the cancellation of the 1-handle labeled 2.

We place a new overcrossing in the isotopy diagram where the arc going down corresponds to an arc of the theta graph.

In the middle right bottom of Figure 12, we can see the handle decomposition after sliding some part of the arc of the theta graph and part of the attaching circle of the only 2-handle in M^3, which turns around the foot of the 1-handle labeled 3. This last one can be pushed as an attaching circle of a 2-handle that joins into a unique arc, the feet of a 1-handle labeled here as 3.

The left bottom of Figure 12 shows the handle decomposition after canceling the last 1-handle labeled 3 by the last 2-handle in the handle decomposition. We again place three new overcrossings in the isotopy diagram, where the arc going down again corresponds to an arc of the knotted theta graph. This diagram has no more information about handles to glue. It represents the boundary of one 0-handle which is attached to a 3-handle by their boundaries. Two 3-balls glued by their boundaries is the handle decomposition of a 3-sphere. To finish, we apply an isotopy of the knotted graph to obtain the right bottom of Figure 12.

We want to remark that we can apply this method to the eight mutually non-isometric compact oriented hyperbolic 3-manifolds with totally geodesic boundaries given by Fujii in [4] in the same way that we did in Fujii's example **a**; since they decomposed into two hyperbolic truncated tetrahedra, whose boundaries are closed surfaces of genus two. We find that the four Fujji's manifolds **a**, **b**, **q** and **r** are the exteriors in the 3-sphere of the four graphs in Figure 1.

The handle decompositions of the other four Fujii's 3-manifolds involve more complex configurations that can not reduce to the most straightforward decomposition of the 3-sphere as the union of two 3-balls by their spherical boundaries. All of them decompose in one k-handle $k = 0, 1, 2, 3$, so when the 1-handle is glued to the 0-handle, we get a canonical pair of disks, but when the 2-handle is glued to the 1-handle turns around more than one time in its handle decomposition diagram, generating a nontrivial element on its first homology group which determines that this manifold is a Lens space. For example, in Figure 13, we show the reduced handle diagram of the Dehn filling along the boundary of the Fujii's 3-manifold called x, which is the handle diagram of the Lens space $L(5, 1)$. For the others Fujii's 3-manifolds called j, u, and v appear in the lens spaces $L(5, 3)$, $L(7, 3)$, and $L(7, 4)$, respectively.

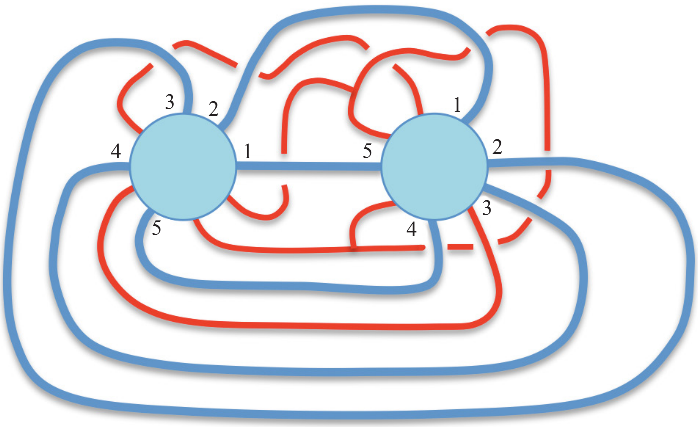

Figure 13. A handle diagram of a Lens space $L(5,1)$ and a knotted theta graph in red

7. Non-compact hyperbolic 3-manifolds with geodesically complete boundary

The goal of this section is to apply Ivanšić's method to a non-compact hyperbolic manifold with a totally geodesic surface boundary of genus two and prove the following.

Theorem 2 The exterior of the knotted theta graph in the complement $\mathbb{S}^3 - C$ of a circle C in the 3-sphere, whose isotopy class is shown in Figure 2, is a non-compact oriented hyperbolic 3-manifold with a totally geodesic closed surface boundary of genus 2.

