Research Article

On the Lie-Algebraic Integrability of the Calogero-Degasperis Dynamical System and Its Generalizations

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Abstract: We studied the Lax type integrability of the Calogero-Degasperis nonlinear dynamical system, possessing only one local conserved quantity. Based on the gradient-holonomic integrability approach there are stated tboth the bi-Hamiltonian structure of the Calogero-Degasperis dynamical system and isomorphism of its symmetries group to the semidirect product of the diffeomorphism group of the circle and the abelian group of functions on it. We also constructed a rich algebra of non-Hamiltonian symmetries, related to the Bäcklund transformed general symmetries of the corresponding linearization of the Calogero-Degasperis dynamical system. There is also analyzed in detail the inverse problem of classifying integrable generalized Calogero-Degasperis type dynamical systems *a priori* possessing a finite number of conserved quantities.

1. Introduction

It was observed [1] still at the very beginning of investigating completely integrable Lax type linearized spatially one-dimensional dynamical systems [2-8] on functional manifolds, that they can be divided into two strongly different S- and C-types, specified, respectively, by either existence of a specially ordered infinite hierarchy of local conservation laws on suitably defined [9-12] jet-manifolds, or existence of only finite amount of local conservation laws. Subject to the dynamical systems of the C-type in many cases there was proved that they could be linearized by means of some, in general, nonlocal transformations of functional manifolds. In particular, the differential-algebraic aspects of the nonlinear C-type dynamical system

$$u_t = (u^3)_x + 3/2(u^2)_{xx} + u_{3x},$$

(1)

being a local symmetry of the well known Burgers [13] evolution flow

$$u_t = 2uu_x + u_{2x}, \tag{2}$$

on a smooth 2π- periodic functional manifold $M \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R})$ and the related linearization mapping were studied in detail in [14-17] by means of the gradientholonomic integrability scheme, devised previously in [3, 4, 18]. In addition, in [15] there was generalized the main result of [16] about the Hamiltonian structure of the Burg-ers evolution flow nonlinear dynamical system (1) was demonstrated to be a completely integrable biHamiltonian system on a suitably constructed infinite-dimensional functional manifold, possessing an infinite hierarchy of commuting to each other nonlocal conservation laws. Another nontrivial case of an integrable and linearizable nonlinear C-type dynamical system

$$u_t = u_{3x} + 3(u^2 u_{xx} + 3uu_x^2) + 3u^4 u_x \tag{3}$$

with a finite number of local conservation laws was presented by F. Calogero and A. Degasperis in [1, 19, 20] and later rederived in [21-24], where, in part, there was shown its relation to the so call completely integrable Krichever-Novikov dynamical system

$$\omega_t = \omega_{3x} - 3/4\omega_{xx}^2/\omega_x + 3\omega_{x\omega xx} + \omega_x^3, \tag{4}$$

also linearizable to the simple evolution form

$$\tilde{u}_t = \tilde{u}_{xx} \tag{5}$$

on a smooth 2π-periodic functional manifold $\tilde{M} \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R})$ by means of the following nonlocal mapping:

$$\tilde{u} = u \exp[\omega(x,t)] \tag{6}$$

of the corresponding functional manifolds, where, by definition,

$$d\omega(x,t) = u^2 dx + (2uu_{xx} - u_x^2 + 6u^3 u_x + u^6) dt \tag{7}$$

generates for all $(x, t) \in \mathbb{R} \times \mathbb{R}$ a conserved quantity to the Calogero-Degasperis dynamical system (3), that is $d\gamma_0/dt = 0$ along the evolution flow (3), where, by definition,

$$\gamma_0 := \int u^2 dx. \tag{8}$$

The rich symmetry properties of the Calogero-Degasperis nonlinear dynamical system (3) were later extensively studied by A. Sergyeyev in [25, 26], where there was constructed an infinite centrally-extended Lie algebra of nonuniform and nonlocal master-symmetries to the vector field (3), whose actions on it are completely equivalent [27] to the usual recursion operator ones. A very detailed and comprehensive analysis of the first integrals to the stationary system of (3) and related exact solutions were some years ago studied by M. Euler, N. Euler and P. Leach in [28] by means of the recursion operator tools, previously found in [29, 30]. Moreover, authors of the mentioned above works [26, 28] stressed on importance of finding a direct analytical approach to studying both the nonlocalities of the constructed symmetries and sorting out local and "weakly nonlocal" symmetries, in particular, by deriving the related differential sequence, which has arisen as a happenstance and whose investigations must inevitably be of some intrinsic interest.

Being experienced in studying integrability properties of diverse nonlinear dynamical systems by means of the

gradient-holonomic approach, as described in manuals [3, 4, 18, 31, 32], we analyze an infinite hierarchy of local and nonlocal conservation laws to the CalogeroDegasperis nonlinear dynamical system (3), its bi-Hamiltonian structure and related local and nonlocal symmetries. We also studied in detail the inverse problem of classifying integrable generalized Calogero-Degasperis type dynamical systems *a priori* possessing a finite number of conserved quantities.

2. Conservation laws analysis and Lax type integrability

Consider the Calogero-Degasperis nonlinear dynamical system (3)

$$u_t = u_{3x} + 3(u^2 u_{xx} + 3uu_2 x) + 3u^4 u_x := K[u] \tag{9}$$

as a vector field $K : M \to T(M)$ on a 2π-periodic smooth functional jet-manifold $M \simeq J^3(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}) \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R})$ with respect to the evolution parameter $t \in \mathbb{R}$ and construct the complexified cotangent space $T^*(M) \otimes \mathbb{C}$. Assume that $\mathcal{L} \in \mathcal{D}(M)$ is an arbitrary smooth, in general, nonlinear functional on the scalar functional manifold M and $\operatorname{grad}\mathcal{L}[u] \in T^*(M)$ denotes its Gateau gradient at point $u \in M$. We now recall the following classical [3, 4, 8, 33, 34] Noether-Lax theorem, which will be applied to the Calogero-Degasperis nonlinear dynamical system (9).

Theorem 2.1 Let an element $\psi \in T^*(M)$ satisfy the following linear Noether-Lax functional equation

$$\partial \psi / \partial t + K'^{,*}[u]\psi = \operatorname{grad}\mathcal{L}[u], \tag{10}$$

where $K'[u] : T(M) \to T(M)$ is the Frechet derivative of the vector field $K : M \to T(M)$ at point $u \in M$, $K'^{,*}[u] : T^*(M) \to T^*(M)$ is its adjoint expression with respect to the usual bilinear form $(\cdot|\cdot) : T^*(M) \times T(M) \to \mathbb{R}$. Then any solution to the functional equation (10) is determined up to solutions $\varphi \in T^*(M)$ to the uniform functional equation

$$\partial \varphi / \partial t + K'^{,*}[u]\varphi = 0, \tag{11}$$

whose "symmetric" elements, satisfying the condition $\varphi'[u] = \varphi'^{,*}[u]$, generate conservation laws $\gamma_\varphi \in \mathcal{D}(M)$ to the dynamical system (9) via the relationship $\varphi[u] = \operatorname{grad}\gamma_\varphi[u] \in T^*(M)$ at each point $u \in M$. Simultaneously, any nontrivial solution $\psi \in T^*(M)$ generates a co-Poisson structure $\Omega[u] := \varphi'[u] - \varphi'^{,*}[u] : T(M) \to T^*(M)$ at $u \in M$, subject to which the dynamical system (9) becomes Hamiltonian:

$$K[u] = -\Omega[u]^{-1} \operatorname{grad}H[u] \tag{12}$$

on the scalar functional manifold M with respect to the Hamiltonian functional $H := (\psi|K) - \mathcal{L}$. Moreover, the uniform functional equation (11), if considered on the complexified cotangent space $T^*(M) \otimes \mathbb{C}$, always possesses an asymptotic as a complex parameter $|\lambda| \to \infty$ solution in the following exponential form:

$$\varphi = \exp[\lambda^{s(\varphi)}t + z(x;\lambda)] \tag{13}$$

for some fixed $s(\varphi) \in \mathbb{Z}_+$, where $z(x; \lambda) \sim \sum_{j \in \mathbb{Z}_+} z_j(x)\lambda^{-j+s(z)}$ as $|\lambda| \to \infty$, $s(z) \in \mathbb{Z}_+$, and whose coefficients $z_j \in C^\infty(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{C})$, $j \in \mathbb{Z}_+$, can be easily calculated by means of the respectively derived recurrent differential-functional relationships, depending exclusively on the derivatives $\partial z_j/\partial t$, $\partial z_j/\partial x$, $j \in \mathbb{Z}_+$.

