Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Optimal Control Model of University-PhD-Postdoc-Industry (UPPI) Migration and Unemployment

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Received: 6 May 2023; **Revised:** 18 July 2023; **Accepted:** 25 July 2023

Abstract: This study explores the impact of the movement of PhD graduates and postdoctoral research fellows between academia and industry on unemployment within the academic and industry sectors. To achieve this, an optimal control model is analyzed which was developed for the population of the university, PhDs holders, postdocs, and industry compartments. The study discovered that, by offering incentives to PhD graduates and postdocs who choose to stay in academia rather than transition to industry, unemployment in the university sector can be reduced. Based on the findings, the authors advise for the governments to concentrate on offering these incentives to PhD holders and postdocs to persuade them to stay in academia. Policymakers can lower unemployment rates in both the academic and industrial sectors by putting in place measures that promote the retention of PhD holders and postdocs in academics and control their migration to the industry.

Keywords: Hamiltonian, optimal control, adjoint variables, PhD holders, postdocs, university, industry

MSC: 34H05, 65L05

1. Preliminaries

Unemployment is a significant challenge facing many countries and is often seen as a negative indicator of the health of an economy. Over the years, researchers have explored various techniques to control unemployment levels and promote economic stability. The share of skilled individuals who want to work but are unable to do so owing to a lack of available jobs is known as the unemployed population, or the active population. In general, unemployment is a risky social situation since some people typically struggle to maintain a minimal standard of living and welfare.

Regarding current research on unemployment by Adeniji et al. [1], the authors worked on unemployment model by considering four compartment i.e. university, PhD, postdocs and industry, with two concentrating key participants of the compartments which is PhD's and postdocs. The authors focused their attention on the migration movement between the university and industry compartments. They provided a scenario in which the population within university compartments remains stable, limiting the rate at which persons with PhDs migrate to industrial compartments and therefore lowering unemployment. Equilibrium of the model was analyzed, using the stability theory for differential equations, and performed a numerical simulations by increasing the rate at which PhD holders are absorbed in the university compartments, and

DOI: https://doi.org/10.37256/cm.5220242986

concluded that higher absorption of PhD holders reduces unemployment in the postdocs and industry compartments and increasing the in migration rate to the university. However, in order for this to occur effectively, the optimal control model was suggested for future work, which was not considered initially, but is investigated in this research and the results is observed.

One approach that has gained popularity in recent times is the use of optimal control methods. This has been implemented by researchers to COVID-19 [2], typhoid [3, 4], and lymphatic filariasis [5]. The use of optimal control has been used to tackle unemployment in several countries such as Bangladesh [6], Columbia [7]. The implementation is not limited only to areas such as controlling the population of criminals in a setting with limited resources [8]. To note, mathematical modelling helps by implementing the effective optimal control model by measuring the control strategies of situations either in measuring the impact of vaccination [9], infectious diseases such as Mpox [10], and measures the population rate [11]. This is to say, it is an imperative knowledge that helps with prediction and understanding the dynamics of situation. A case study on the approach of optimal control on unemployment was investigated in Portugal to discuss the suitable policies to avoid unemployment [12].

Other researchers have introduced the use of optimal control methods to investigate unemployment in different research area and countries. The optimal control method has been used to understudy the financial and used as an approach to reduce unemployment through public investment [13]. Authors like Munoli et al. [14], strive to identify an ideal control strategy for unemployment through two metrics which are the government policies that aim to create new openings and government policies that are focused directly on hiring unemployed people. Optimal control is a mathematical framework that aims to find the best way to control a system or process over a given time horizon. In the context of unemployment, optimal control methods can be used to determine the optimal policies that can minimize the unemployment rate. Mallick et al. [15] and Biswas et al. [16], used the optimal control strategy to minimize unemployment by describing a nonlinear mathematical model of unemployment and analyze the impact of creating new vacancies and providing skilled manpower. They use Pontryagin's maximum principle to evaluate different constant control strategies and determine the best approach to minimize the number of unemployed persons. It is also imperative to note that the control methods or strategy is not limited to the unemployment model but is also used in the analysis of infectious models [17, 18].

It has been found that, optimal control methods have been used to implement the best policies to reduce unemployment such as investment in education and training, improvement of labor market flexibility, and support for entrepreneurship. This has informed several decisions by governments and entrepreneurs that these policies would lead to a substantial reduction in each country's unemployment rate. Optimal control methods have been used to determine the optimal unemployment benefits policy in a dynamic model of labor market, benefits, and policy. The model is balanced by the use or introduction of incentives to create new jobs which reduces unemployment. However, it is imperative to note that the optimal policy would lead to lower unemployment and higher employment, and therefore, greater economic stability and it helps in fields such as infectious disease and vaccination [19]. The use of optimal control methods has been investigated in order to demonstrate the relationship between unemployment and inflation in an open economy, which would aid in the reduction of unemployment and the improvement of economic stability.

In conclusion, the use of optimal control methods in the context of unemployment has been shown to be effective in finding the best policies to reduce unemployment and promote economic stability. The paper is structured as follows: Section 2 talks about the model formulation that includes optimal control parameters, Section 2.1.1 talks about the impacts of the incentives on the compartments, Section 3 discusses the solution to the optimal control problem, and Section 4 investigates the simulation and discusses the results.

2. Model formulation

The previous study in [1] developed and investigated a mathematical model on the university, PhDs, postdocs, and industry dynamics. Precisely, the following non-linear mathematical model was considered and the parameters are described in [1]:

$$\frac{dU(t)}{dt} = \Lambda - \beta \cdot P_h(t) \cdot U(t) + \alpha_4 \cdot I(t) + \alpha_5 \cdot P_h(t) + \alpha_6 P_D(t) - \mu \cdot U(t),$$

$$\frac{dP_h(t)}{dt} = \beta \cdot P_h(t) \cdot U(t) - \alpha_1 \cdot P_h(t) - \alpha_2 \cdot P_h(t) - \alpha_5 \cdot P_h(t) - \mu \cdot P_h(t),$$

$$\frac{dI(t)}{dt} = \alpha_2 \cdot P_h(t) + \alpha_3 \cdot P_D(t) - \alpha_4 \cdot I(t) - \mu \cdot I(t). \tag{1}$$

The model in (1) is motivated as follows: The population in the university community increases through the admission of PhD students at the rate Λ, migration from the industry, and employment of PhD graduates and postdoctoral scholars at the rates of α_4, α_5, α_6, respectively. PhD students graduate and move to the PhD compartment at the rate β and progress to the postdoctoral, industry, and university classes at the rate of α_1, α_2, and α_5 respectively. Individuals in the postdoctoral class progress to the industry and university at the rate α_3 and α_6, respectively. The number of individuals in the industry class increased through the migration from the PhD graduates compartment at the rate of α_2 and the postdoc candidates at the rate α_3, while the number of individuals in the industry moved back to academia at the rate α_4. The study assumes that individuals in all compartments can move to sectors other than academia and the industry at the rate of μ. The study in [1] conclude that to manage the migration of PhD holders and postdocs into the industry while guaranteeing that the university's research capacities are not hampered. Thus, the authors highlighted the causes and impacts of migration. However, the study recommends investigating incentives as a form of control measure to mitigate the migration of researchers from university to the industry and observe the results. Incentives are widely used in various fields to encourage behaviors that align with specific goals, whether in business, education, healthcare, or public policy. However, it is important to carefully design and implement incentives to ensure that they are effective, fair, and aligned with the desired outcomes. Incentives can be financial, non-financial, material, intrinsic, social, cooperative, and performance-based. Incentives can play a role in mitigating migration from academia to the industry by addressing some of the factors that attract researchers and academics to leave their academic positions. Here are a few ways in which incentives can mitigate migration from academia to the industry:

1. Competitive salaries and benefits: Offering competitive salaries and benefits in the academic sector can help retain talented researchers and academics who might be enticed by higher-paying industry positions. Adequate compensation can alleviate financial concerns and make staying in academia more appealing.