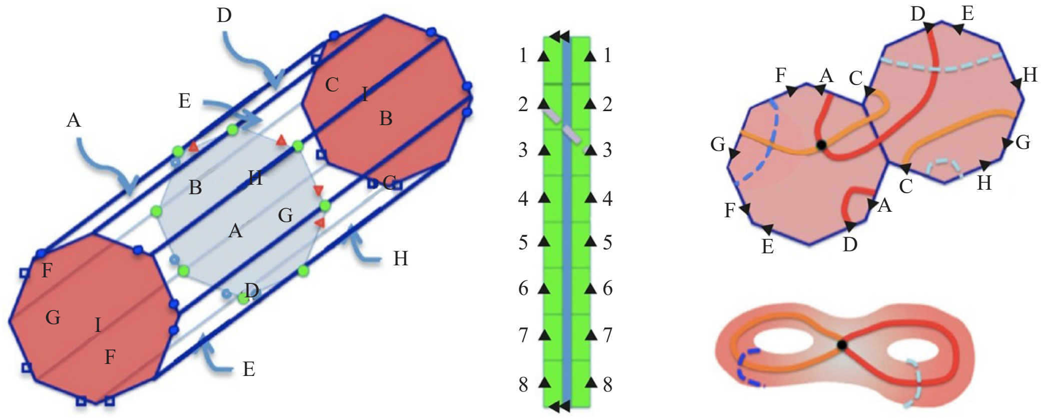

Figure 14. Left: There is a hyperbolic polyhedron with eight ideal vertices with a side-pairing. Middle: A torus getting from the truncation of the ideal vertices by horoballs. Right: A surface of genus two as the boundary

Proof. Notice that in the non-compact case, we must close each end of the hyperbolic 3-manifold with a solid torus or a handlebody of genus 1; so, we can apply the Ivanšić's method given in [5] to the toroidal ends. We will use this method in the example Fujii gave in Section 1, p.245 in [6], which we will describe below (see Figure 14).

On the left of Figure 14, there is a polyhedron $\hat{P}^3$ which appears as two octagonal prisms glued so that the top and bottom of $\hat{P}^3$ are two geodesically complete hyperbolic octagons, which are right-angled with the internal sides of the prism. In the middle of $\hat{P}^3$ is an ideal octagon with eight ideal vertices of $\hat{P}^3$. $\hat{P}^3$ has eighteen geodesically complete faces: Two octagonal faces and sixteen quadrilateral faces with two ideal vertices each.

In [6], Fujii gives a side-pairing of the central sixteen squared faces in such a way that $\hat{P}^3$, with these identifications, gets a hyperbolic structure as a non-compact hyperbolic 3-manifold $\hat{M}^3$, whose boundary is a surface of genus two assembled by gluing the top octagonal face with the bottom of $\hat{P}^3$ which are not pairing at the middle. The central octagon, with its edge-pairing, is an embedded hyperbolic surface of genus two without one point. The links of the eight ideal vertices are isometric flat squares that assemble to get a torus (see the middle of Figure 14). At the right of Figure 14, we have two glued octagonal faces with a side-pairing, representing the closed, 2-genus surface which is the boundary of the hyperbolic 3-manifold M^3.

We start with the handle decomposition diagram of the hyperbolic manifold M^3 given by the side-pairing of the polyhedron, and we place the attaching circles of 2-handles. Next, we draw one attaching circle of a 2-handle for the solid torus as a meridian of the assembled torus composed of the links of ideal vertices. We place two attaching circles of the 2-handles of the handlebody attached by its boundary as meridians on the 2-genus surface assembled by the two octagonal faces. In the middle of Figure 14, we have the longitude of the torus, which is a curve parallel in $\overline{M}^3$ to the soul of the solid torus attached to M^3. We paint a theta graph in the boundary of M^3, which is homotopic to the surface.

Now, we draw the arcs of the theta graph in the two octagons. We use it to construct the handle decomposition diagram with additional information about the isotopy class of the ends as link complements and the isotopy class of the boundary components as knotted graphs exteriors in $\mathbb{S}^3$.

Finally, we perform handle moves to obtain a hyperbolic knotted graph which is the union of an embedded theta graph and one circle (see Figure 2). The exterior of the knotted theta graph in the complement of the circle in the 3-sphere is a hyperbolic manifold with a geodesically complete boundary a surface of genus 2.

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Conflict of interest

Authors declare there is no conflict of interest at any point with reference to research findings.

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