Proof. (Sketch) The statements above follow from classical properties [2, 33, 35, 36] both of smooth vector fields on Poisson manifolds and asymptotic solutions to linear evolution differential equations [3, 5-7, 18, 37, 38], generated by the corresponding Lax type spectral problems with periodic coefficients and meromorphically depending on the spectral parameter $\lambda \in \mathbb{C}$. The spectral and variational analysis of the Floquet monodromy matrix as a generating

functional for conservation laws directly ensues to the Noether-Lax equation (9) and its solution structure in the exponential form (13). □

Remark 2.2 It is worth to remark here that if, in addition, the coefficients $z_j \in C^\infty(J(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}); \mathbb{C}), j \in \mathbb{Z}_+$, then there exist a priori local quantities $\partial z_j/\partial x := \sigma_j[u] \in C^\infty(J(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}); \mathbb{C}), j \in \mathbb{Z}_+$, such that $\partial \sigma_j[u]/\partial t = \partial (\partial z_j/\partial t)/\partial x$, simply meaning that the quantities $\gamma_j := \int_0^{2\pi} \sigma_j[u]dx, j \in \mathbb{Z}_+$, are conserved, that is $\frac{d}{dt}\int_0^{2\pi} \sigma_j[u]dx = 0$ for all $j \in \mathbb{Z}_+$. The latter condition, as the first step, makes it possible to regularly check the existence of mentioned above hidden symmetry structures, in particular local conserved quantities . If obtained this way, they prove to be suitably ordered, this case will strictly certify the evolution flow (9) as a Lax type linearized completely integrable dynamical system on the functional manifold M.

Applying the theorem above to the Calogero-Degasperis nonlinear dynamical system (9), we calculate the operator $K'^{,*}[u] : T^*(M) \to T^*(M)$ as

$$K'^* = -6uu_{xx} - 3u_x^2 - 6uu_x\partial - 3u^4\partial - 3u^2\partial^2 - \partial^3 \tag{14}$$

and construct an exponential solution $\varphi \in T^*(M) \otimes \mathbb{C}$ to the Noether-Lax equation (11) in the form

$$\varphi(x,t;\lambda) = \exp[\lambda^3 t + \partial^{-1}\sigma(x;\lambda)], \tag{15}$$

where the asymptotic as $|\lambda| \to \infty$ expansion

$$\sigma(x;\lambda) \sim \sum_{j=-1}^{\infty} \sigma_j[u]\lambda^{-j} \tag{16}$$

holds. Having substituted the solution (15) into the linear evolution equation (11) and equated coefficients at different powers of the spectral parameter $\lambda \in \mathbb{C}$, we obtain easily an infinite recurrent sequence of functional relationships

$$\delta_{j,-3} + \partial^{-1}\sigma_{j,t} - (6uu_{xx} + 3u_x^2)\delta_{j,0} - (6uu_x + 3u^4)\sigma_j - \sigma_{j,xx}$$

$$-3u^2(\sigma_{j,x} + \sum \sigma_{j-k}\sigma_k) - 3\sum \sigma_{j-k}\sigma_{k,x} - \sum \sigma_{j-k}\sigma_k\sigma_{k-s} = 0, \tag{17}$$

which should be compatible for all $j \in \mathbb{Z}_+ \cup \{-2, -1\}$. By means of simple enough calculations of (17) one finds successively the following local functionals on the functional manifold M:

$$\sigma_{-1} = 1, \ \sigma_0 = -u^2, \ \sigma_1 = -2(u^2)_x, \ \sigma_2 = 2uu_{xx} + 2u_x^2 = (u^2)_{xx},$$

$$\sigma_3 = (-2uu_{xx} - 4u^3u_x - 2/3u^6)_x, \ ..., \ \sigma_n = (...)_x, \ ... \tag{18}$$

and so on for all natural $n \in \mathbb{N}$. As all functionals $\gamma_j := \int \sigma_j[u]dx, j \in \mathbb{Z}_+$, by construction, are conserved quantities, we can derive the only nontrivial local conservation low

$$\gamma_0 = -\int u^2 dx \tag{19}$$

to the Calogero-Degasperis nonlinear dynamical system (9), coinciding with that (8), mentioned before in Introduction. Moreover, the sequence of the conserved densities (18) clearly demonstrates that the solution $\varphi \in T^*(M) \otimes \mathbb{C}$ to the

evolution equation (11) reduces to the following functional form:

$$\varphi(x;\lambda) = \tilde{u}(x;\lambda)\exp[-\omega(x,t)], \quad \omega(x,t) := \partial^{-1}u^2, \tag{20}$$

where the element $\tilde{u} \in \tilde{M} \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R})$ satisfies the following linear evolution equation:

$$\tilde{u}_t = \tilde{u}_{3x}, \tag{21}$$

exactly coinciding with that (5). Moreover, as the solution $\varphi \in T^*(M) \otimes \mathbb{C}$ to the determining linear evolution equation (11) is related, in part, with the conserved quantity $\varphi = 1/2\,\mathrm{grad}\gamma_0[u] = u \in M$, one derives right away the corresponding nonlocal [15, 17] linearizing Bäcklund type transformation $M \ni u \to B[u] := \tilde{u} \in \tilde{M}$, where, by definition,

$$\tilde{u} = B[u] := u\exp(\partial^{-1}u^2), \tag{22}$$

being equivalent for all $(x, t) \in \mathbb{R} \times \mathbb{R}$ with that of (6), mentioned in Introduction.

Return now to the expression (15) in the following equivalent form:

$$\varphi(x,t;\lambda) = f(x;\lambda)\exp(\lambda x - \partial^{-1}u^2 + \lambda^3 t), \tag{23}$$

where, by definition, $f(x; \lambda) := \tilde{u}(x; \lambda)\exp(-\lambda x - \lambda^3 t)$, for all $(x, t) \in \mathbb{R} \times \mathbb{R}$, and take once more into account that the conserved quantity $\varphi = 1/2\mathrm{grad}\gamma_0[u] = u \in M$. Then we easily obtain that

$$f_x = (u^2 + u_x/u - \lambda)f, \tag{24}$$

representing a scalar linear Lax type spectral problem for the Calogero-Degasperis nonlinear dynamical system (9), where one assumes that the element $f \in L_\infty(\mathbb{R}; \mathbb{C})$. The corresponding temporal linear evolution equation on the element $f \in L_\infty(\mathbb{R}; \mathbb{C})$ is obtained upon substituting the expression (23) into the determining Noether-Lax equation (11):

$$f_t = f_{3x} + 3\lambda f_{xx} + 3\lambda^2 f_x, \tag{25}$$

equivalent, evidently, to the linearized relationship (21). The compatibility of the obtained above linear Lax type equations (24) and (25) for all $\lambda \in \mathbb{C}$ gives rise exactly to the Calogero-Degasperis nonlinear dynamical system (9). The described above results can be formulated as the following proposition.