2. Research funding and resources: Providing ample research funding and resources is crucial for conducting meaningful research and academic work. Lack of funding can be a significant factor driving academics to seek industry positions. By offering sufficient resources and research grants, academia can create an environment that supports impactful research and innovation.

3. Career development opportunities: Establishing clear career development paths and opportunities within academia can incentivize researchers to remain in the academic sector. This can include mentorship programs, leadership training, opportunities for collaboration, and chances to work on high-profile research projects.

4. Recognition and prestige: Academic researchers often value recognition for their contributions and the prestige associated with their work. Providing recognition through awards, honors, and public acknowledgment can motivate academics to stay within academia, where their accomplishments are highly valued.

5. Work-life balance and flexibility: Offering a healthy work-life balance and flexibility in work arrangements can be appealing to academics who may be seeking a better work-life integration. Flexible schedules, remote work options, and supportive policies can help retain talented individuals in academia.

6. Collaboration and interdisciplinary opportunities: Encouraging interdisciplinary collaborations and providing platforms for knowledge exchange can make academia more attractive. By fostering collaboration with industry partners or other academic institutions, researchers can engage in projects that have real-world impact while still working within an academic environment.

7. Intellectual freedom and autonomy: Preserving intellectual freedom and providing autonomy in research pursuits are crucial incentives for academics. Allowing researchers to explore their interests, choose their research topics, and

maintain academic independence will make staying in academia more fulfilling.

8. Strong institutional support: Creating a supportive institutional environment is vital for retaining academics. This includes providing mentorship, guidance, and administrative support, as well as fostering a culture of inclusivity, respect, and collaboration.

It is important to note that incentives alone may not be sufficient to mitigate the migration from academia to industry. Addressing factors such as the availability of industry opportunities, job security, and the perception of academia as a career will also play a significant role. It requires a comprehensive approach involving multiple stakeholders, including academic institutions, funding agencies, policymakers, and industry partners, to create an environment that values and supports academic careers. Our main focus in this study is to investigate how incentives can mitigate the migration from academia to the industry. Consequently, an optimal control model with time-dependent control $0 < u(t) < 1$ as an extension of (1) is formulated, where $u(t)$ is the time control using incentives. Thus, the optimal control model is described by the following nonautonomous system :

$$\begin{aligned}
\frac{dU(t)}{dt} &= \Lambda - \beta \cdot P_h(t) \cdot U(t) + u(t) \cdot \alpha_4 \cdot I(t) + u(t) \cdot \alpha_5 \cdot P_h(t) + u(t) \cdot \alpha_6 P_D(t) - \mu \cdot U(t), \\
\frac{dP_h(t)}{dt} &= \beta \cdot P_h(t) \cdot U(t) - \alpha_1 \cdot P_h(t) - \left(1 - u(t)\right) \cdot \alpha_2 \cdot P_h(t) - u(t) \cdot \alpha_5 \cdot P_h(t) - \mu \cdot P_h(t), \\
\frac{dP_D(t)}{dt} &= \alpha_1 \cdot P_h(t) - \left(1 - u(t)\right) \cdot \alpha_3 \cdot P_D(t) - \mu \cdot P_D(t) - u(t) \cdot \alpha_6 \cdot P_D(t), \\
\frac{dI(t)}{dt} &= \left(1 - u(t)\right) \cdot \alpha_2 \cdot P_h(t) + \left(1 - u(t)\right) \cdot \alpha_3 \cdot P_D(t) - u(t) \cdot \alpha_4 \cdot I(t) - \mu \cdot I(t),
\end{aligned} \tag{2}$$

subject to the initial conditions $U(0) > 0$, $P_h(0) > 0$, $P_D(0) > 0$, and $I(0) > 0$.

2.1 *Model analysis*

The qualitative analysis of the model, Equation 2, is carried out by considering the time-dependent controls $u(t)$ as bounded control parameters i.e., $u \in [0, 1]$. Since system (2) is similar to system (1) except for the control $u(t)$, the positivity and boundedness of solution of 2 follows from [1].

2.1.1 *Equilibrium points, migration basic reproduction number and stability*

In this section, the equilibrium points and migration basic reproduction number will be discussed. Consider the system of equations in (2) at equilibrium, then

$$\Lambda - \beta \cdot P_h(t) \cdot U(t) + u(t) \cdot \alpha_4 \cdot I(t) + u(t) \cdot \alpha_5 \cdot P_h(t) + u(t) \cdot \alpha_6 P_D(t) - \mu \cdot U(t) = 0 \tag{3}$$

$$\beta \cdot P_h(t) \cdot U(t) - \alpha_1 \cdot P_h(t) - \left(1 - u(t)\right) \cdot \alpha_2 \cdot P_h(t) - u(t) \cdot \alpha_5 \cdot P_h(t) - \mu \cdot P_h(t) = 0 \tag{4}$$

$$\alpha_1 \cdot P_h(t) - \left(1 - u(t)\right) \cdot \alpha_3 \cdot P_D(t) - \mu \cdot P_D(t) - u(t) \cdot \alpha_6 \cdot P_D(t) = 0 \tag{5}$$

$$\left(1 - u(t)\right) \cdot \alpha_2 \cdot P_h(t) + \left(1 - u(t)\right) \cdot \alpha_3 \cdot P_D(t) - u(t) \cdot \alpha_4 \cdot I(t) - \mu \cdot I(t) = 0 \tag{6}$$

It is easy to see from (5) that,

$$P_D = \frac{\alpha_1}{A_1} P_h. \tag{7}$$

The substitution of (7) in (4) and (6) while solving for U and I respectively yield

$$U = \frac{A_0}{\beta} \tag{8}$$

and

$$I = \frac{(1-u)\alpha_2\left((1-u)\alpha_3 + u\alpha_6 + \mu\right)}{A_1 A_2} P_h. \tag{9}$$

We finally solve for P_h by substituting (7) and (8) in (3) gives

$$P_h = \frac{(\mu A_0 - \Lambda\beta)}{\beta A_4}. \tag{10}$$

where $A_0 = \left(\alpha_1 + (1-u)\alpha_2 + u\alpha_5 + \mu\right)$, $A_1 = \left((1-u)\alpha_3 + u\alpha_6 + \mu\right)$, $A_2 = \left(u\alpha_4 + \mu\right)$, $A_3 = \dfrac{(1-u)\left(\alpha_2 A_1 + \alpha_3\alpha_1\right)}{A_1 A_2}$, $A_4 = -\dfrac{\left(1-u\right)\mu\left(\alpha_2\alpha_3(1-u) + u\alpha_2\alpha_6 + u\alpha_3\alpha_4 + \mu\alpha_2 + \mu\alpha_3 + \alpha_1\alpha_3\right) + (\alpha_4 u + \mu)(\mu u\alpha_6 + \mu^2 + \mu\alpha_1)}{\alpha_4 u + \mu}$.

