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Research Article

Fixed Point Iterations for Functional Equations and Split Feasibility Problems in CAT(0) Spaces

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Abstract: Among other things, finding solutions for functional as well as other types of problems (including differential and integral) by suggesting fixed-point procedures is a difficult task, especially when we study the approximation techniques in the absence of linearity in the domain of definition. In this paper, an effective iterative approximation procedure is successfully applied to find fixed points of a general class of operators in the nonlinear setting of CAT(0) spaces. The results are illustrated with the help of some examples. Some numerical computations are also provided. Eventually, we prove that our new results are applicable to solving split feasibility problems. Our results are new and complement some recently published results from the literature.

***Keywords*:** fixed point technique, functional equation, convergence result, condition (I), CAT(0) space, split feasibility problem

MSC: 47H09, 47H10

1. Introduction

Consider a nonempty subset $\mathcal{H}$ of a metric space $\mathcal{D}$. Suppose $\mathcal{A}$ is a self-map of $\mathcal{H}$. In this work, we consider the following fixed-point problem:

$$\text{Find } y_0 \in \mathcal{H} \text{ such that } \mathcal{A}y_0 = y_0. \tag{1}$$

We shall denote by $F_{\mathcal{A}}$, the set of all fixed points of $\mathcal{A}$ in $\mathcal{H}$. In recent years, the Problem (1) has been studied by many authors. For example, Sahu et al. [1] studied this problem in the setting of quasi-nonexpansive mappings and provided some applications in convex programming and feasibility problems. Also, Usurelu and Postolache [2] studied Problem (1) in the setting of hybrid mappings. They provided some applications of their main results to splitting equilibrium problems. Very recently, Yao et al. [3] studied Problem (1) in the setting of monotone mappings and provided applications in split equilibrium problems. Motivated by the above, we study Problem (1) in the setting of a

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generalized class of nonlinear mappings and provide applications to split feasibility problems.

Once a fixed-point result in a linear domain (e.g., Hilbert or Banach space) is established, its extension to a nonlinear domain (e.g., metric space) is often desirable. However, to do this, we often need the notion of convexity. Towards this success, Takahashi [4] was the first to provide the concept of convexity on nonlinear domains and pioneered the study of fixed point theory and related topics in various nonlinear domains. For more details, see [5-8].

Definition 1.1. [9] Suppose that a self-map of a subset of a metric space endowed with the distance ρ. $\mathcal{A}$ is called

(a) nonexpansive if $\rho(\mathcal{A}y, \mathcal{A}z) \le \rho(y, z)$ holds for all $y, z \in \mathcal{H}$,

(b) Suzuki map if $\frac{\rho(y, \mathcal{A}y)}{2} \le \rho(y, z)$ implies $\rho(\mathcal{A}y, \mathcal{A}z) \le \rho(y, z)$ for all $y, z \in \mathcal{H}$.

Remark 1.2. By observing the above definition, we note that all nonexpansive mappings are eventually contained in the class of Suzuki nonexpansive maps. The converse of this statement is nevertheless not valid in general, as shown by Suzuki in [9].

There are many results available in the literature related to the classes of nonexpansive and Suzuki maps. In particular, Browder [10] and Göhde [11] proved that nonexpansive maps under some assumption in a uniformly convex Banach space setting always admit a fixed point. Suzuki proved that this result is valid for the class of Suzuki maps. Kirk proved the nonlinear space version of the Browder [10] and Göhde [11] results.

Recently, a new concept of mapping, the so-called Reich-Suzuki-type nonexpansive maps, has been reported as follows:

Definition 1.3. [12] Let $\mathcal{H}$ be a subset of a metric space endowed with the distance ρ, and $\mathcal{A}$ be a self-map on $\mathcal{H}$. Then, $\mathcal{A}$ is called Reich-Suzuki-type nonexpansive if there is a $c \in [0,1)$, such that $\frac{\rho(y, \mathcal{A}y)}{2} \le \rho(y, z)$ implies $\rho(\mathcal{A}y, \mathcal{A}z) \le c\rho(y, \mathcal{A}z) + c\rho(z, \mathcal{A}y) + (1-2c)\rho(y, z)$ for all $y, z \in \mathcal{H}$.

It is important to note that Reich-Suzuki-type nonexpansive operators share the following properties:

Proposition 1.4. [13] Let $\mathcal{H}$ be a subset of a metric space and $\mathcal{A}$ be a self-map on $\mathcal{H}$ with $F_{\mathcal{A}} = \{y_0 \in \mathcal{H} : y_0 = \mathcal{A}y_0\} \ne \varnothing$. Then, the following properties are valid.

(a) When $\mathcal{A}$ is Riech-Suzuki-type nonexpansive, then the inequality $\rho(\mathcal{A}y, \mathcal{A}y_0) \le \rho(y, y_0)$ is valid for all $y \in \mathcal{H}$ and any $y_0 \in F_{\mathcal{A}}$.

(b) If $\mathcal{A}$ is Suzuki map, then $\mathcal{A}$ is Reich-Suzuki-type nonexpansive.

The following iterative scheme has been introduced by Hassan et al. [14]:

$$\left.\begin{aligned} &r_1 \in \mathcal{H}, \\ &q_i = \mathcal{A}((1-d_i)r_i + d_i\mathcal{A}r_i), \\ &p_i = \mathcal{A}((1-c_i)q_i + b_i\mathcal{A}q_i), \\ &s_i = \mathcal{A}((1-b_i)p_i + b_i\mathcal{A}p_i), \\ &r_{i+1} = \mathcal{A}((1-a_i)s_i + a_i\mathcal{A}s_i), i \ge 1, \end{aligned}\right\}, \tag{2}$$

where $a_i, b_i, c_i, d_i \in (0,1)$.