Proposition 2.3 The Calogero-Degasperis nonlinear dynamical system (9) presents a completely Lax type integrable evolution flow on the functional manifold M, linearized by means of two compatible for all $\lambda \in \mathbb{C}$ differential relationships (24) and (25) on the Banach space $L_\infty(\mathbb{R}; \mathbb{C})$.

It is natural now to analyze the corresponding general algebraic structure which is responsible for the linearized form (21) of the Calogero-Degasperis nonlinear dynamical system (9). To do this, let us take the linearizing mapping (22) and assume that it satisfies some linear evolution differential relationship

$$\partial\tilde{u}/\partial t = \sum_{j=0}^{n} c_{n-j}[v]\partial^j\tilde{u}/\partial x^j, \tag{26}$$

of the n-th order, where smooth scalar coefficients $c_j : J(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R}) \to \mathbb{R}$, $j = \overline{0,n}$, can depend on some additional functional parameter $v \in N \subset C(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R})$. From simple enough calculations we obtain the next equivalent differential relationship:

$$\partial u / \partial t + u \partial^{-1} \left[\partial (u^2) / \partial t \right] = \exp(-\partial^{-1} u^2) \sum_{j=0}^{n} c_{n-j}[v] \partial^j [u \exp(\partial^{-1} u^2)] / \partial x^j, \qquad (27)$$

which can be solved with respect to some still unknown adjoint evolution flows

$$\partial u / \partial t = K[u, v], \, \partial v / \partial t = F[u, v] \qquad (28)$$

on the corresponding functional manifolds M and N, if the following integrability condition

$$u \partial u / \partial t = \partial A[u, v] / \partial x \qquad (29)$$

for some smooth mapping $A : J(\mathbb{R} / \{2\pi \mathbb{Z}\}; \mathbb{R}^2) \to \mathbb{R}$ holds. The latter makes it possible to reduce the above differential relationship (27) to the next pure differential form:

$$\partial A / \partial x + 2u^2 A = u \exp(-\partial^{-1} u^2) \sum_{j=0}^{n} c_{n-j}[v] \partial^j [u \exp(\partial^{-1} u^2)] / \partial x^j, \qquad (30)$$

which can be easily solved with respect to the expression $A[u, v]$ as

$$A[u, v] = \exp(-2\partial^{-1} u^2) \partial^{-1} \left(\sum_{j=0}^{n} c_{n-j}[u, v][u \exp(\partial^{-1} u^2)] \partial^j [u \exp(\partial^{-1} u^2)] / \partial x^j \right). \qquad (31)$$

The obtained above integral-differential expression on the functional manifold $M \times N$ will be local as a smooth mapping $A : J(\mathbb{R} / \{2\pi \mathbb{Z}\}; \mathbb{R}^2) \to \mathbb{R}$ if the nonlocal algebraic subintegral expression in (31) is a complete derivative with respect to the spatial variable $x \in \mathbb{R} / \{2\pi \mathbb{Z}\}$, that is

$$\sum_{j=0}^{n} c_{n-j}[v][u \exp(\partial^{-1} u^2)] \partial^j [u \exp(\partial^{-1} u^2)] / \partial x^j = \partial B[u, v] / \partial x, \qquad (32)$$

where, by definition, the nonlocal expression

$$B[u, v] = \sum_{(j \leq k)} b_{j,k}[u, v] \partial^j [u \exp(\partial^{-1} u^2)] / \partial x^j \partial^k [u \exp(\partial^{-1} u^2)] / \partial x^k \qquad (33)$$

with smooth functional coefficients $b_{j,k} : J(\mathbb{R} / \{2\pi \mathbb{Z}\}; \mathbb{R}^2) \to \mathbb{R}, k \geq j \geq 0, j + k = \overline{0, n-1}$, allowing recurrently to retrieve all the coefficients of the determining equation (26). Having substituted the expression (33) into the relationship (31), we finally obtain the searched local vector field

$$K[u, v] = u^{-1} \frac{\partial}{\partial x} \left(\exp(-2\partial^{-1} u^2) \sum_{(j \leq k)} b_{j,k}[u, v] \right.$$

$$\left. \times \partial^j [u \exp(\partial^{-1} u^2)] / \partial x^j \partial^k [u \exp(\partial^{-1} u^2)] / \partial x^k \right) \qquad (34)$$

on the functional manifold M. Concerning the second vector field component $F : M \times N \to T(N)$, it can be determined from the condition that the joint dynamical system (28) allows for some smooth scalar elements $f, g : J(\mathbb{R} / \{2\pi \mathbb{Z}\}; \mathbb{R}^2)$

$\to \mathbb{R}$ the following asymptotic as $|\lambda| \to \infty$ vector solution $\varphi[u, v] \in T^*(M \times N)$

$$\varphi[u, v] = \begin{pmatrix} f[u, v] \\ g[u, v] \end{pmatrix} \exp\left(-\lambda^n t + \lambda x - \partial^{-1}(u^2 + \lambda^{-1}\sigma_1[v])\right) \tag{35}$$

to the corresponding Noether-Lax linear equation

$$\frac{\partial \varphi[u, v]}{\partial t} + \begin{pmatrix} K_u'^{,*}[u, v] & F_u'^{,*}[u, v] \\ K_v'^{,*}[u, v] & F_v'^{,*}[u, v] \end{pmatrix} \varphi[u, v] = 0 \tag{36}$$

on the cotangent space $T^*(M \times N)$, where we have assumed *a priori* that the joint dynamical system (28) possesses only two nontrivial conserved quantities: $\gamma_0 = \int u^2 dx$ and arbitrarily chosen $\gamma_1 = \int \sigma_1[v]dx$, thus passing through the gradient-holonomic integrability criterion.

As an example, let us take the following coefficients: $c_0 = b_{02}[v]$, $c_1 = \partial b_{02}[v]/\partial x$, $c_2 = 2b_{00}[v] - \partial^2 b_{11}[v]/\partial x^2$, $c_3 = \partial b_{00}[v]/\partial x$; the corresponding vector field (34) then gives rise to the following evolution equation

$$\partial u / \partial t = u^{-1}\frac{\partial}{\partial x}\left(b_{00}[v]u^2 + b_{01}[v](uu_x + u^4)\right.$$

$$\left. +b_{02}[v](uu_{xx} + 4u^3u_x + u^6) + b_{11}[v](u_x + u^3)^2\right) \tag{37}$$

on the functional manifold, linearizable via the mapping (22) to the linear equation

$$\partial \tilde{u} / \partial t = \sum_{j=0}^{3} c_{3-j}[v]\partial^j \tilde{u} / \partial x^j, \tag{38}$$

on the related functional manifold $\tilde{M}$. In particular, at $c_0 = 1$, $c_1 = c_2 = c_3 = 0$ the evolution flow (37) reduces to the Calogero-Degasperis nonlinear dynamical system (9) on the functional manifold M. If one puts $b_{00}[v] = v$, $b_{02}[v] = 1$, $b_{11}[v] = 0 = b_{01}[v]$, the resulting vector field (37) reduces to

$$\partial u / \partial t = u^{-1}\frac{\partial}{\partial x}(vu^2) + uu_{xx} + 4u^3u_x + u^6 \tag{39}$$

jointly with the linearizing relationship (38) as

$$\partial \tilde{u} / \partial t = \partial^3 \tilde{u} / \partial x^3 + 2v\partial \tilde{u} / \partial x + v_x \tilde{u}. \tag{40}$$