Precisely, (11) can be expressed compactly as

$$P_h = \frac{\mu A_0}{\beta A_4}\left(1 - R_0(u)\right), \tag{11}$$

where $R_0(u)$ is the university-PhD-postdoc-industry (UPPI) effective migration reproduction number, defined as

$$R_0(u) = \frac{\beta\Lambda}{\mu A_0}. \tag{12}$$

Similarly, (7), (8), and (9) can be written as a function of $R_0(u)$ as follows:

$$\begin{cases} U = \dfrac{\Lambda}{\mu R_0(u)}, \\ P_D = \dfrac{\mu\alpha_1 A_0}{\beta A_1 A_4}\left(1 - R_0(u)\right), \\ I = \dfrac{\mu A_0 A_3}{\beta A_4}\left(1 - R_0(u)\right). \end{cases} \tag{13}$$

Observe that the expressions in (11) and (14) are unique and positive if $R_0(u) > 1$. The above discussion can be summarized in the following theorem.

Theorem 1 Consider system (1) at equilibrium, then there exists a positive unique university-PhD-postdoc-industry migration equilibrium (UPPDIME) $E^\star$ whenever $R_0(u) > 1$, otherwise. Where,

$$E^\star = \begin{cases} U^\star = \dfrac{\Lambda}{\mu R_0(u)}, \\ P_h^\star = \dfrac{\mu A_0}{\beta A_4}\left(1 - R_0(u)\right), \\ P_D^\star = \dfrac{\mu \alpha_1 A_0}{\beta A_1 A_4}\left(1 - R_0(u)\right), \\ I^\star = \dfrac{\mu A_0 A_3}{\beta A_4}\left(1 - R_0(u)\right). \end{cases} \tag{14}$$

Corollary 1 System (2) has a university-PhD-postdoc-industry migration-free equilibrium (UPPDIMFE), E_0 if $R_0(u) = 1$. Then,

$$E_0 = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right) \tag{15}$$

Proof. The proof is straightforward by substituting $R_0(u)) = 1$ in (11) and (14) yield

$$E_0 = (U_0, P_{h0}, P_{D0}, I_0) = (\frac{\Lambda}{\mu}, 0, 0, 0)$$

Remark 1 From Theorem 1 and Corollary 1, system (2) has two equilibrium points namely:

(i) UPPDIMFE, E_0: This is the equilibrium state where there are no PhD students, hence, there is no migration to the postdoc and industry. E_0 is feasible and locally asymptotically stable if $R_0(u) < 1$, otherwise.

(ii) UPPDIME, $E^\star$: This is a scenario where there are PhD students and postdoctoral scholars who can migrate to either academia or industry. $E^\star$ is feasible and locally asymptotically stable if $R_0(u) > 1$, otherwise.

2.2 *Impact of incentives on $R_0(u)$*

The impact of incentives u as intervention is measured by considering the effect on the UPPI effective migration reproduction number, $R_0(u)$. Consider the partial derivative of $R_0(u)$ with respect to incentive u as follows:

$$\frac{\partial}{\partial u} R_0(u) = -\frac{\beta \Lambda(-\alpha_2 + \alpha_5)}{\mu\left(\alpha_1 + (1-u)\alpha_2 + u\alpha_5 + \mu\right)^2}. \tag{16}$$

It follows from (16) that $\frac{\partial}{\partial u} R_0(u) < 0$, that it is a decreasing function of incentives u if $\alpha_5 > \alpha_2$. This indeed will be the case if favourable incentives that can stimulate the migration of PhD graduates to academia are in place. However, if $\alpha_5 < \alpha_2$, then $\frac{\partial}{\partial u} R_0(u) > 0$, implies that the incentives u, are not attractive enough to prevent the migration of PhD graduates to the industry. See Figure 1 for the numerical simulation. Figure 2 depicts the impact of good incentives to mitigate the migration of PhD graduates and postdocs scholars from academia to the industries. Figure 2(a) shows that when incentives (u) are above the critical value $u^\star$, more PhD graduates and postdoc scholars prefer academia to the industry compared to the case when $u < u^\star$. When $u > u^\star$, it is observed from Figure 2(b) and (c) that the migration of PhD graduates and postdocs scholars decreased significantly as against the increase in the number of PhD graduates and postdocs scholars migration when $u < u^\star$. Figure 2(d) suggests that when $u > u^\star$, very few individuals from academia may migrate to the industry when compared to the case when $u < u^\star$.

Let $u^\star$ be the critical value of incentives (u) when $R_0(u) = 1$, then

$$R_0(u) = \frac{\beta\Lambda}{\mu A_0} = 1. \tag{17}$$

It follows from (17) that the critical threshold for incentives, $u^\star$ is

$$u^\star = \frac{\mu(\mu + \alpha_1 + \alpha_2) - \beta\Lambda}{\mu(\alpha_2 - \alpha_5)}. \tag{18}$$

Note from (1) above that if $R_0(u^\star) > 1$, then $u > u^\star$ and $R_0(u) < 1$. Further, if $R_0(u^\star) < 1$, then $u < u^\star$ and $R_0(u) > 1$. This result is summarized as follows.

Lemma 1 Suppose that the effective migration reproduction number $R_0(u) > 1$, then there exists

$$u^\star = \frac{\mu(\mu + \alpha_1 + \alpha_2) - \beta\Lambda}{\mu(\alpha_2 - \alpha_5)},$$

such that $R_0(u^\star) = 1$, then $R_0(u) < 1(R_0(u) > 1)$ whenever $u > u^\star(u < u^\star)$.