Hassan et al. [14] proved that this scheme is better than many other iterative schemes like Picard [15], Mann [16], Ishikawa [17], Noor [18], Agarwal et al. [19], Abbas and Nazir [20], Thakur et al. [21], and Ullah and Arshad [22] iterations and so on. Recently, Ullah et al. [13] generalized their results to the setting of Reich-Suzuki-type nonexpansive mappings; for more details in this direction, see [23, 24]. Fixed point schemes play an important role in applied analysis and share many useful applications in image-related problems and computer science (see, e.g., [25] and others). Here, we apply these results to the more general framework of nonlinear spaces called CAT(0) spaces.

2. Preliminaries

We assume a bounded closed interval [0, r] of reals, that is, in the $\mathbb{R}$. Let y, z be two points in a metric space $\mathcal{D} = (\mathcal{D}, \rho)$. If there is a mapping $q : [0, r] \to \mathcal{D}$ with $y = q(0), z = q(r)$ and $\rho(q(y_1), q(y_2)) = |y_1 - y_2|$ for all two points $y_1, y_2 \in [0, r]$, then it is called a geodesic map (or simply a geodesic), and the image of q is often called geodesic segment connecting the point y to the point z. This geodesic segment is denoted by $[y, z]$, provided that it is unique. A metric space $\mathcal{D}$ we called a geodesic space, provided that any two given elements of $\mathcal{D}$ can be connected by a geodesic. The space $\mathcal{D}$ is said to be uniquely geodesic if and only if any two elements $y, z \in \mathcal{D}$, there is a unique geodesic connecting them. Any subset, namely, $\mathcal{H}$ of $\mathcal{D}$ is said to be convex set if and only if for each y, z, there is a geodesic in $\mathcal{D}$ that connects them. The image of every geodesic is essentially contained in $\mathcal{H}$.

If $\mathcal{D}$ is a geodesic space, then a geodesic triangle $\Delta(t_1, t_2, t_3)$ in $\mathcal{D}$ is consists of three points, namely, t_1, t_2, t_3 in $\mathcal{D}$, and a choice of three geodesic segments $[t_1, t_2], [t_2, t_3], [t_3, t_1]$ connecting them.

The triangle $\overline{\Delta}(\overline{t_1}, \overline{t_2}, \overline{t_3})$ in the plane $\mathbb{R}^2$ is said to be a comparison triangle for $\Delta(t_1, t_2, t_3)$ if and only if

$$\rho_{\mathbb{R}^2}(\overline{t_1}, \overline{t_2}) = \rho(t_2, t_3), \rho_{\mathbb{R}^2}(\overline{t_2}, \overline{t_3}) = \rho(t_2, t_3) \text{ and } \rho_{\mathbb{R}^2}(\overline{t_3}, \overline{t_1}) = d(t_3, t_1).$$

The point $\overline{e} \in [\overline{t_1}, \overline{t_2}]$ is said to be a comparison point for $e \in [t_1, t_2]$ if and only if $\rho(t_1, e) = \rho_{\mathbb{R}^2}(\overline{t_1}, \overline{e})$. Comparison points for $[\overline{t_2}, \overline{t_3}]$ and $[\overline{t_3}, \overline{t_1}]$ are defined in the same manner.

Definition 2.1. [26] Suppose that a geodesic triangle $\Delta(t_1, t_2, t_3)$ is a metric space $(\mathcal{D}, \rho)$. $\Delta(t_1, t_2, t_3)$ is said to be equipped with the CAT(0) property if and only if for all $t, t' \in \Delta(t_1, t_2, t_3)$, and for their comparison points $\overline{t}, \overline{t'} \in \overline{\Delta}(\overline{t_1}, \overline{t_2}, \overline{t_3})$, satisfy the following:

$$\rho(t, t') = \rho_{\mathbb{R}^2}(\overline{t}, \overline{t'}).$$

Hence, a geodesic metric space $\mathcal{D}$ is said to be CAT(0) space if and only if any geodesic triangle in it is equipped with the CAT(0) property. If anyone needs some more definitions of the CAT(0) space, then we recommend the work in [27]. Notice that every CAT(0) space is essentially uniquely geodesic. The important examples of CAT(0) spaces are pre-Hilbert spaces and metric trees. For detailed study on this topic, we suggest the readers to see [26-29] and others.

The following are some interesting facts about CAT(0) spaces:

Lemma 2.2. [30] Consider a CAT(0) space $(\mathcal{D}, \rho)$.

(a_1) For any pair of elements $y, z \in \mathcal{D}$ and a scalar $a \in [0,1]$, one has a unique point $v \in [y, z]$ satisfying

$$\rho(y, v) = a\rho(y, z) \text{ and } \rho(z, v) = (1-a)\rho(y, z). \tag{3}$$

To denote the unique point v that satisfies (3), we shall write $(1-a)y \oplus az$.

(a_2) For any points $y, z, u \in \mathcal{D}$ and a scalar $a \in [0,1]$, one has

$$\rho(u, ay \oplus (1-a)z) \le a\rho(u, y) + (1-a)z.$$

Now, to establish the main outcome, we also need some other notions and facts. For example, if $\mathcal{D}$ is a CAT(0) space, and $\varnothing \ne \mathcal{H} \subseteq \mathcal{D}$ denote a closed and convex subset of $\mathcal{D}$. Then, for a bounded sequence $\{r_i\} \subseteq \mathcal{D}$ and a fixed element r_0 of $\mathcal{D}$, we may set

$$r(r_0, \{r_i\}) := \limsup_{i \to \infty} \rho(r_i, r_0).$$

As many know, the asymptotic radius of $\{r_i\}$ as concerns with the set $\mathcal{H}$ is read as follows:

$$r(\mathcal{H}, \{r_i\}) = \inf\{r(r_0, \{r_i\}) : r_0 \in \mathcal{H}\}.$$

The asymptotic center of $\{r_i\}$ w.r.t. $\mathcal{H}$, as concerns with the set $\mathcal{H}$ is read as follows:

$$A(\mathcal{H}, \{r_i\}) = \{r_0 \in \mathcal{H} : r(r_0, \{r_i\}) = r(\mathcal{H}, \{r_i\})\}.$$

Interestingly, the set $A(\mathcal{H}, \{r_i\})$ always contains a unique point in a CAT(0) space setting (see, e.g., [31] and others).