Concerning the adjoint to (34) vector field $F : M \times N \to T(N)$ we need to choose the functional form of the vector (35) and to take into account that $\varphi[u, v] = (1/2\mathrm{grad}\gamma_0, 1/2\mathrm{grad}\gamma_1)^\top$, or equivalently,

$$u = f \exp\left(-\lambda^n t + \lambda x - \partial^{-1}(u^2 + \lambda^{-1}\sigma_1[v])\right), \tag{41}$$

from which and the relationship (22) one obtains easily the relationship

$$\tilde{u} = f \exp(\lambda x - \lambda^n t - \lambda^{-1}\partial^{-1}\sigma_1[v]). \tag{42}$$

Having substituted (42) into the linearizing relationship (26), we derive a linear differential relationship

$$\partial f / \partial t = \sum_{j=0}^{n} q_{n-j}[v;\lambda]\partial^j f / \partial x, \tag{43}$$

specified by some meromorphic in $\lambda \in \mathbb{C}$ coefficients $q_j : J(\mathbb{R}/\{2\pi\mathbb{Z}\};\mathbb{R})\times\mathbb{C}\to\mathbb{C}, \ j = \overline{0,n},$ being naturally compatible with the next linear differential relationship:

$$\partial f / \partial x = (u_x / u + u^2 + \sigma_1[v]\lambda^{-1} - \lambda)f, \tag{44}$$

easily following from (41). Thus, within the analytic construction above, the system of linear differential relationships (43) and (44) should be *a priori* compatible, giving rise both to the constructed above vector field (34) and to the searched vector field $F : M \times N \to T(N)$ on the functional manifold N, herewith solving answering the question posed above.

3. Bi-Hamiltonian structure and complete integrability

Observe first that the linear evolution equation (21) presents a canonically bi-Hamiltonian flow on the related functional manifold $\tilde{M}$:

$$\tilde{u}_t = -\frac{\partial}{\partial x}\mathrm{grad}\int \tilde{u}_x^2 / 2dx, \ \tilde{u}_t = \frac{\partial^3}{\partial x^3}\mathrm{grad}\int \tilde{u}^2 / 2dx \tag{45}$$

with respect to the a priori compatible Poisson structures $\tilde{\vartheta}, \tilde{\eta} : T^*(\tilde{M}) \to T(\tilde{M})$ on $\tilde{M}$, where

$$\tilde{\vartheta} := \partial / \partial x, \tilde{\eta} := \partial^3 / \partial x^3. \tag{46}$$

Thus, the constructed above nonlocal Burgers type linearizing transformation $M \ni u \to B[u] := \tilde{u} \in \tilde{M}$ (22) of the Calogero-Degasperis nonlinear dynamical system (9) to the linear evolution equation (21) makes it possible to study the Hamiltonian properties of our flow (9) on the functional M by means of the corresponding Bäcklund type [3, 18, 27] transformation. This and related aspects we will analyze in detail below.

Consider the nonlocal Burgers type linearizing transformation $M \ni u \to B[u] := \tilde{u} \in \tilde{M}$, (22), $B[u] = u\exp(\partial^{-1}u^2)$, and calculate the corresponding Bäcklund transformed Poisson operators $\vartheta, \eta : T^*(M) \to T(M)$

$$\vartheta = B'[u]^{-1}\tilde{\vartheta}\left(B'^{,*}[u]\right)^{-1} = u^{-1}\partial \exp(-2\partial^{-1}u^2)\partial^{-1}\exp(\partial^{-1}u^2)u$$

$$\times\partial u \exp(\partial^{-1}u^2)\partial^{-1}\exp(-2\partial^{-1}u^2)\partial u^{-1},$$

$$\eta = B'[u]^{-1}\tilde{\eta}\left(B'^{,*}[u]\right)^{-1} = u^{-1}\partial \exp(-2\partial^{-1}u^2)\partial^{-1}\exp(\partial^{-1}u^2)u$$

$$\times\partial^3 u \exp(\partial^{-1}u^2)\partial^{-1}\exp(-2\partial^{-1}u^2)\partial u^{-1}, \tag{47}$$

where we have used the inverse mapping $B'[u]^{-1} = u^{-1}\partial \exp(-2\partial^{-1}u^2)\partial^{-1} \exp(\partial^{-1}u^2)u : T(\tilde{M}) \to T(M)$. Subject to the Poisson operators (47) the Calogero-Degasperis nonlinear dynamical system (9) is represented as the bi-Hamiltonian system

$$u_t = -\vartheta \text{grad} H_\vartheta = \eta \text{grad} H_\eta \tag{48}$$

with the Hamiltonians

$$H_\vartheta = \frac{1}{2}\int (u_x + u^2)^2 \exp(2\partial^{-1}u^2)dx, \ H_\eta = \frac{1}{2}\int u^2 \exp(2\partial^{-1}u^2)dx, \tag{49}$$

naturally following from the corresponding representations (45).

Another very interesting consequence of the analysis carried out above consists in the fact that the Bäcklund transformed Poisson operators (47) are compatible [2-4, 27, 31, 38] to each other, that is the affine sum $\eta + \lambda\vartheta$ is a Poisson operator all $\lambda \in \mathbb{R}$. The latter, in particular, means that all expressions

$$\vartheta_n := \vartheta(\vartheta^{-1}\eta)^n = u^{-1}\partial \exp(-2\partial^{-1}u^2)\partial^{-1} \exp(\partial^{-1}u^2)u$$

$$\times \partial u \exp(\partial^{-1}u^2)\partial^{-1} \exp(-2\partial^{-1}u^2)\partial u^{-1}$$

$$\times [u^{-1}\partial \exp(-2\partial^{-1}u^2)\partial^{-1} \exp(\partial^{-1}u^2)u\partial^2$$

$$\times u\partial^{-1} \exp(2\partial^{-1}u^2)\partial \exp(\partial^{-1}u^2)u^{-1}]^n \tag{50}$$

are for all $n \in \mathbb{Z}$ Poisson operators on the manifold M, generating an infinite hierarchy of commuting to each other uniform symmetries

$$u_{t_n} = -\vartheta \text{grad}\gamma_n[u] := K_n[u] \tag{51}$$

to the Calogero-Degasperis nonlinear dynamical system (9), where grad $\gamma_n[u] := (\vartheta^{-1}\eta)^n \text{grad}\gamma_0[u] \in T^*(M)$ are nonlocal conserved quantities, generating its infinite hierarchy of nonlocal conservation laws

$$\gamma_n = (-1)^n / 2\int \left(\left[u\exp(\partial^{-1}u^2)\right]_{nx}\right)^2 dx \tag{52}$$

for all $n \in \mathbb{Z}$. Equivalently, all of the symmetries (51) can be constructed [2, 3, 27, 38] by means of the symmetry recursion operator $\Phi := \eta\vartheta^{-1} : T(M) \to T(M)$ action on the uniform seed symmetry $K_0[u] = u_x$:

$$K_n[u] = \Phi^n u_x \tag{53}$$

for all $n \in \mathbb{Z}$, where

$$\Phi = u^{-1} \exp(-\partial^{-1}u^2)\partial \exp(2\partial^{-1}u^2)\partial^{-1}u$$

$$\times \partial^2 u^{-1}\partial \exp(-2\partial^{-1}u^2)\partial^{-1} \exp(\partial^{-1}u^2)u. \tag{54}$$