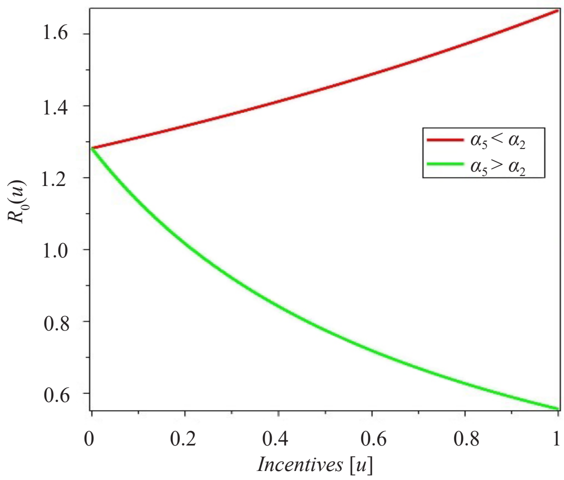
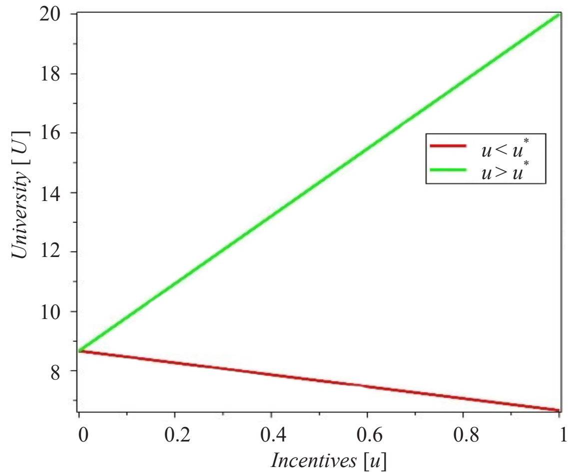
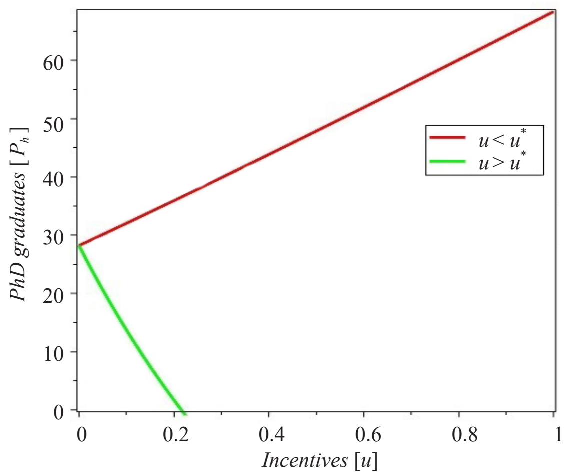


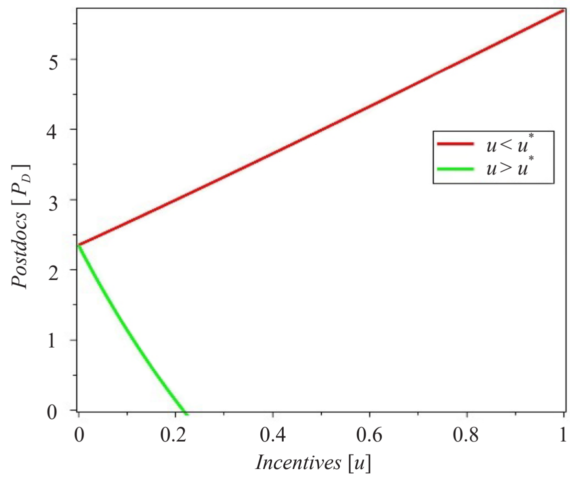
Figure 1. Graph of the impact of incentives (u) on the effective migration reproduction number $R_0(u)$ for $\alpha_5 < \alpha_2\,(\alpha_5 > \alpha_2)$



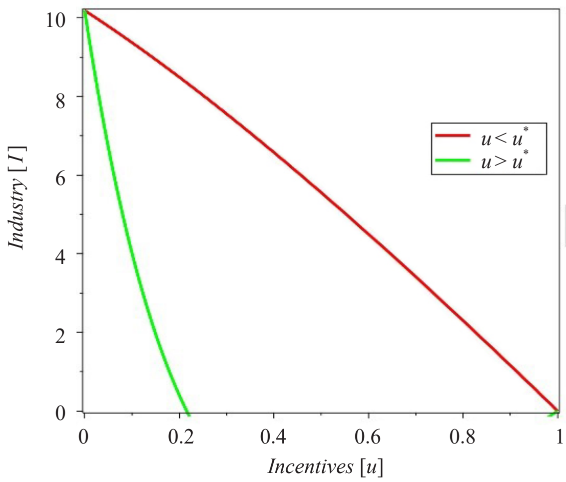
(a) University population



(b) PhD graduates



(c) Postdocs



(d) Industry

Figure 2. Simulation showing the impact of incentives (u) on (a)university population (b)PhD graduates (c)postdocs (d)industry with parameters set given in Table 2

In Table 1 and Table 2, we define the parameters in the model formation in Equation (2).

Table 1. Variable descriptions in the model

Parameter	Interpretation
$U(t)$	University population
$P_h(t)$	Population of PhD holders
$P_D(t)$	Population of PhD holders with postdoc placements
$I(t)$	Industry population

Table 2. Interpretation of parameters in the model

Parameter	Interpretation	Value	Source
Λ	Rate at which PhD students are admitted into the university	0.5	[1]
β	Progression rates of students to PhD class	0.0075	[1]
μ	Rate at which individuals leaves the academia and industry for other endeavors	0.045	[1]
α_1	Progression rate of individual with a PhD to postdoctoral class	0.0005	[1]
α_2	Progression rate of individual with a PhD directly to the industry	0.015	[1]
α_3	Progression rate of individual with a postdoctoral position to the industry	0.015	[1]
α_4	Progression rate of individual from the industry to the academia	0.05	[1]
α_5	Rate at which individual with a PhD secure employment in the university	[0,1]	[1]
α_6	Rate at which individual with a postdoctoral experience secure employment in the university	0.015	[1]

3. Optimal control problem

The optimal control analysis of the non-autonomous system (2) is considered in this section. The Pontryagins maximum principle is used in the analysis of the optimal control problem. This method has been used by many researchers in previous studies (see [17-18], [20-23]).

3.1 *Objective functional*

The main objective of the optimal control problem is to minimize the migration of PhDs and postdocs to industries. Thus, an objective functional J is defined as follows:

$$J\left(u(t)\right)=\int_0^{tf}\left(W_1P_h(t)+W_2P_D(t)+W_3u^2(t)\right)dt \tag{19}$$

subject to the non-autonomous system 2 where t_f is the final time for control implementation, and the control set $\mathcal{U}$. The control u is bounded to Lebesgue integrable function. The aim is to find an optimal control (u^*) such that

$$J(u^*)=\min_{J(u)}(u\in U), \tag{20}$$

with $\{u(t):0\leq u(t)\leq 1\}$.

Weight functions are mathematical functions that assign weights or importance to different elements or variables within a system. The purpose of weight functions is to regulate the influence of different components or data points in a given context. A weight function can emphasize or de-emphasize certain aspects based on their relevance or significance in a particular analysis or calculation by assigning varying weights to individual elements. In the literature [20], quadratic objective functions have been used to measure the intervention costs. Hence, a similar quadratic function is adopted here. A carefully chosen positive coefficients W_1, W_2, and W_3 to balance the weights. In the following section, the existence of the optimal solution which minimizes the objective functional J would be investigated.