Definition 2.3. [32] Suppose that $\mathcal{D}$ is a CAT(0) space and $\{r_i\} \subseteq \mathcal{D}$. The point $y_0 \in \mathcal{D}$ is called a Δ limit of $\{r_i\}$ if and only if y_0 is the only asymptotic center for $\{s_i\}$ where $\{s_i\}$ denotes any subsequence of $\{r_i\}$.

The following is the CAT(0) space version of the Opial's [33] property.

Definition 2.4. A CAT(0) space $\mathcal{D}$ is said to satisfy the Opial's property, if $\{r_i\} \subseteq \mathcal{D}$ is any Δ-convergent sequence to $y_0 \in \mathcal{D}$, then one has

$$\limsup_{i\to\infty} \rho(r_i, y_0) < \limsup_{i\to\infty} \rho(r_i, z_0),$$

for all $z_0 \in \mathcal{D} - \{y_0\}$.

It is known that each CAT(0) space satisfies this property.

Lemma 2.5. [34] Consider a complete CAT(0) space $\mathcal{D}$ such that the sequence $\{r_i\} \subseteq \mathcal{D}$ is bounded. In this case, the sequence $\{r_i\}$ admits a Δ-convergent subsequence.

Lemma 2.6. [33] Consider a complete CAT(0) space $\mathcal{D}$ such that $\mathcal{H} \subseteq \mathcal{D}$ is closed and convex. If the sequence $\{r_i\} \subseteq \mathcal{D}$ is bounded, then consequently, the asymptotic center of $\{r_i\}$ contains in $\mathcal{H}$.

The following lemma follows from the definition of Reich-Suzuki-type nonexpansive maps.

Lemma 2.7. [13] Consider any subset $\mathcal{H}$ of a CAT(0) space and set $\mathcal{A} : \mathcal{H} \to \mathcal{H}$. If $\mathcal{A}$ is Riech-Suzuki-type nonexpansive and $y, z \in \mathcal{H}$ are any, then

$$\rho(y, \mathcal{A}z) \le \frac{(c+3)}{(1-c)} \rho(y, \mathcal{A}y) + \rho(y, z).$$

A very key property of a complete CAT(0) space that is stated below can be found in [14].

Lemma 2.8. Let us consider a complete CAT(0) space $\mathcal{D}$ such that $0 < d \le \theta_i \le e < 1$ and the sequences $\{r_i\}$, $\{s_i\}$ in the space $\mathcal{D}$, obey the conditions $\limsup_{i\to\infty} d(r_i, a) \le \eta$, $\limsup_{i\to\infty} d(s_i, a) \le \eta$, and $\lim_{i\to\infty} d(\theta_i r_i \oplus (1-\theta_i)s_i, a) = \eta$, where a stand for any real number that is greater or equal to zero. Subsequently, we have $\lim_{i\to\infty} d(r_i, s_i) = 0$.

3. Convergence results

We essentially obtain several convergence results in the CAT(0) space setting. However, we first modify the scheme (2) to CAT(0) spaces as follows:

$$\left.\begin{aligned} r_1 &\in \mathcal{H}, \\ q_i &= \mathcal{A}((1-d_i)r_i \oplus d_i \mathcal{A}r_i), \\ p_i &= \mathcal{A}((1-c_i)q_i \oplus b_i \mathcal{A}q_i), \\ s_i &= \mathcal{A}((1-b_i)p_i \oplus b_i \mathcal{A}p_i), \\ r_{i+1} &= \mathcal{A}((1-a_i)s_i \oplus a_i \mathcal{A}s_i), i \ge 1, \end{aligned}\right\}, \tag{4}$$

where $a_i, b_i, c_i, d_i \in (0,1)$. It should be noted that in this part, $\mathcal{D}$ stands for a complete CAT(0) space. We begin our main outcome with the following lemma.

Lemma 3.1. Let $\mathcal{A}$ be a self-map of a closed nonempty convex subset $\mathcal{H}$ of $\mathcal{D}$. If $\mathcal{A}$ is Reich-Suzuki-type nonexpansive with $F_{\mathcal{A}} \neq \varnothing$ and $\{r_i\}$ is a sequence obtained from (4), then $\lim_{i\to\infty} \rho(r_i, y_0)$ exists for any choice of $y_0 \in F_{\mathcal{A}}$.