The constructed above uniform symmetries (51) satisfy *a priori* the following vector field commutator relationships

$$[K, K_n] = 0 = [K_m, K_n] \tag{55}$$

for all $n, m \in \mathbb{Z}$ and compile together an abelian Lie algebra $\mathcal{G}_0 := \text{span}_{\mathbb{R}}\{K_n : M \to T(M) : n \in \mathbb{Z}\}$. Recall now that an arbitrary completely integrable Lax type integrable dynamical system on a functional manifold M possesses [3, 39, 40] an infinite Lie algebra $\mathcal{G}$ of its symmetries, being the semidirect sum $\mathcal{G} = \mathcal{G}_1 \ltimes \mathcal{G}_0$, where the Lie algebra $\mathcal{G}_1 := \{T_j : M \to T(M) : j \in \mathbb{Z}\}$ is compiled from the nonuniform symmetries, satisfying the current Lie algebra [41, 40] relationships

$$[T_j, T_s] = (s - j)T_{j+s}, \quad [T_j, K_s] = (j + s)K_s, \tag{56}$$

being isomorphic to the Lie algebra of the semidirect product group $G := \text{Diff}(\mathbb{S}^1) \ltimes F(\mathbb{S}^1)$ of the diffeomorphism group $\text{Diff}(\mathbb{S}^1)$ and the abelian group $F(\mathbb{S}^1)$ of smooth functions on the circle $\mathbb{S}^1$. The related symmetry generators of the Lie algebra $\mathcal{G}$ one can easily construct if to take into account that they coincide with those ones, Bäcklund transformed from the nonuniform symmetries to the linearized flow (21), that was already mentioned in [26]. Namely, it is easy to check that the following expressions

$$\tilde{T}_s := \tilde{\Phi}^s(x\tilde{u}_x + \tilde{u}/2), \tag{57}$$

for all $s \in \mathbb{Z}$, where, by definition, the symmetry recursion operator $\tilde{\Phi} := \tilde{\eta}\tilde{\vartheta}^{-1} = \partial^2/\partial x^2 : T(\tilde{M}) \to T(\tilde{M})$ generates the nonuniform Lie subalgebra $\tilde{\mathcal{G}}_1$ of symmetries to the linear flow (21) on the functional manifold $\tilde{M}$. Having applied to this symmetry algebra $\tilde{\mathcal{G}}_1$ the related Bäcklund transformation $B[u] = u\exp(\partial^{-1}u^2)$, one easily obtains the nonuniform symmetry subalgebra $\mathcal{G}_1$ to the Calogero-Degasperis nonlinear dynamical system (9):

$$\mathcal{G}_1 = \text{span}_{\mathbb{R}}\{T_j = \Phi^j(xu_x + u/2) \in T(M) : j \in \mathbb{Z}\}, \tag{58}$$

isomorphic to the Lie algebra $\mathcal{G}$ of the diffeomorphism group $G := \text{Diff}(\mathbb{S}^1)$ of the circle $\mathbb{S}^1$, where we took into account that the nonuniform seed symmetry $T_0 = xu_x + u/2 \in T(M)$ is the Bäcklund transform of the nonuniform seed symmetry $\tilde{T}_0 = x\tilde{u}_x + \tilde{u}/2 \in T(\tilde{M})$. In particular, one can check that the moniliform symmetry

$$T_1 = x(u_{3x} + 3u^2u_{xx} + 9uu_x^2 + 3u^4u_x) + 3/2u_{xx} + 5u_xu^2 + 1/2u^5 \tag{59}$$

to the nonlinear dynamical system (9) exactly coincides with that before calculated in [26].

It is now worth to observe that the vector field

$$\tilde{u}_\tau = \tilde{Q}[\tilde{u}], \tag{60}$$

where $\tilde{Q} : \tilde{M} \to T(\tilde{M})$ is an arbitrary constant pseudo-differential expression $\tilde{Q} = \sum_{j \ll \infty} \tilde{q}_j \partial^j/\partial x^j$, represents a symmetry to the linear vector field (21) on the functional manifold $\tilde{M}$, whose Bäcklund transformation

$$u_\tau = B'[u]\tilde{Q}[\tilde{u}]\big|_{\tilde{u}=B[u]} \tag{61}$$

naturally generates additional, yet already abelian Lie algebra nonlocal symmetries to the Calogero-Degasperis nonlinear dynamical system (9). In particular, for the trivial choice $\tilde{Q}[\tilde{u}] = 0$, $\tilde{u} \in \tilde{M}$, one derives the nonlocal symmetry

$$u_\tau = u\exp(-2\partial^{-1}u^2) := q_0[u] \tag{62}$$

to (9), belonging to the Lie algebra $\mathcal{G}_0$, and which was found before in [26] by means of direct calculations. This means that the Lie algebra $\mathcal{G}$ of the semidirect product $G = \text{Diff}(\mathbb{S}^1) \ltimes F(\mathbb{S}^1)$ is equivalent to the corresponding semi-direct sum $\mathcal{G}_1 \ltimes \mathcal{G}_0$, as was suggested in [25], completely solving the difficulty there mentioned: "*Even the description of nonlocalities that occur in thus constructed symmetries, as well as sorting out local and "weakly nonlocal" (depending only on !) symmetries, is quite a difficult task, and we intend to analyze this and related problems elsewhere*". The obtained above results can be formulated as the following proposition.

Proposition 3.1 The Calogero-Degasperis nonlinear dynamical system (9) is a Lax type linearized and completely integrable bi-Hamiltonian evolution flow on the functional manifold M, possessing an infinite hierarchy of nonlocal conservation laws, generating the related commuting to each other local Hamiltonian symmetries (51). The direct sum extended Lie algebra $\mathcal{G}_{ext} := \mathcal{G} \oplus q_0[u]\mathbb{R}$ of uniform and nonuniform symmetries is isomorphic to the Lie algebra of the semidirect product group $G = \text{Diff}(\mathbb{S}^1) \ltimes F(\mathbb{S}^1)$ of the diffeomorphism group $\text{Diff}(\mathbb{S}^1)$ and the abelian group $F(\mathbb{S}^1)$ of smooth functions on the circle $\mathbb{S}^1$.

The functional structure of the constructed above Lie algebra $\mathcal{G} = \mathcal{G}_1 \ltimes \mathcal{G}_0$ of symmetries to the Calogero-Degasperis nonlinear dynamical system (9) on manifold M makes it possible to classify them subject to their dependence on the nonlocal factor $\exp(-2\partial^{-1}u^2)$, strictly entering the corresponding conservation laws. Namely, it is easy to observe that the symmetry recursion operator (54) is a completely local on the functional manifold M, thus producing via the relationships (52) and (58) strictly local both uniform and nonuniform symmetries to (9). The latter answers a question posed in [26] concerning the related sorting out local and nonlocal symmetries. Another logically natural problem, related with the carried out above integrability analysis of the Calogero-Degasperis nonlinear dynamical system (9), consists in solving the inverse problem of classification of all smooth nonlinear Lax type integrable dynamical systems on the functional manifolds, possessing only a finite number of conservation laws and allowing the corresponding Burgers type linearization. This problem was in part already analyzed in the works [15, 42] and will be solved in the next Section.

4. Inverse problem for the Lax integrable and Burgers type linearized dynamical systems

Following a differential-algebraic approach devised in the work [42], we consider the ring $\mathcal{K} := C^\infty(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathbb{R})$ of real-valued 2π-periodic smooth functions and construct the associated differential quotient ring $\mathcal{K}\{u\} := Quot(\mathcal{K}[\Theta u])$ with respect to a chosen functional vector-variable $u \in \mathcal{K}^m$, where Θ denotes [43-48] the standard monoid of the derivation $\partial/\partial x, x \in \mathbb{R}/2\pi\mathbb{Z}$. An ideal $I\{u\} \subset \mathcal{K}\{u\}$ is called differential if the condition $I\{u\} = \Theta I\{u\}$ holds.