3.2 *Existence of an optimal control*

The objective function and the adjoint variables are related to the control u, through the following Hamiltonian:

$$H = W_1 \cdot P_h(t) + W_2 \cdot P_D(t) + W_3 \cdot u^2(t) + \lambda_1 \cdot \frac{dU(t)}{t} + \lambda_2 \cdot \frac{dP_h(t)}{t} + \lambda_3 \cdot \frac{dP_D(t)}{t} + \lambda_4 \cdot \frac{dI(t)}{t}, \tag{21}$$

where λ_1, λ_2, λ_3, and λ_4 are the adjoint variables. The adjoint variables are obtained by taking the partial derivatives of the Hamiltonian. The adjoint variables and the control characterization are presented in the following theorem.

Theorem 2 Given an optimal control $u^\star$ and its corresponding state variables solutions (U, P_h, P_D, I) of system 2, then there exists adjoint variables λ_1, λ_2, λ_3, and λ_4 satisfying the following equations:

$$\frac{d\lambda_1}{dt} = -\frac{\partial}{\partial U} H,$$

$$\frac{d\lambda_1}{dt} = -\frac{\partial}{\partial P_h} H,$$

$$\frac{d\lambda_1}{dt} = -\frac{\partial}{\partial P_D} H,$$

$$\frac{d\lambda_1}{dt} = -\frac{\partial}{\partial I} H, \tag{22}$$

with the transversality conditions $\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = \lambda_4(t_f) = 0$. The control $u^\star$ satisfies the following optimal condition:

$$u^\star = \frac{\left(-\lambda_1\alpha_6 + \lambda_4\alpha_3 - \lambda_3(\alpha_3 - \alpha_6)\right)P_\mathrm{D} + \left(-\lambda_1\alpha_5 + \lambda_4\alpha_2 - \lambda_2(\alpha_2 - \alpha_5)\right)P_h - i\alpha_4(\lambda_1 - \lambda_4)}{2W_3}. \tag{23}$$

Proof. The adjoint variable is obtained as the differential of the associated adjoint variables, obtained by differentiating the Hamiltonian function by the corresponding state variables. Thus,

$$\frac{d\lambda_1}{dt} = (-\beta P_h - \mu)\lambda_1 - \beta P_h \lambda_2,$$

$$\frac{d\lambda_2}{dt} = (-\beta U + u\alpha_5)\lambda_1 - (\beta U - \alpha_1 - (1-u)\alpha_2 - u\alpha_5 - \mu)\lambda_2 - \alpha_1\lambda_3 - (1-u)\alpha_2\lambda_4 - W_1,$$

$$\frac{d\lambda_3}{dt} = \left((1-u)\alpha_3 + u\alpha_6 + \mu\right)\lambda_3 + \lambda_4(u-1)\alpha_3 - u\alpha_6\lambda_1 - W_2,$$

$$\frac{d\lambda_4}{dt} = -u\alpha_4\lambda_1 - (-u\alpha_4 - \mu)\lambda_4, \tag{24}$$

subject to the transversality conditions $\lambda_i(t_f)$ for i =1, 2, ..4. Further, the optimality condition is obtained by differentiating the Hamiltonian with respect to u at a steady state, hence,

$$\frac{\partial}{\partial u}H \;= 2W_3 u + (\alpha_4 I + \alpha_6 P_{\mathrm{D}} + \alpha_5 P_h)\lambda_1 + (\alpha_2 P_h - \alpha_5 P_h)\lambda_2 + (\alpha_3 P_{\mathrm{D}} - \alpha_6 P_{\mathrm{D}})\lambda_3 + (-\alpha_4 I - \alpha_3 P_{\mathrm{D}} - \alpha_2 P_h)\lambda_4 = 0. \tag{25}$$

By solving Equation (25) with respect to u, we obtain

$$u = \frac{\left(-\lambda_1\alpha_6 + \lambda_4\alpha_3 - \lambda_3(\alpha_3 - \alpha_6)\right)P_{\mathrm{D}} + \left(-\lambda_1\alpha_5 + \lambda_4\alpha_2 - \lambda_2(\alpha_2 - \alpha_5)\right)P_h - i\alpha_4(\lambda_1 - \lambda_4)}{2W_3}. \tag{26}$$

Thus, u satisfy the optimality condition

$$u^\star = \max\left\{0, \min(1, B)\right\}, \tag{27}$$

where $B = \dfrac{\left(-\lambda_1\alpha_6 + \lambda_4\alpha_3 - \lambda_3(\alpha_3 - \alpha_6)\right)P_{\mathrm{D}} + \left(-\lambda_1\alpha_5 + \lambda_4\alpha_2 - \lambda_2(\alpha_2 - \alpha_5)\right)P_h - i\alpha_4(\lambda_1 - \lambda_4)}{2W_3}$.

Theorem 3 [20] There exists an optimal control given the objective functional J is defined on the control $\mathcal{U}$ and subject to the state system (2) with positive initial conditions at $t = 0$, u^* such that $J(u^*) = min\left\{J(u) : u \in \mathcal{U}\right\}$ holds whenever the following properties are satisfied:

(i) The permissible control set $\mathcal{U}$ is convex and closed.
(ii) The state system is constrained by a linear function in the states and control variables.
(iii) The integral of the objective functional J in Equation (19) is convex with respect to the control.
(iv) The Lagrange is bounded below by

$$d_0(|u|^2)^{\frac{d_2}{2}} - d_1,$$

for constants d_0, $d_1 > 0$, and $d_2 > 1$.

Proof. Consider the control set $\mathcal{U} = [0,1]$, $u \in \mathcal{U}, X = (U, P_h, P_D, I)$ and $g_0(t, X, u)$ be the right hand side of the non-autonomous system in Equation (2), is given by

$$g_0(t, X, u) = \begin{pmatrix} \Lambda - \beta \cdot P_h \cdot U + u \cdot \alpha_4 \cdot I + u \cdot \alpha_5 \cdot P_h + u \cdot \alpha_6 P_D - \mu \cdot U \\ \beta \cdot P_h \cdot U - \alpha_1 \cdot P_h - (1-u) \cdot \alpha_2 \cdot P_h - u \cdot \alpha_5 \cdot P_h - \mu \cdot P_h \\ \alpha_1 \cdot P_h - (1-u) \cdot \alpha_3 \cdot P_D - \mu \cdot P_D - u \cdot \alpha_6 \cdot P_D \\ (1-u) \cdot \alpha_2 \cdot P_h + (1-u) \cdot \alpha_3 \cdot P_D - u \cdot \alpha_4 \cdot I - \mu \cdot I \end{pmatrix}, \tag{28}$$

for $u \in \mathcal{U}$. Next, we verify the properties in Theorem 3:

(i) Suppose the set $\mathcal{U} = [0,1]$. By definition, $\mathcal{U}$ is closed. Further, let $v_1, v_2, \in \mathcal{U}$, where v_1, and v_2 are any two arbitrary points. From the definition of a convex set, it follows that:

$$\lambda v_1 + (1-\lambda)v_2 \in [0,1], \forall \lambda \in [0,1].$$

Consequently, $\lambda v_1 + (1-\lambda)v_2 \in \mathcal{U}$ implies the convexity of $\mathcal{U}$.