Proof. Let $y_0 \in F_{\mathcal{A}}$. Then, using (4) along with Proposition 1.4 (i), we have

$$\begin{aligned}\rho(q_i, y_0) &= \rho(\mathcal{A}((1-d_i)r_i \oplus d_i\mathcal{A}r_i), y_0)\\ &\le \rho((1-d_i)r_i \oplus d_i\mathcal{A}r_i, y_0)\\ &\le (1-d_i)\rho(r_i, y_0) + d_i\rho(\mathcal{A}r_i, y_0)\\ &\le (1-d_i)\rho(r_i, y_0) + d_i\rho(r_i, y_0)\\ &\le \rho(r_i, y_0). \end{aligned} \tag{5}$$

Similarly,

$$\begin{aligned}\rho(p_i, y_0) &= \rho(\mathcal{A}((1-c_i)q_i \oplus c_i\mathcal{A}q_i), y_0)\\ &\le \rho((1-c_i)q_i \oplus c_i\mathcal{A}q_i, y_0)\\ &\le (1-c_i)\rho(q_i, y_0) + c_i\rho(\mathcal{A}q_i, y_0)\\ &\le (1-c_i)\rho(q_i, y_0) + c_i\rho(q_i, y_0)\\ &\le \rho(q_i, y_0). \end{aligned} \tag{6}$$

Also,

$$\begin{aligned}\rho(s_i, y_0) &= \rho(\mathcal{A}((1-b_i)p_i \oplus b_i\mathcal{A}p_i), y_0)\\ &\le \rho((1-b_i)p_i \oplus b_i\mathcal{A}p_i, y_0)\\ &\le (1-b_i)\rho(p_i, y_0) + b_i\rho(\mathcal{A}p_i, y_0)\\ &\le (1-b_i)\rho(p_i, y_0) + b_i\rho(p_i, y_0)\\ &\le \rho(p_i, y_0). \end{aligned} \tag{7}$$

Now (5), (6), and (7) imply that

$$\begin{aligned}\rho(r_{i+1}, y_0) &= \rho(\mathcal{A}((1-a_i)s_i \oplus a_i\mathcal{A}s_i), y_0)\\ &\le \rho((1-a_i)s_i \oplus a_i\mathcal{A}s_i, y_0)\\ &\le (1-a_i)\rho(s_i, y_0) + a_i\rho(\mathcal{A}s_i, y_0)\\ &\le (1-a_i)\rho(s_i, y_0) + a_i\rho(s_i, y_0)\\ &\le \rho(s_i, y_0) \le \rho(p_i, y_0) \le \rho(q_i, y_0)\\ &\le \rho(r_i, y_0). \end{aligned} \tag{8}$$

Thus, for all y_0 in $F_{\mathcal{A}}$, we proved $\rho(r_{i+1}, y_0) \le \rho(r_i, y_0)$. So, $\{\rho(r_i, y_0)\}$ is essentially bounded and also nonincreasing. Accordingly, $\lim_{i\to\infty}\rho(r_i, y_0)$ exists for all y_0 in $F_{\mathcal{A}}$.

Now, we are in the position to provide a key theorem of the paper, which will help us in the convergence theorems of the section.

Theorem 3.2. Let $\mathcal{A}$ be a self-map of a closed nonempty convex subset $\mathcal{H}$ of $\mathcal{D}$. If $\mathcal{A}$ is Reich-Suzuki-type nonexpansive and $\{r_i\}$ is a sequence obtained from (4). Then, $F_{\mathcal{A}} \neq \varnothing$ if and only if $\{r_i\}$ is bounded as well as $\lim_{i\to\infty}\rho(r_i, \mathcal{A}r_i) = 0$.

Proof. Consider that $F_{\mathcal{A}} \neq \varnothing$, and the aim is to show that $\{r_i\}$ is bounded with $\lim_{i\to\infty}\rho(r_i, \mathcal{A}r_i) = 0$. Suppose $y_0 \in F_{\mathcal{A}}$

be any point, then according to the Lemma 3.1, $\{r_m\}$ is bounded with $\lim_{i\to\infty}\rho(r_i, y_0)$ exists. It is now remaining to prove that $\lim_{i\to\infty}\rho(r_i, \mathcal{A}r_i) = 0.$ Suppose

$$\lim_{i\to\infty}\rho(r_i, x_0) = \eta, \tag{9}$$

where η is any constant in $[0, \infty)$. We consider only the case $\eta > 0$. Now, according to (5),

$$\rho(q_i, y_0) \leq \rho(r_i, y_0),$$

$$\Rightarrow \limsup_{i\to\infty}\rho(q_i, y_0) \leq \limsup_{i\to\infty}\rho(r_i, y_0) = \eta. \tag{10}$$

Also, by Proposition 1.4 (a), we have

$$\rho(\mathcal{A}r_i, y_0) \leq \rho(r_i, y_0),$$

$$\Rightarrow \limsup_{i\to\infty}\rho(\mathcal{A}r_i, y_0) \leq \limsup_{i\to\infty}\rho(r_i, y_0) = \eta. \tag{11}$$

Now from (8), we have

$$\rho(r_{i+1}, y_0) \leq \rho(q_i, y_0).$$

Using this together with (9), we obtain

$$\eta \leq \liminf_{i\to\infty}\rho(q_i, y_0). \tag{12}$$

From (10) and (12), we obtain

$$\lim_{i\to\infty}\rho(q_i, y_0) = \eta. \tag{13}$$

Using (13), we get

$$\begin{aligned}
\eta &= \lim_{i\to\infty}\rho(q_i, y_0) = \lim_{i\to\infty}\rho(\mathcal{A}((1-d_i)r_i \oplus d_i\mathcal{A}r_i), y_0) \\
&\leq \lim_{i\to\infty}\rho((1-d_i)r_i \oplus d_i\mathcal{A}r_i, y_0) \\
&\leq \lim_{i\to\infty}(1-d_i)\rho(r_i, y_0) + \lim_{i\to\infty}d_i\rho(\mathcal{A}r_i, y_0) \\
&\leq \lim_{i\to\infty}(1-d_i)\rho(r_i, y_0) + \lim_{i\to\infty}d_i\rho(r_i, y_0)) \\
&= \lim_{i\to\infty}\rho(r_i, y_0) \\
&= \eta.
\end{aligned}$$

Consequently, we have

$$\eta = \lim_{i\to\infty}d_i((1-d_i)r_i \oplus d_i\mathcal{A}r_i, y_0)). \tag{14}$$