Consider the differential ring $\mathcal{K}\{u\}, u \in \mathcal{K}^m$, and the exterior derivation $d : \mathcal{K}\{u\} \to \Lambda^1(\mathcal{K}\{u\}), ..., d : \Lambda^p(\mathcal{K}\{u\}) \to \Lambda^{p+1}(\mathcal{K}\{u\})$ for $p \in \mathbb{Z}_+$, acting on the freely generated Grassmann algebras $\Lambda(\mathcal{K}\{u\}) = \oplus_{p \in \mathbb{Z}_+}\Lambda^p(\mathcal{K}\{u\})$, where by definition,

$$\Lambda^1(\mathcal{K}\{u\}) := \mathcal{K}\{u\}dx + \sum_{k=\overline{1,m}, j \in \mathbb{Z}_+} \mathcal{K}\{u\}du_k^{(j)},$$

$$\Lambda^2(\mathcal{K}\{u\}) := \mathcal{K}\{u\}d\Lambda^1(\mathcal{K}\{u\}), ...,$$

$$\Lambda^{p+1}(\mathcal{K}\{u\}) := \mathcal{K}\{u\}d\Lambda^p(\mathcal{K}\{u\}), \tag{63}$$

and $u_k^{(j)} := \partial^j u_k / \partial x^j, k = \overline{1,m}, j \in \mathbb{Z}_+$. The triple $\mathcal{A} := (\mathcal{K}\{u\}, \Lambda(\mathcal{K}\{u\}), d)$ is usually called the *Grassmann differential algebra* [46], generated by a fixed vector-variable $u \in \mathcal{K}^m$. In the algebra $\mathcal{A}$ one naturally defines the actions of derivations $\partial := \partial/\partial x$ and $D := dx \wedge \partial / \partial x : \mathcal{A} \to \mathcal{A}, k = \overline{1,m}, j \in \mathbb{Z}_+$, as follows:

$$\partial u_k^{(j)} = u_k^{(j+1)}, \partial du_k^{(j)} = du_k^{(j+1)}$$

$$D\left(\sum_{j\geq 0}\langle P_j[u]\,|\,du^{(j)}\rangle\right) = dx \wedge \sum_{j\geq 0}\langle\langle P'_{j,u}[u]\,|\,u^{(1)}\rangle\,|\,du^{(j)}\rangle$$

$$+\sum_{j\geq 0}\langle P_j[u]\,|\,dx \wedge du^{(j+1)}\rangle, \tag{64}$$

where the sign "$\wedge$" denotes the standard [47] exterior multiplication in the differential Grassmann algebra $\Lambda(\mathcal{K}\{u\})$, and for any $P \in \Lambda(\mathcal{K}\{u\})$ the mapping

$$P'_u[u] : \Lambda(\mathcal{K}\{u\}) \to \Lambda(\mathcal{K}\{u\})^m, \tag{65}$$

denotes the usual Frechet derivative with respect to the vector $u \in \mathcal{K}^m$. In addition, the following commutation properties

$$\partial d = d\partial, \; Dd + dD = 0 \tag{66}$$

hold on the Grassmann differential algebra $\mathcal{A}$.

The following remark [46] proves to be important.

Remark 4.1 The Lie derivative $L_V : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$ with respect to a vector field $V : \mathcal{K}\{u\} \to T(\mathcal{K}\{u\})$, satisfying the condition $L_V : \mathcal{K} \subset \mathcal{K}$, can be uniquely extended to the derivation $L_V : \mathcal{A} \to \mathcal{A}$, satisfying the commutation condition $L_V d = dL_V$.

The following lemma, stated for a special case in [46] is also useful for applications.

Lemma 4.2 The next relationship

$$\mathrm{Ker}\,\partial\,/\,(\mathrm{Im}\,\partial \oplus \mathbb{R}) \simeq H^1(\Lambda(\mathcal{K}\{u\})) :=$$

$$\mathrm{Ker}\{d : \Lambda^1(\mathcal{K}\{u\}) \to \Lambda^2(\mathcal{K}\{u\})\}\,/\,d\Lambda^0(\mathcal{K}\{u\}), \tag{67}$$

is a canonical isomorphism, where $H^1(\mathcal{A})$ is the corresponding cohomology class of the Grassmann complex $\Lambda(\mathcal{K}\{u\})$.

It is well known that in the case of the free differential ring $\mathcal{K}\{u\}$ all of the cohomology classes $H^j(\mathcal{A})$, $j \in \mathbb{Z}_+$, are trivial, giving rise to the well known [46, 47] classical Poincare type relationship

$$\mathrm{Ker}\,d = \mathrm{Im}\,d \oplus \mathbb{R}. \tag{68}$$

Moreover, if to take into account that the anti-derivation $d = d_u \oplus d_x$, subject to which the splitting $\Lambda^p(\mathcal{K}\{u\}) = \Lambda^{p,0}(\mathcal{K}\{u\}) \oplus \Lambda^{p-1,1}(\mathcal{K}\{u\})$, $p \in \mathbb{Z}_+$, generates two exact complexes

$$\Lambda^{0,0}(\mathcal{K}\{u\}) \xrightarrow{d_u} \Lambda^{1,0}(\mathcal{K}\{u\}) \xrightarrow{d_u} \Lambda^{2,0}(\mathcal{K}\{u\}) \xrightarrow{d_u} \ldots \xrightarrow{d_u} \Lambda^{n,0}(\mathcal{K}\{u\}) \xrightarrow{d_u} \ldots,$$

$$\Lambda^{0,1}(\mathcal{K}\{u\}) \xrightarrow{d_u} \Lambda^{1,1}(\mathcal{K}\{u\}) \xrightarrow{d_u} \Lambda^{2,1}(\mathcal{K}\{u\}) \xrightarrow{d_u} \ldots \xrightarrow{d_u} \Lambda^{n,1}(\mathcal{K}\{u\}) \xrightarrow{d_u} \ldots, \tag{69}$$

since $d_u^2 = 0$. If to define the anti-derivation $D := dx \wedge \partial/\partial x : \Lambda(\mathcal{K}\{u\}) \to \Lambda(\mathcal{K}\{u\})$, then the for any $f[x;u] \in \Lambda^{p,k}(\mathcal{K}\{u\})$, $k = \overline{0,1}$, one can define the gradient mapping grad : $\Lambda^p(\mathcal{K}\{u\}) \in \Lambda^p(\mathcal{K}\{u\})^m$ by means of factorization as grad $f[x;u] := d_u f[x;u]/D\Lambda^{p,k-1}(\mathcal{K}\{u\}) = f_u'^*[x,u]$ (1), subject to which the following relationship

$$\mathrm{grad}\,(\mathrm{Im}\,\partial) = 0, \tag{70}$$

holds. Based on Lemma 4.2 one can define the following equivalence class $\Lambda^{p,0}/\{\mathrm{Im}\partial \oplus \mathbb{R}\} := \mathcal{D}(\mathcal{A}; dx)$, whose elements will be called *functionals*, as any element $\gamma \in \mathcal{D}(\mathcal{A}; dx)$ can be represented as a suitably defined integral element $\gamma := \int dx \gamma[u] \in \mathcal{D}(\mathcal{A}; dx)$ for some $\gamma[u] \in \Lambda^{p,0}(\mathcal{K}\{u\})$ with respect to the Lebesgue measure dx on $\mathbb{R}/\{2\pi\mathbb{Z}\}$.