(ii) Since $g_0(t, X, u)$ can be expressed as

$$g_0(t, X, u) = g_1(t, X) + g_2(t, X)u, \tag{29}$$

where

$$g_1(t,X)=\begin{pmatrix}\Lambda-\beta\cdot P_h\cdot U-\mu\cdot U\\ \beta\cdot P_h\cdot U-\alpha_1\cdot P_h-\alpha_2\cdot P_h-\mu\cdot P_h\\ \alpha_1\cdot P_h-\alpha_3\cdot P_D-\mu\cdot P_D\\ \alpha_2\cdot P_h+\alpha_3\cdot P_D-\mu\cdot I\end{pmatrix}, \tag{30}$$

$$g_2(t,X)=\begin{pmatrix}0 & \alpha_5\cdot P_h & \alpha_6\cdot P_D & \alpha_4\cdot I\\ 0 & (\alpha_2-\alpha_5)\cdot P_h & 0 & 0\\ 0 & 0 & (\alpha_3-\alpha_6)P_D & 0\\ 0 & -\alpha_2\cdot P_h & -\alpha_3\cdot P_D & -\alpha_4 I\end{pmatrix}. \tag{31}$$

Let V be a vector space defined over real or complex numbers, then the norm on V denoted by $\|.\|$ for all vectors $x, y\in V$ and scalar c is defined as $\|x+y\|\le\|x\|+\|y\|$. Thus, the norm of the Equation (29) is

$$\|g_1(t,X,u)\|\le\|g_1(t,X)\|+\|g_2(t,X)\|\|u\|, \tag{32}$$

$$\le d_0+d_1\|u\|, \tag{33}$$

where the positive constants d_0 and d_1 are determined by the superposition approach, where the components of the upper bounds of $g_1(t, X)$ is obtained to be in Equation (34):

$$d_0^2=\begin{pmatrix}\Lambda^2\\ \beta^2\cdot P_h^2\cdot U^2\\ \alpha_1^2\cdot P_h^2\\ (\alpha_2\cdot P_h+\alpha_3\cdot P_D)^2\end{pmatrix}$$

$$=\Lambda^2+\beta^2\cdot P_h^2\cdot U^2+\alpha_1^2\cdot P_h^2+\alpha_2^2\cdot P_h^2+2\alpha_2\cdot\alpha_3\cdot P_h\cdot P_D+\alpha_3^2\cdot P_D^2. \tag{34}$$

Suppose the upper bound on the state variable is N such that

$$N=max\{N_U, N_{p_h}, N_{P_D}, N_I\}, \tag{35}$$

then,

$$d_0^2=\Lambda^2+\beta^2N^4+\alpha_1^2N^2+\alpha_2^2N^2+2\alpha_1\alpha_3N^2+\alpha_1^2N^2$$

$$=\beta^2N^4+(\alpha_1^2+\alpha_2^2+2\alpha_2\alpha_3+\alpha_3^2)N^2+\Lambda^2 \tag{36}$$

By using the inequality concept,

$$a\cdot x+b\cdot y\le max\{a,b\}(x+y) \tag{37}$$

yields

$$d_0 = \sqrt{max\{A_1, A_2, A_3\}(N^4 + N^2 + \Lambda^2)} \tag{38}$$

where $A_1 = \beta^2$, $A_2 = \alpha_1^2 + \alpha_2^2 + 2\alpha_2\alpha_3 + \alpha^3$, $A_3 = 1$. Furthermore, the sum of the squares of each component of the $g_2(t, X)$ is therefore,

$$d_1^2 = \alpha_5^2 P_h^2 + \alpha_6^2 P_D^2 + \alpha_4^2 I^2 + (\alpha_2 - \alpha_5)^2 P_h^2 + (\alpha_3 - \alpha_6)^2 P_D^2 + \alpha^2 P_h^2 + \alpha_3^2 P_D^2 + \alpha_4^2 I^2. \tag{39}$$

Please note that the upper bound for the state variable, i.e. $U = P_h = P_D = I = \dfrac{\Lambda}{\mu}$. Thus, by replacing each variable with its upper bound in Equation (39) gives

$$d_1^2 = \frac{\alpha_5^2\Lambda^2}{\mu^2} + \frac{\alpha_6^2\Lambda^2}{\mu^2} + \frac{\alpha_4^2\Lambda^2}{\mu^2} + \frac{(\alpha_2 - \alpha_5)\Lambda^2}{\mu^2} + \frac{(\alpha_3 - \alpha_6)\Lambda^2}{\mu^2} + \frac{\alpha_2^2\Lambda^2}{\mu^2} + \frac{\alpha_3^2\Lambda^2}{\mu^2} + \frac{\alpha_4^2\Lambda^2}{\mu^2}$$

$$= A_4 \frac{\Lambda^2}{\mu^2},$$

where $A_4 = \alpha_5^2 + \alpha_6^2 + 2\alpha_4^2 + \alpha_3^2 + \alpha_2^2 + (\alpha_2 - \alpha_5)^2 + (\alpha_3 - \alpha_6)^2$

$$\therefore \quad d_1 = \sqrt{A_4} \frac{\Lambda}{\mu}. \tag{40}$$

(iii) Recall from the objective function written in Equation (19),

$$G(t, X, u) = G_1(t, X) + G_2(t, u), \tag{41}$$

where $G_1(t, X) = W_1 P_h + W_2 P_D$ and $G_2(t, u) = W_3 u^2$. We only need to prove that $G_2(t, u)$ is convex on u i.e.

$$G_2\left(t, (1-\lambda)v_1 + \lambda v_2\right) \le (1-\lambda)G_2(t, v_1) + \lambda G_2(t, v_2) \tag{42}$$

for $v_1, v_2 \in \mathcal{U}$ and $\lambda \in [0.1]$. By definition, it implies that

$$G_2\left(t, v_1\right) = W_3 V_1^2;$$

$$G_2\left(t, v_2\right) = W_3 V_2^2;$$

$$G_2\left(t, (1-\lambda)v_1 + \lambda v_2\right) = W_3\left((1-\lambda)v_1 + \lambda v_2\right)^2. \tag{43}$$

Thus, using Equation (43) in Equation (42) yields

$$G_2\left(t, (1-\lambda)v_1 + \lambda v_2\right) - (1-\lambda)G_2(t, v_1) - \lambda G_2(t, v_2) = -W_3\lambda(1-\lambda)(v_1 - v_2)^2 \le 0, \tag{44}$$

since $\lambda \in [0,1]$, it implies that the integrand $G\ (t, X, u)$ of the objective functional J is convex.

(iv) We prove the fourth property as follows:

$$G(t, X, u) = G_1(t, X) + G_2(t, u)$$

$$\geq G_2(t, u) = W_3 u^2$$

$$= d_0 \left(u^2\right)^{\frac{d_2}{2}} - d_1 \tag{45}$$

where $d_0 = W_3$, $d_1 > 0$, and $d_2 = 2$. This completes the proof.