By using (9), (11), and (14), and applying Lemma 2.8, the following facts are obtained

$$\lim_{i\to\infty}\rho(r_i,\mathcal{A}r_i)=0.$$

In the converse case, we suppose $\{r_i\}$ is bounded sequence with $\lim_{i\to\infty}\rho(r_i,\mathcal{A}r_i)=0$, and show that $F_{\mathcal{A}}\neq\varnothing$. To do this, we assume any $y_0\in A(\mathcal{D},\{r_i\})$, and apply Lemma 2.7 as follows:

$$\begin{aligned}A(\mathcal{A}y_0,\{r_i\}) &= \limsup_{i\to\infty}\rho(r_i,\mathcal{A}y_0)\\ &\le \frac{(c+3)}{(1-c)}\limsup_{i\to\infty}\rho(r_i,\mathcal{A}r_i)+\limsup_{i\to\infty}\rho(r_i,y_0)\\ &= \limsup_{i\to\infty}\rho(r_i,y_0)\\ &= A(y_0,\{r_i\}).\end{aligned}$$

Hence, we got $\mathcal{A}y_0\in A(\mathcal{D},\{r_i\}$, and also the set $A(\mathcal{D},\{r_i\}$ contains only one point. Therefore, $y_0=\mathcal{A}y_0$, that is, $y_0\in F_{\mathcal{A}}$. Accordingly, the fixed-point set $F_{\mathcal{A}}$ is nonempty.

We are going to give the first convergence theorem of the section.

Theorem 3.3. Let $\mathcal{A}$ be a self-map of a compact nonempty convex subset $\mathcal{H}$ of $\mathcal{D}$. If $\mathcal{A}$ and the sequence $\{r_i\}$ are as stated in Theorem 3.2, and $F_{\mathcal{A}}\neq\varnothing$. Then, $\{r_i\}$ converges strongly to a point of $F_{\mathcal{A}}$.

Proof. Since $\{r_i\}$ contained in the compact set $\mathcal{H}$, we have a subsequence $\{r_{i_l}\}$ of $\{r_i\}$, and some $q_0\in\mathcal{H}$ such that $\rho(r_{i_l},q_0)\to 0$. It suffice to prove q_0 is the strong limit of the $\{r_i\}$. To do this, we use lemma to obtain with

$$\rho(r_{i_l},\mathcal{A}q_0)\le\frac{(c+3)}{(1-c)}\rho(r_{i_l},\mathcal{A}r_{i_l})+\rho(r_{i_l},q_0) \tag{15}$$

Since Theorem 3.2 suggests that $\rho(r_{i_l},\mathcal{A}r_{i_l})\to 0$. Hence, (15) gives us $\lim_{l\to\infty}\rho(r_{i_l},\mathcal{A}q_0)=0$. It follows that $\mathcal{A}q_0=q_0$, that is, $q_0\in F_{\mathcal{A}}$. Since $\lim_{i\to\infty}\rho(r_i,q_0)$ exists according to Lemma 3.1, we have q_0 is the strong limit of $\{r_i\}$.

Theorem 3.4. Let $\mathcal{A}$ be a self-map of a closed nonempty convex subset $\mathcal{H}$ of $\mathcal{D}$. If $\mathcal{A}$ and the sequence $\{r_i\}$ are as stated in Theorem 3.2, and $F_{\mathcal{A}}\neq\varnothing$. Then, $\{r_i\}$ converges strongly to a point of $F_{\mathcal{A}}$ provided that $\liminf_{i\to\infty}\rho(r_i,F_{\mathcal{A}})=0$.

Proof. We do not include the details of this result. Because its proof is not very difficult.

A convergence theorem without compactness of the domain is now desirable. The following condition is in fact needed.

Definition 3.5. [35] Let $\mathcal{H}$ be closed and nonempty subset of $\mathcal{D}$. The map $\mathcal{A}:\mathcal{H}\to\mathcal{H}$ is said to satisfy the condition (I), if there is a μ with $\mu(a)=0$ if and only if $a=0$, $\mu(a)>0$ for all $a>0$, and $\rho(y,\mathcal{A}y)\ge\mu(\rho(y,F_{\mathcal{A}}))$ for all elements $y\in\mathcal{H}$.

Theorem 3.6. Let $\mathcal{A}$ be a self-map of a closed nonempty convex subset $\mathcal{H}$ of $\mathcal{D}$. If $\mathcal{A}$ and the sequence $\{r_i\}$ are as stated in Theorem 3.2, and $F_{\mathcal{A}}\neq\varnothing$. Then, $\{r_i\}$ converges strongly to a point of $F_{\mathcal{A}}$, provided that $\mathcal{A}$ has condition (I).

Proof. The conclusion of Theorem 3.2 provides us

$$\liminf_{i\to\infty}\rho(r_i,\mathcal{A}r_i)=0. \tag{16}$$

The condition (I) of $\mathcal{A}$ suggests

$$\rho(r_i,\mathcal{A}r_i)\ge\mu\rho(r_i,F_{\mathcal{A}})).$$

But the map $\mu:[0,\infty)\to[0,\infty)$ is nondecreasing with $\mu(0)=0,\mu(p)>0$ for each choice of $p>0$, so

$$\liminf_{i\to\infty}\rho(r_i,F_{\mathcal{A}})=0.$$

It is seen that all the needed hypotheses of Theorem 3.4 are satisfied. Hence, by its conclusions, $\{r_i\}$ is strongly convergent in the set $F_{\mathcal{A}}$.

The final result is the following which establishes the Δ-convergence of $\{r_i\}$ generated by (4).