Proceed now to treating the case when the following uniform in the variables $x \in \mathbb{R}/\{2\pi\mathbb{Z}\}$ differential constraint

$$\partial u / \partial t = K[u] \tag{71}$$

is imposed on a vector-variable $u \in \mathcal{K}^m$, where a smooth element $\mathcal{K}[u] \in \mathcal{K}\{u\}^m$ can be considered as a vector field $\partial/\partial t$: $\mathcal{K}\{u\} \to T(\mathcal{K}\{u\})$ on $\mathcal{K}\{u\}$, defined by means of the following differential-algebraic condition:

a) the vector field (71) is for some natural $r \in \mathbb{N}$ an r-differential order expression,

b) it is Lax linearizable on the Banach space $L_\infty(\mathbb{R}; \mathbb{R}^m)$ and

c) it possesses only a finite number $s \in \mathbb{N}$ of suitably ordered functionally independent conserved quantities $\gamma_j[u] \in \mathcal{D}(\mathcal{A}; dx)$, $j = \overline{1, s}$, for which $\partial \gamma_j[u]/\partial t = 0 \in \mathcal{D}(\mathcal{A}; dx)$, $j = \overline{1, s}$.

Taking into account the Noether-Lax Theorem 2.1, applied to the vector field (71) on the differential ring $\mathcal{K}[u]$, we assume that the corresponding complexified vector $\varphi(x; \lambda) \in T^*(\mathcal{K}\{u\}) \otimes \mathbb{C}$ satisfies the Noether-Lax determining relationship

$$\partial \varphi / \partial t + K'^{,*}[u]\varphi = 0 \tag{72}$$

for all $x \in \mathbb{R}/\{2\pi\mathbb{Z}\}$ and $\lambda \in \mathbb{C}$. A general solution to (72), following the reasonings, devised in [3, 4, 18, 31], can be representable in the following exponential vector form:

$$\varphi(x; \lambda) = (1, a(x; \lambda))^\top \exp[\lambda^r t + \partial^{-1}\sigma(x; \lambda)], \tag{73}$$

depending on the spectral parameter $\lambda \in \mathbb{C}$ for some natural $r \in \mathbb{N}$, subject to which the following asymptotic as $|\lambda| \to \infty$ expansions

$$a(x; \lambda) \sim \sum_{j \in \mathbb{Z}_+} a_j[u]\lambda^{-j}, \quad \sigma(x; \lambda) \sim \sum_{j \in \mathbb{Z}_+ \cup \{-1\}} \sigma_j[u]\lambda^{-j} \tag{74}$$

hold for $k = \overline{1, m-1}$ and $x \in \mathbb{R}/\{2\pi\mathbb{Z}\}$. Taking into account that *a priori* all functionals $\sigma_j[u] \in \mathcal{K}\{u\}, j \in \mathbb{Z}_+ \cup \{-1\}$, are conserved quantities with respect to vector field (71) and, moreover, their amount is finite, we can assume with no loss of generality that the conservation laws are ordered as

$$\sigma_{-1}[u] = \gamma_1[u], \sigma_0[u] = \gamma_2[u], \sigma_1[u] = \gamma_3[u], ..., \sigma_{s-2}[u] = \gamma_s[u], \tag{75}$$

yet, simultaneously,

$$\sigma_{j+s-2}[u] = 0 \in \mathcal{D}(\mathcal{A}; dx) \tag{76}$$

for all $j \in \mathbb{N}$. Based now on the relationships (73), (75) and (76) one obtains the following gradient representation

$$\mathrm{grad}\gamma[u; \lambda] = \tilde{u}(x; \lambda) \exp\left(\partial^{-1}\gamma[u; \lambda]\right), \tag{77}$$

where we put, by definition,

$$\gamma[u;\lambda] := \sum_{k=1,s} \lambda^{2-k} \partial^{-1} \gamma_k[u],$$

$$\tilde{u}(x;\lambda) := (1, a(x;\lambda)^\mathsf{T} \exp\left(\lambda^r t + \sum_{j \geq s-1} \partial^{-1} \sigma_j[u] \lambda^{-j} \right) \tag{78}$$

Taking now into account that $\varphi(x;\lambda) = \text{grad}\,\gamma(x;\lambda)$ for all $x \in \mathbb{R}/\{2\pi\mathbb{Z}\}$ and $\lambda \in \mathbb{C}$ satisfies the determining Noether-Lax differential relationship (72), we can *a priori* obtain the resulting linear differential equations on the vectors $\tilde{u} \in \mathcal{K}\{u; \partial^{-1}\gamma_1[u], \partial^{-1}\gamma_2[u], ..., \partial^{-1}\gamma_s[u]\}$, $k = \overline{1,s}$:

$$\partial\tilde{u} / \partial t = \tilde{q}[u;\lambda]\tilde{u}, \tag{79}$$

where the linear matrix differential expressions

$$\tilde{q}[u;\lambda] = \sum_{k=0,r} \tilde{q}_k[u;\lambda]\partial^k / \partial x^k, \tag{80}$$

are equipped with coefficients $\tilde{q}_k[u;\lambda] \in \mathcal{K}\{u\}$, $k = \overline{1,s}$, depending, in general, on the vector-variable $u \in \mathcal{K}^m$.

The obtained above result can be naturally reinterpreted as the canonical Bäcklund type linearization mapping $M \ni u[u;\lambda] = \tilde{u} \in \tilde{M}$, ensued from the functional relationship (77) in the special exponential form

$$\tilde{u} = \text{grad}\,\gamma[u;\lambda]\exp(-\partial^{-1}\gamma[u;\lambda]) \tag{81}$$

and applied to the differential evolution constraint (71), considered on the functional manifold $M \simeq \mathcal{K}\{u\}$. The derived a priori linear dynamical system (79) is naturally considered on the constructed above functional manifold $\tilde{M} \simeq \mathcal{K}\{\tilde{u}\}$, which can be retrieved under assumption that Bäcklund type mapping $M \ni u \rightarrow B[u;\lambda] = \tilde{u} \in \tilde{M}$ is invertible.

Respectively, the inverse problem of constructing the differential constraints like (71) is successively solved the following algorithmic way: one takes an arbitrary linear differential expression (80) on the functional manifold $\tilde{M}$ with coefficients $\tilde{q}_k[u;\lambda] \in \mathcal{K}\{u\}$, $k = \overline{0,r}$, which should be found from the Bäcklund type relationship (81) and conditions that the *a priori* given quantities $\gamma_k[u] \in \mathcal{D}(\mathcal{A}; dx)$, $k = \overline{1,s}$, are conserved, that is $\partial\gamma_k[u] / \partial t \big|_{u=B^{-1}[\tilde{u}]} = 0 \in \mathcal{D}(\mathcal{A}; dx)$, $k = \overline{1,s}$. As the next step one needs to calculate the ensued from (77) the quasi-linear spectral problem

$$\partial\tilde{u} / \partial x = -\gamma[u;\lambda]\tilde{u} + \exp(\partial^{-1}\gamma[u;\lambda])\partial\text{grad}\,\gamma[u;\lambda] / \partial x \tag{82}$$

on the vector $\tilde{u} \in \tilde{M}$.

Remark 4.3 It is worth remarking here that the obtained systems of quasi-linear equations (79) and (82) should be *a priori* compatible, equivalently reducing to the searched for differential relationship (71) on the functional vector-variable $u \in M$.