3.3 *Solving the optimal control system*

In this section, attention is focused on solving the optimality system, which typically involves finding the control inputs that optimize a given objective function while satisfying the system's dynamics and constraints. The optimality system combines the state system in (2) and the adjoint system in (24). The state system is subject to the initial conditions, and the adjoint system is subject to the transversality conditions. Thus, combining systems (2) and (24) leads to the following optimality system:

$$\frac{dU(t)}{dt} = \Lambda - \beta \cdot P_h \cdot U(t) + u \cdot \alpha_4 \cdot I(t) + u \cdot \alpha_5 \cdot P_h(t) + u \cdot \alpha_6 \cdot P_D(t) - \mu \cdot U(t),$$
$$\frac{dP_h(t)}{dt} = \beta \cdot P_h(t) \cdot U(t) - \alpha_1 \cdot P_h(t) - (1-u) \cdot \alpha_2(t) \cdot P_h(t) - u \cdot \alpha_5 \cdot P_h(t) - \mu \cdot P_h(t),$$

$$\frac{dP_D(t)}{dt} = \alpha_1 \cdot P_h(t) - (1-u) \cdot \alpha_3 \cdot P_D(t) - \mu \cdot P_D(t) - u \cdot \alpha_6 \cdot P_D(t),$$

$$\frac{dI(t)}{dt} = (1-u) \cdot \alpha_2 \cdot P_h(t) + (1-u) \cdot \alpha_3 \cdot P_D(t) - u \cdot \alpha_4 \cdot I(t) - \mu \cdot I(t),$$

$$\frac{d\lambda_1}{dt} = -\left(-\beta \cdot P_h(t) - \mu\right) \cdot \lambda_1 - \beta \cdot P_h(1) \cdot \lambda_2,$$

$$\frac{d\lambda_2}{dt} = -\left(-\beta \cdot U(t) + u \cdot \alpha_5\right) \cdot \lambda_1 - \left(\beta \cdot U(t) - \alpha_1 - (1-u) \cdot \alpha_2 - u \cdot \alpha_5 - \mu\right) \cdot \lambda_2,$$

$$-\alpha_1 \cdot \lambda_3 - (1-u) \cdot \alpha_2 \cdot \lambda_4 - W_1,$$

$$\frac{d\lambda_3}{dt} = -\alpha_6 \cdot \lambda_1 - \left(-(1-u) \cdot \alpha_3 - \mu - u \cdot \alpha_6\right) \cdot \lambda_3 - (1-u) \cdot \alpha_3 \cdot \lambda_4 - W_2,$$

$$\frac{d\lambda_4}{dt} = -u \cdot \alpha_4 \cdot \lambda_1 - \left(-u \cdot \alpha_4 - \mu\right) \cdot \lambda_4. \tag{46}$$

The solution to the optimality system (46) will be obtained using the forward and backward Runge-Kutta scheme at the initial conditions for the states equations and the transversality condition for the adjoint equations.

4. Simulations and discussion of the results

The section details the numerical simulation and the varying effect of the optimal control in the case of the UPPI model with the key participants to be PhD and postdoc population. Although parameter values were not available from the literature or estimation, the simulation assumed realistic values. The optimal control is simulated using MATLAB with the following parameters values $\Lambda = 2.614828.10^6$, $\alpha_1 = 0.3606022$, $\alpha_2 = 0.1$, $\alpha_3 = 0.2655162$, $\alpha_4 = 0.08976775$, $\alpha_5 = 0.3$, $\alpha_6 = 0.7212043$, $\beta = 8.992457.10^{-16}$, $\beta = 0.01305483$. To perform the simulation, we assumed the weight functions to be $W_i' s \; \forall \, i = 1, 2, 3$ and the values are assumed to be $W_1 = W_2 = W_3 = 1$, which are associated with the controls. To completely implement the simulation of the numerical analysis, the model is investigated with the initial conditions, U_0, P_h, P_D, I = [15312289, 119580, 42858, 208169996]. The computing results can be seen in Figures 3-7 respectively and representation of the various scenarios of the compartment are described. It is imperative to note that the control parameter is denoted as u_1. Figure 3 represent the control parameter and likelihood effect on other compartments. The control parameter was simulated with other compartment to investigate what the outcome will be with or without the optimal control implemented.

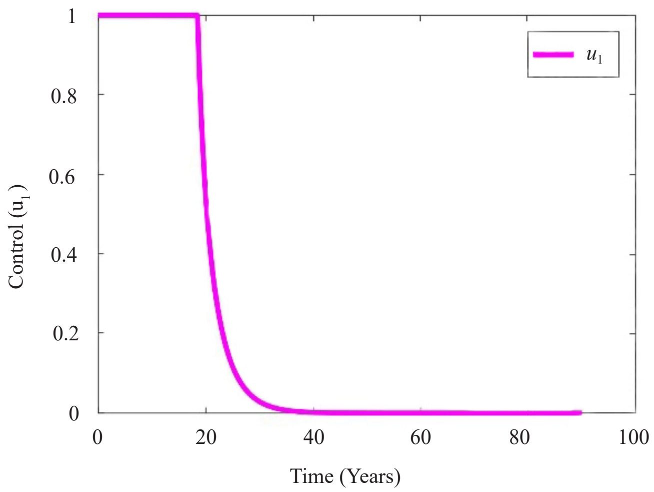


Figure 3. Graphical representation of the control profile (u)

The simulated graph in Figure 4-7 provided insights into the behavior of the system under the influence of optimal control.

The graph plotted in Figure 4, clearly indicates that without the optimal control, the university population will decrease due to the migration of PhD and postdocs to the industry, which may lead to brain drain in the university population. This is how the PhD and postdoc unemployment will reduce the impact of the research arm or output on the university population. This scenario in Figure 4 depicts that the industry compartment will be at an advantage to the effect without the control. From the graph in Figure 4, the effect of the optimal control encourages the university population to be on the rise i.e., the university has become attractive to researchers to work and the presence of financial stability can as well be obtained. To infer this, it can be seen that with the control measure, the first 20 years show an increase in the employment rate at the university compartment compared to without the control measure. It can be seen that the university population maintained a steady population after 20 years, which can be attributed to a stable optimal control policy.