Theorem 3.7. Let $\mathcal{A}$ be a self-map of a closed nonempty convex subset $\mathcal{H}$ of $\mathcal{D}$. If $\mathcal{A}$ and the sequence are as stated in Theorem 3.2, and $F_{\mathcal{A}} \neq \varnothing$. Then, $\{r_i\}$ Δ-converges to a point of $F_{\mathcal{A}}$

Proof. Since $\mathcal{H}$ is convex, so $\{r_i\}$ is in $\mathcal{H}$ and so according to Theorem 3.2, $\{r_i\}$ is bounded and satisfies $\lim_{i\to\infty} \rho(r_i, \mathcal{A}r_i) = 0$. Let $\omega_\Delta(\{r_i\}) = \bigcup A(\{s_i\})$, where $\{s_i\}$ is any sub-sequences of $\{r_i\}$. The target is to show that $\omega_\Delta(\{r_i\}) \subseteq F_{\mathcal{A}}$. For $s \in \omega_\Delta(\{r_i\})$, we may choose a sub-sequence, namely, $\{s_i\}$ of the sequence $\{r_i\}$ such that $A(\{s_i\}) = \{s\}$. According to Lemmas 2.5 and 2.6, one can find a sub-sequence $\{e_i\}$ of $\{s_i\}$ endowed with the Δ-limit e in B. But, Theorem 3.2 suggests that $\lim_{i\to\infty} \rho(e_i, \mathcal{A}e_i) = 0$. By Lemma 2.7,

$$\rho(e_i, \mathcal{A}e) \leq \frac{(3+\gamma)}{(1-\gamma)} \rho(e_i, \mathcal{A}e_i) + \rho(e_i, e). \tag{17}$$

Applying lim sup on both of the sides of (17), it follows that $e \in F_{\mathcal{A}}$. Lemma 3.1 suggests that $\lim_{i\to\infty} \rho(e_i, e)$ exists. We need to show that $s = e$. Assume on contrary, that s is different from the e. So, by uniqueness of asymptotic centers, one has

$$\begin{aligned} \limsup_{i\to\infty} \rho(e_i, e) &< \limsup_{i\to\infty} \rho(e_i, s) \leq \limsup_{i\to\infty} \rho(s_i, s) \\ &< \limsup_{i\to\infty} \rho(s_i, e) = \limsup_{i\to\infty} \rho(r_i, e) \\ &= \limsup_{i\to\infty} \rho(e_i, e). \end{aligned}$$

Accordingly, we proved $\limsup_{i\to\infty} \rho(e_i, e) < \limsup_{i\to\infty} \rho(e_i, e)$. It follows that $s = e \in F_{\mathcal{A}}$ and $\omega_\Delta(\{r_i\}) \subseteq F_{\mathcal{A}}$.

We want to show that $\{r_i\}$ essentially Δ-convergences in $F_{\mathcal{A}}$. To achieve the goal, we shall prove that $\omega_\Delta(\{r_i\})$ consists of a one point. Suppose $\{s_i\}$ is a given sub-sequence of $\{r_i\}$, then according to Lemmas 2.5 and 2.6, we have a Δ-convergent sub-sequence, namely, $\{e_i\}$ of $\{s_i\}$ with the $\Delta - \lim e$ in $\mathcal{H}$. Assume that $A(\{s_i\}) = \{s\}$ and $A(\{r_i\}) = \{g\}$. Keep in mind that we have already proved that $s = e$ and $e \in F_{\mathcal{A}}$. We want to prove $g = e$. Suppose not, then $\lim_{i\to\infty} \rho(r_i, e)$ exists, and also the asymptotic centers are singleton, we have

$$\limsup_{i\to\infty} \rho(e_i, e) < \limsup_{i\to\infty} \rho(e_i, g) \leq \limsup_{i\to\infty} \rho(r_i, g) < \limsup_{i\to\infty} \rho(r_i, e) = \limsup_{i\to\infty} \rho(e_i, g),$$

which is a contradiction, and so, $g = r \in F_{\mathcal{A}}$. Thus, $\omega_\Delta(\{s\}) = \{g\}$. This establishes the proof.

We finish this section with the following example.

Example 3.8. Consider $\mathcal{D} = \mathbb{H}^3 = \{y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : \langle y, y \rangle = -1, y_4 > 0\}$, a three-dimensional space enriched with Lorentz distance $\langle .,. \rangle$ of $\mathbb{R}^4$ as

$$y.z = y_1 z_1 + y_2 z_2 + y_3 z_3 - y_4 z_4,\ y = (y_1, y_2, y_3, y_4),\ z = (z_1, z_2, z_3, z_4) \in \mathbb{H}^3.$$

In this case, the geodesic map q is defined by $q(r) = \cosh(r)y + \sinh(y)v$, where v is a unit unit vector. Set a mapping $\mathcal{A}$ on $\mathbb{H}$ by the formula:

$$\mathcal{A}(y_1, y_2, y_3, y_4) = (-y_1, -y_2, -y_3, y_4).$$

We see that $\mathcal{A}$ is Reich-Suzuki-type nonexpansive with a fixed point $y_0 = (0,0,0,1)$. By our main results, the sequence $\{r_i\}$ converges to this y_0.

4. Illustrative examples

The following examples support the theoretical results:

Example 4.1. We now consider an operator $\mathcal{A}$ that connects the values of the domain $\mathcal{H}$ = [7,9], as $\mathcal{A}9 = 6$ and $\mathcal{A}y = \frac{y+42}{7}$ if $7 \leq y < 9$. Put $c = \frac{1}{2}$, consider the various cases below.