From practical point of view we need to ensure that a priori chosen quantities $\gamma_k[u] \in \mathcal{D}(\mathcal{A}; dx)$, $k = \overline{1,s}$, are really conserved with respect to some evolution derivative $\partial/\partial t$ on the ring $\mathcal{K}\{u\}$. Having applied the constructed above inverse Bäcklund type linearizing mapping (81) to the chosen above set of conserved quantities, we obtain a new set $\tilde{\gamma}_k[\tilde{u}] := \gamma_k[u]\big|_{u=B^{-1}[\tilde{u}]} \in \mathcal{D}(\tilde{\mathcal{A}}; dx), k = \overline{1,s}$, of conserved quantities subject to a suitably chosen linear differential equation (79) on the related functional manifold $\tilde{M} \simeq \mathcal{K}\{\tilde{u}\}$. By means of direct checking the conservation conditions $\partial\tilde{\gamma}_k[\tilde{u}] / \partial t = 0 \in \mathcal{D}(\tilde{\mathcal{A}}; dx), k = \overline{1,s}$, we successively retrieve all unknown coefficients (80) entering the chosen linear differential equation (79).

5. Examples: modified Burgers and Calogero-Degasperis type integrable dynamical system

5.1 *A modified Burgers type dynamical system*

As a simple, yet interesting, example we put, by definition, that a searched integrable dynamical system (71) on the differential ring $\mathcal{K}\{u_1, u_2\}$ possesses only two nontrivial conserved quantities $\gamma_2[u] = u_1$, $\gamma_3[u] = u_2 \in \mathcal{D}(\mathcal{A}; dx)$, that is $\partial \gamma_k[u] / \partial t = 0 \in \mathcal{D}(\mathcal{A}; dx)$, $k = \overline{2, 3}$, subject to the corresponding searched flow

$$\frac{\partial}{\partial t}(u_1, u_2)^\top = \mathcal{K}[u_1, u_2]. \tag{83}$$

We will put, a priori, that the differential order of the flow (83) equals $r = 2$. The related canonical linearizing Bäcklund type transformation (81) is reduced to the following relationship

$$\tilde{u} = \exp(\lambda x + \partial^{-1} u_1), \tag{84}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, and whose evolution with respect to the temporal derivation $\partial/\partial t : \mathcal{K}\{u_1, u_2\} \to \mathcal{K}\{u_1, u_2\}$ can be chosen as a linear differential relationship

$$\partial \tilde{u} / \partial t = \tilde{u}_{xx} + \tilde{q}_1[u]\tilde{u}_x + \tilde{q}_2[u]\tilde{u}, \tag{85}$$

guaranteeing the existence of two conserved quantities $\gamma_2[u] = u_1$ and $\gamma_2[u] = u_2$. The latter gives rise to the coefficients $\tilde{q}_1[u] = 0$ and $\tilde{q}_2[u] = u_2$ under condition that the following Burgers type evolution flow

$$\left. \begin{array}{l} \partial u_1 / \partial t = \partial^2 u_1 / \partial x^2 + 2u_1 \partial u_1 / \partial x + \partial u_2 / \partial x, \\ \partial u_2 / \partial t = u_1 \partial u_2 / \partial x + u_1 \partial u_2 / \partial x \end{array} \right\} = K[u_1, u_2] \tag{86}$$

holds on the ring $\mathcal{K}\{u_1, u_2\}$. Thus, we stated that the nonlinear evolution flow (86) on the functional manifold $M \simeq \mathcal{K}\{u_1, u_2\}$ is a compatibility condition of the following Lax type linear system:

$$\partial \tilde{u} / \partial t = \tilde{u}_{xx} + u_2\tilde{u}, \; \partial \tilde{u} / \partial x = (\lambda + u_1)\tilde{u} \tag{87}$$

for any $\lambda \in \mathbb{C}$. As a simple consequence from the results above we derive that the modified Burgers type dynamical system (86) is also a bi-Hamiltonian flow on the functional manifold M, possessing only a one local and no nonlocal conserved quantities. As the Burgers type evolution flow (86) possesses a symmetry recursion operator

$$\Phi = \begin{pmatrix} \partial + \partial u \partial^{-1} & 1 \\ \partial v \partial^{-1} & 0 \end{pmatrix}, \tag{88}$$

it generates an infinite hierarchy of commuting to each other generalized Burgers type evolution flows

$$K_n[u_1, u_2] := \Phi^n \; (\partial u_1 / \partial x, \partial u_2 / \partial x)^\top \tag{89}$$

on the functional manifold M, also linearizable by means of the linearizing relationship (84) for all $n \in \mathbb{N}$.

5.2 *A modified Calogero-Degasperis type dynamical system*

Take the unique conserved quantity $\gamma_1[u] = u^2 \in \mathcal{D}(\mathcal{A}; dx)$ and look for a nonlinear Lax type integrable dynamical system

$$\partial u / \partial t = K[u], \qquad (90)$$

of the differential order $r = 1$, subject to which the related canonical Bäcklund type transformation $M \ni u \to B[u] = \tilde{u} = \exp(\partial^{-1}u^2) \in \tilde{M}$ reduces (90) to the *a priori* linear second order evolution differential relationship

$$\partial \tilde{u} / \partial t = q_0[u]\, \partial \tilde{u} / \partial x + q_1[u]\tilde{u} \qquad (91)$$

on the functional manifold $\tilde{M}$ with still unknown coefficients $q_j[u] \in \mathcal{K}\{u\}$, $j = \overline{0,1}$. Having imposed on the evolution relationship (91) the conserved quantity constraint $\partial u^2/\partial t = 0 \in \mathcal{D}(\mathcal{A}; dx)$, one easily obtains that $q_0[u] = 1$, $q_1[u] = 0$,

$$\partial \tilde{u} / \partial t = \partial^2 \tilde{u} / \partial x^2 \qquad (92)$$

if the following evolution flow

$$\partial u / \partial t = \partial u / \partial x \qquad (93)$$

holds on the functional manifold M. A similar problem, posed for an even order $r = 2k$, $k \in \mathbb{N}$, the related nonlinear Lax type integrable dynamical systems (90), prove to be incompatible, yet if its order $r = 2k + 1$, $k \in \mathbb{N}$, is odd, then the corresponding linearizing equations become compatible and reducing to the integrable hierarchy (53), constructed above in Section 2 and possessing an infinite hierarchy both local and nonlocal conserved quantities.

We can now summarize the results obtained above as the following proposition.

Proposition 5.1 The modified Burgers and Calogero-Degasperis type dynamical systems (86) and (53) are Lax integrable bi-Hamiltonian flows on the functional manifold M and possess only a one local but an infinite hierarchy of nonlocal conserved quantities.

Similar way can be constructed other Lax type linearizable nonlinear dynamical systems on functional manifolds, possessing only a finite number of conserved quantities but an infinite hierarchy of nonlocal conserved quantities, including both the Burgers type flow (1), the Calogero-Degasperis (3) and the modified Burgers and Calogero-Degasperis type flows (86), (93), analyzed above.

6. Conclusions

We presented a detailed symmetry and integrability analysis of the Calogero-Degasperis nonlinear dynamical system, possessing only one local conserved quantity and Lax type linearized. In particular, based on the gradient-holonomic integrability scheme, we have stated its bi-Hamiltonian structure and relationship of local symmetries to the semidirect product of the diffeomorphism group of the circle and the abelian group of functions on it. We classified all nonlocal symmetries, generated by a special abelian Hamiltonian symmetry. There was constructed a rich algebra of non-Hamiltonian symmetries, related with the Bäcklund transformed general symmetries of the corresponding linearization of the Calogero-Degasperis dynamical system. We have also analyzed the inverse problem of classifying Lax type integrable nonlinear dynamical systems *a priori* possessing a finite number of conserved quantities and presented some interesting examples of modified Burgers and Calogero-Degasperis type flows on functional manifolds.

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Conflict of interest

The authors declare that they have no conflict of interest.

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