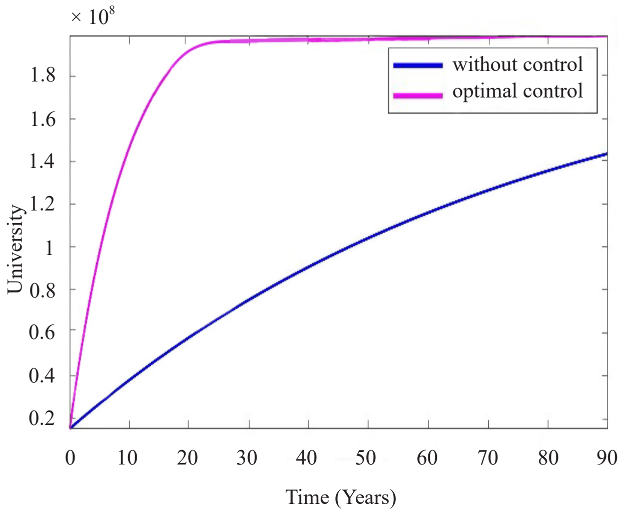


Figure 4. Graphical effect of the optimal control on the university population

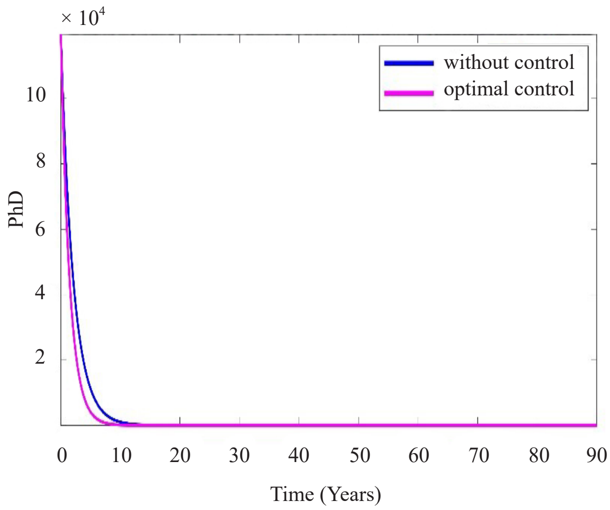


Figure 5. Graphical effect of the optimal control on the PhD population

We can see from Figure 5, that the impact of incentives is denoted to be the control measure on the distribution of PhD holders between the university and industry sectors. It suggests that incentives that provide satisfaction and prestige will cause more PhD holders to choose the university sector over the industry sector. This, in turn, will lead to a greater number of postdoc positions being created within the university system, causing the university sector to become more populated with PhD holders, while the industry compartment will have fewer.

Figure 6 shows that without the control in the postdoc population, we can observe increment at some point there but it begins to decrease as time passes. Furthermore, when individuals leave the postdoc compartment, they are more likely to move to the industry sector rather than the university sector due to lack of incentives and financial stability. However, with the optimal control measure, individuals have the choice to stay in the postdoc compartment or move to the university compartment to gain both prestige (and passion) and financial stability. This choice is facilitated by the incentives provided. As a result, the postdoc population becomes more stable, or individuals may choose to move to the university, resulting in a more populated university compartment.

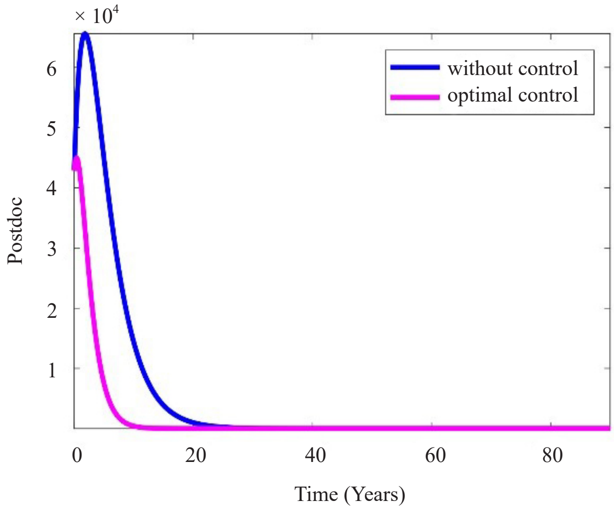


Figure 6. Graphical effect of the optimal control on the postdoc population

Figure 7 shows the optimal control measure has a significant impact on the distribution of PhD and postdoc populations between the university and industry sectors. Without optimal control, the simulation suggests that more individuals will be present in the industry compartment, leading to a reduced university population. This indicates that the incentives present in the industry sector are more attractive to individuals in the PhD and postdoc compartments than those offered by the university sector. In contrast, the presence of optimal control leads to a reduction in the PhD and postdoc populations and favors individuals in these compartments whose interests lie solely in research or academia to go back to the university compartment. This suggests that the optimal control measure provides incentives that are more attractive to individuals who are solely interested in research or academia than those offered by the industry sector.

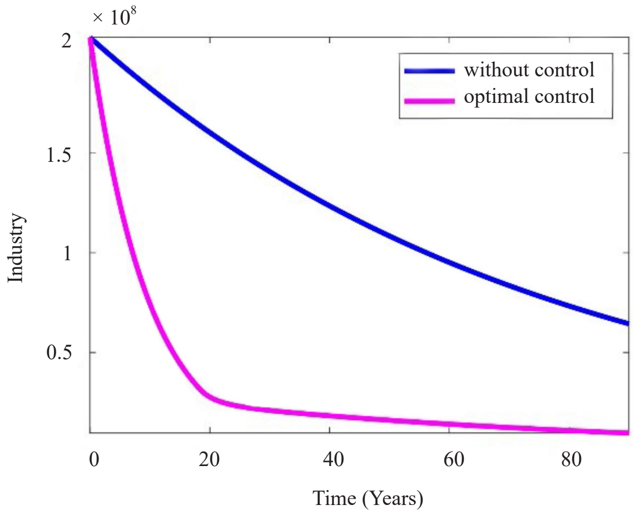


Figure 7. Graphical effect of the optimal control on the industry population

The huge gap between the with and without optimal control measures indicates that incentives can have a significant impact on the interest of individuals and the distribution of talent across different sectors. The results suggest that with the right incentives, individuals can be encouraged to pursue academic careers, leading to a more populated university compartment. Without a loss in generality, some researchers will decide to stay in the industry compartment notwithstanding any incentives offered by the institution to the research community.

5. Conclusion

The use of optimal control simulation to explore the dynamics and the effect on other compartments provided insights into the behavior of the system under the influence of optimal control. The simulation results suggest that without optimal control, the university population will decrease, leading to unemployment and incapacity in the research arm of the university population. This indicates that other compartments such as the industry will be at an advantage without the control. However, with the optimal control measure, the university population becomes attractive to researchers, leading to its rise, driven by both passion and financial stability. Furthermore, the impact of incentives, denoted as the control measure, on the distribution of PhD holders between the university and industry sectors was also observed. The incentives that provide satisfaction and prestige cause more PhD holders to choose the university sector over the industry sector, leading to a greater number of postdoc positions being created within the university system. This, in turn, causes the university sector to become more populated with PhD holders, while it is fewer in the industry. Moreover, the simulation also suggests that without the control in the postdoc population, there will be an initial increase in their population that eventually decreases with time. When individuals leave the postdoc compartment, they are more likely to move to the industry sector rather than the university sector due to a lack of incentives and financial stability. However, with the optimal control measure, individuals have the choice to stay in the postdoc compartment or move to the university compartment to gain both prestige (and passion) and financial stability. This choice is facilitated by the incentives provided. As a result, the postdoc population will be stable, or individuals may choose to move to the university, resulting in a more populated university compartment.

It is good to know that the optimal control measure has a significant impact on the behavioral pattern of the PhD and postdoc populations, and the distribution of researchers between the university and industry. The use of incentives to encourage academic careers can lead to a more populated university compartment, which can have positive effects on research, innovation, and overall economic growth. The results point to the importance of incentives in shaping the decisions of individuals and the distribution of researchers across different sectors, indicating the need for policies to optimize the system's behavior over time.

Acknowledgements

The authors wish to thank Tshwane University of Technology for their financial support and the Department of Higher Education and Training, South Africa.

Conflict of interest

The authors declare no conflicting interest in the study.

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