(i) When $y, z \in [7, 9)$. Then, $\mathcal{A}y = \frac{y+42}{7}$ and $\mathcal{A}z = \frac{z+42}{7}$. Using triangle inequality, we have

$$\begin{aligned}
c\,|y-\mathcal{A}y| + c\,|z-\mathcal{A}z| + (1-2c)\,|y-z| &= \frac{1}{2}|y-(\frac{y+42}{7})| + \frac{1}{2}|z-(\frac{z+42}{7})| \\
&= \frac{1}{2}|\frac{6y-42}{7}| + \frac{1}{2}|\frac{7z-42}{7}| \\
&\geq \frac{1}{2}|(\frac{6y-42}{7}) - (\frac{6z-42}{7})| \\
&= \frac{1}{2}|\frac{6y-6z}{7}| \\
&= \frac{3}{7}|y-z| \\
&\geq \frac{1}{7}|y-z| \;=\; |\mathcal{A}y-\mathcal{A}z|.
\end{aligned}$$

(ii) When $y \in [7, 9)$ and $z \in \{9\}$. Then, $\mathcal{A}y = \frac{y+42}{7}$ and $\mathcal{A}z = 6$. Accordingly, we have

$$\begin{aligned}
c\,|y-\mathcal{A}y| + c\,|z-\mathcal{A}z| + (1-2c)\,|y-z| &= \frac{1}{2}|y-(\frac{y+42}{7})| + \frac{1}{2}|9-6| \\
&= \frac{1}{2}|\frac{6y-42}{7}| + \frac{1}{2}|3| \\
&\geq \frac{1}{2}|3| \\
&> \frac{9}{7} \\
&\geq \;|\frac{y}{7}| \;=\; |\mathcal{A}y-\mathcal{A}z|.
\end{aligned}$$

(iii) When $y \in [7, 9)$ and $z \in \{9\}$. Then, $\mathcal{A}z = \frac{z+42}{7}$ and $\mathcal{A}z = 6$. Accordingly, we have

$$c\,|y-\mathcal{A}y|+c\,|z-\mathcal{A}z|+(1-2c)\,|y-z| = \frac{1}{2}|9-6|+\frac{1}{2}|z-(\frac{z+42}{7})|$$

$$= \frac{1}{2}|3|+\frac{1}{2}|\frac{6z-42}{5}|$$

$$\geq \frac{1}{2}|3|$$

$$> \frac{7}{5}$$

$$\geq |\frac{z}{5}| = |\mathcal{A}y-\mathcal{A}z|.$$

(iv) When $y, z \in \{9\}$. Then, $\mathcal{A}y = \mathcal{A}z = 6$. Accordingly, we have

$$c\,|y-\mathcal{A}y|+c\,|z-\mathcal{A}z|+(1-2c)\,|y-z| \geq 0$$

$$= |\mathcal{A}y-\mathcal{A}z|.$$

Keeping above cases in mind, one can conclude that F is Reich-Suzuki-type map with $c = \frac{1}{2}$.

On the other hand, $\mathcal{A}$ is not Suzuki type. Because, one can select the values $y = 8$ and $z = 9$, then it is easy to show that $\frac{1}{2}|y-\mathcal{A}y| < 1 = |y-z|$ and $|\mathcal{A}y-\mathcal{A}z| > 1 = |y-z|$. For every natural number $i \geq 1$, select the value of $a_i = 0.95$, $b_i = 0.65$, and $c_i = 0.85$, and starting values is $r_1 = 8.9$. Then we can see in Tables 1 and 2, and Figure 1, that the studied iteration is good as compared to the earlier iterative schemes.

Table 1. Convergence of Hassan, Ullah, and Thakur schemes to the fixed point $y_0 = 7$ of the self-map $\mathcal{A}$ as given in Example 4.1

i	Hassan	Ullah	Thakur
1	8.9	8.9	8.9
2	7.00003282209953	7.00553935860058	7.038775510204088
3	7.00000000106117	7.00003691367664	7.000621764740880
4	7.00000000000003	7.00000023061439	7.000010272109420
5	7	7.00000000132068	7.000000175943570
6	7	7.00000000000702	7.000000003098250
7	7	7.00000000000003	7.000000000055700
8	7	7	7.000000000001020
9	7	7	7.000000000000020
10	7	7	7

Table 2. Convergence of Agarwal, Ishikawa, and Mann schemes to the fixed point $y_0 = 7$ of the self-map $\mathcal{A}$ as given in Example 4.1

i	Agarwal	Ishikawa	Mann
1	8.9	8.9	8.9
2	7.27142857142857	7.27142857142857	7.27142857142857
3	7.03046647230321	7.14679300291545	7.15510204081633
4	7.00352333353166	7.10085777751334	7.11078717201166
5	7.00042244049997	7.07692978437880	7.08704706372345
6	7.00005207225755	7.06223462556685	7.07212470994229
7	7.00000655331132	7.05228555277215	7.06182117995053
8	7.00000083792860	7.04509928354482	7.05425123954842
9	7.00000010848183	7.03966320918897	7.04843860673967
10	7.00000001418545	7.0354060846955	7.04382540609779

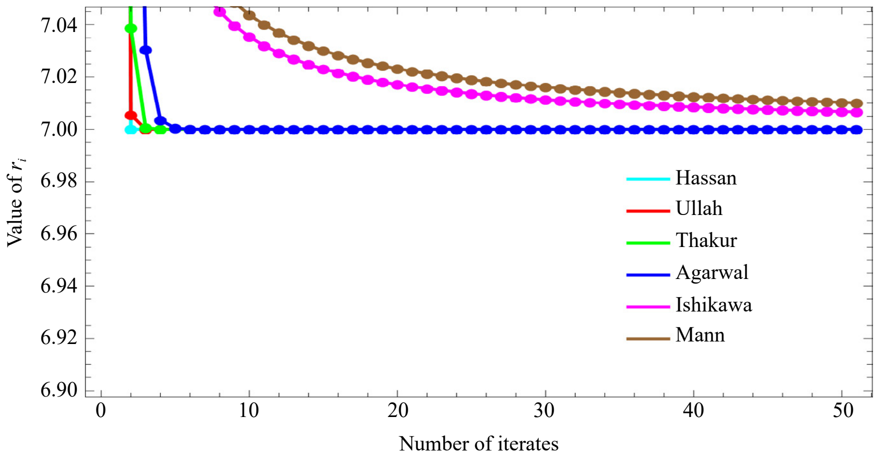

Figure 1. Behaviors of iterates for the schemes due to different authors for Example 4.1, $r_1 = 8.9$, and $i = 50$

Now, we provide a CAT(0) space example, which is not a linear Banach space. After this, we provide an example of Reich-Suzuki-type nonexpansive mapping, which will illustrate our main results.

Example 4.2. Consider the space $\mathbb{R}^2$, and d be its metric induced by the usual norm. Then, $(\mathbb{R}^2, d)$ forms a complete metric space. We now replace d by another type of metric ρ, where ρ is defined by the following formula:

$$\rho((y_1, y_2), (z_1, z_2)) = \begin{cases} |y_1 - z_1| \text{ if } y_2 = 0 = z_2 \\ |y_2 - z_2| \text{ if } y_1 = 0 = z_1 \\ |y_1| + |z_2| \text{ if } y_2 = 0 = z_1. \end{cases}$$

In this case, $(\mathbb{R}^2, \rho)$ forms only a geodesic space CAT(0) but does not form a Banach space. Hence, we take $\mathcal{H}_1 = \{(y, 0) : y \in \mathbb{R}\}$ and $\mathcal{H}_2 = \{(0, z) : z \in \mathbb{R}\}$. Put $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$. It follows that $\mathcal{H}$ is a nonempty closed convex subset of the complete CAT(0) space $(\mathbb{R}^2, \rho)$. Now, let $\mathcal{A}$ be the metric projection on $\mathcal{H}$, then by a well-known result (see, p.178 in [29]) that $\mathcal{A}$ is nonexpansive, and hence, it is Reich-Suzuki-type nonexpansive. By our main results, the sequence of iterates produced in (4) converges to a fixed point of $\mathcal{A}$.

5. Application

We now show that our new outcome is applicable in solving a split feasibility problem (SFP). For this goal, we assume that $\mathcal{D}_1$ and $\mathcal{D}_2$ are two Hilbert spaces, and the subsets $C \subseteq \mathcal{D}_1$ and $Q \subseteq \mathcal{D}_2$ are both closed and convex at the same time. We also assume a bounded and linear map $\mathcal{T}$ on $\mathcal{D}_1$ to $\mathcal{D}_2$. In this case, we define the SFP [36] as a problem such that:

$$\text{Compute } y_0 \in C : \mathcal{T}y_0 \in Q. \tag{18}$$

In our case, we assume that the SFP (18) admits at least one solution, and we denote the set of all solutions by t S. Accordingly, if P^C and P^Q are the nearest point projections (NPP), respectively, on the set C and on the set Q, respectively, the real number θ is greater than zero and $\mathcal{T}^*$ is essentially the adjoin operator for $\mathcal{T}$ It is easy to check that the point $y_0 \in C$ is a solution of the SFP (18) if and only if this y_0 is a solution for the following equation (see, also [37] and others):

$$y = P^C(I - \theta\mathcal{T}^*(I - P^Q)\mathcal{T})y.$$

Since for nonexpansive mappings, the Picard iteration is not well applicable, in the paper [38], the author connected the SFP (18) with the class of nonexpansive mappings and constructed the following iterative scheme for finding its sought solution:

$$r_{i+1} = P^C(I - \theta\mathcal{T}^*(I - P^Q)\mathcal{T})r_i.$$

However, he gets only the weak convergence of the above scheme. But we know that once a weak convergence of an iterative scheme for a certain problem is established, one always requests to obtain a strong convergence [37]. In this article, we carry out a strong convergence based on Reich-Suzuki-type nonexpansive mappings, as opposed to Byrne [38], who used the concept of nonexpansive mappings, and we also use a more general iterative scheme to obtain a strong convergence. Our main result from this section is the following:

Theorem 5.1. Consider the SFP (18) with $S \neq \varnothing, 0 < \theta < \frac{2}{\delta}$, and $P^C(I - \theta\mathcal{T}^*(I - P^Q)\mathcal{T})$ is any Reich-Suzuki-type nonexpansive self-map that satisfies the condition (I). Then, $\{r_i\}$ produced by (4) converges strongly to a solution of the SFP (18).

Proof. It is well-known in the literature that each Hilbert space is a CAT(0) space. So, put $\mathcal{A} = P^C(I - \theta\mathcal{T}^*(I - P^Q)\mathcal{T})$, that is, the map $\mathcal{A}$ is essentially a Reich-Suzuki-type nonexpansive self-map. Thus, one can now apply the Theorem 3.6 to obtain the strong convergence of $\{r_i\}$ in the set $F_{\mathcal{A}}$. But $F_{\mathcal{A}} = S$, so we can say that $\{r_i\}$ converges to a solution of the SFP (18).

6. Conclusion

The existence as well approximation of fixed point for any operator whose domain is a subset of a nonlinear domain like CAT(0) space has its own merit. As mentioned early in the paper that Takahashi [4] was the first person who essentially introduced the idea of convexity in the nonlinear setting of metric space for the study of fixed point theory of nonexpansive operators. This paper essentially introduced the CAT(0) space version of a newly suggested iterative scheme due to Hassan et al. [14]. We studied several convergence theorem in CAT(0) setting for the larger class of nonlinear operators. The main outcome is eventually supported by new examples. Finally, an application to a solve SFP in the more general setting of mappings is obtained.

Availability of data and materials

The data used to support the findings of this study are available from the corresponding author upon request.

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Conflict of interest

The authors declare that they have no competing interests.

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