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Research Article

On Nontriviality of a Product in the Classical Adams Spectral Sequence

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Abstract: Let $p \geq 11$ be an odd prime and $q = 2(p-1)$. Suppose that $n \geq 1$ with $n \neq 5$. Let $0 \leq s < p-4$ and $t = s+2+(s+2)p+(s+3)p^2+(s+4)p^3+p^n$. This paper shows that the product element $\tilde{\delta}_{s+4}h_0b_{n-1} \in \mathrm{Ext}_{\mathcal{A}}^{s+7,tq+s}(\mathbb{Z}/p,\mathbb{Z}/p)$ is a nontrivial permanent cycle in the classical Adams spectral sequence, where $\tilde{\delta}_{s+4}$ denotes the 4th Greek letter element.

Keywords: stable homotopy groups of sphere, Adams spectral sequences, May spectral sequences

MSC: 55Q45

1. Introduction

Throughout this paper, we let p be an odd prime and fix $q = 2(p-1)$. The notation $\mathcal{A}$ denotes the mod p Steenrod algebra. For more details about the Steenrod algebra, please refer to [1-3]. We let S be the sphere spectrum, which is localized at p. It is well known that determining the structure of the stable homotopy groups of spheres is one of the most important problems in homotopy theory. In order to approach it, we mainly use the classical Adams spectral sequence (ASS), whose E_2-term is $E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{Z}/p,\mathbb{Z}/p)$, the cohomology of the Steenrod algebra $\mathcal{A}$. The differential of this spectral sequence is $d_r: E_r^{s,t} \to E_r^{s+r,t+r-1}$. According to the reference [4], we know that the first filtration $\mathrm{Ext}_{\mathcal{A}}^{1,*}(\mathbb{Z}/p,\mathbb{Z}/p)$ has a $\mathbb{Z}/p$-basis, including $a_0 \in \mathrm{Ext}_{\mathcal{A}}^{1,1}(\mathbb{Z}/p,\mathbb{Z}/p), h_i \in \mathrm{Ext}_{\mathcal{A}}^{1,p^iq}(\mathbb{Z}/p,\mathbb{Z}/p)$ for all $i \geq 0$, and the second filtration $\mathrm{Ext}_{\mathcal{A}}^{2,*}(\mathbb{Z}/p,\mathbb{Z}/p)$ has a $\mathbb{Z}/p$-basis including $\tilde{a}_2, a_0^2, a_0h_i(i>0), g_i(i\geq 0), k_i(i\geq 0), b_i(i\geq 0)$, and $h_ih_j(j\geq i+2, i\geq 0)$ with degrees as $2q+1, 2, p^iq+1, (p^{i+1}+2p^i)q, (2p^{i+1}+p^i)q, p^{i+1}q$ and $(p^i+p^j)q$, respectively.

Generally, if a generator $x_i \in E_2^{s,*}$ is a nontrivial permanent cycle in the ASS and converges to an element $f_i \in \pi_*S$, we say that f_i can be represented by x_i in the ASS. Although much work has been done on this problem, a huge number of homotopy elements in π_*S have not been determined. In the following, we state some known results on this problem. In [5], a family of nontrivial elements $\zeta_n \in \pi_{p^nq+q-3}S (n\geq 2)$ has been determined. It is already known that ζ_n has filtration 3 and can be represented by a permanent cycle $h_0b_{n-1} \in \mathrm{Ext}_{\mathcal{A}}^{3,p^nq+q}(\mathbb{Z}/p,\mathbb{Z}/p)$ in the ASS. In [6], a family of nontrivial elements $\xi_n \in \pi_{(p^n+p)q-3}$ ($n\geq 3$ and $p\geq 5$) has been determined. This element has filtration 3 and can be represented by a permanent cycle $(b_0h_n + h_1b_{n-1}) \in \mathrm{Ext}_{\mathcal{A}}^{3,(p^n+p)q}(\mathbb{Z}/p,\mathbb{Z}/p)$ in the ASS.

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We know that the periodic elements of the stable homotopy groups of spheres are very important. There is a close relationship between the existence of the periodic elements and the existence of the Toda-Smith spectra. We use the notation *BP* to denote the Brown-Peterson spectrum, which is localized at *p*. It is known that *BP* is a *p*-local ring spectrum whose coefficient ring is $BP_* = BP_*S = \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$, where $\mathbb{Z}_{(p)}$ denotes the *p*-localization of $\mathbb{Z}$ and the generator v_i has degree $2(p^i - 1)$. In [7], it is shown that if we are given a spectrum *X*, then BP_*X admits a comodule structure over the Hopf algebroid BP_*BP. We use the notation I_{n+1} to denote an ideal generated by the elements p, $v_1, \cdots$, v_n. In [8], Toda studied the existence of some finite spectra $V(n)$ whose *BP*-homology is

$$BP_*V(n) \cong BP_* / I_{n+1} (\text{as } BP_*\text{-module, hence as } BP_*BP\text{-comodule})$$

In [8], it is shown that the spectrum $V(n)$ exists under the condition $p > 2n$ for $0 \le n \le 3$. Toda [8] also verified the existence of some Greek letter maps

$$\alpha^{(n)} : \Sigma^{2(p^n-1)} V(n-1) \to V(n-1),$$

where $\alpha^{(n)} = p, \alpha, \beta, \gamma$ when n = 0, 1, 2, 3, respectively. Here, we mention that the notation $V(-1)$ denotes the sphere spectrum *S*. Furthermore, we have a cofiber sequence

$$\Sigma^{2(p^n-1)} V(n-1) \xrightarrow{\alpha^{(n)}} V(n-1) \xrightarrow{i_n} V(n) \xrightarrow{j_n} \Sigma^{2p^n-1} V(n-1),$$

where $V(n)$ is the cofiber of $\alpha^{(n)}$. In what follows, we write the three notations

$$\alpha_s = j_0(\alpha^{(1)})^s i_0,\ \beta_s = j_0 j_1 (\alpha^{(2)})^s i_1 i_0, \text{ and } \gamma_s = j_0 j_1 j_2 (\alpha^{(3)})^s i_2 i_1 i_0 \text{ v}$$

that denote the first, second, and third periodic elements in π_*S, respectively (refer to [9]).

In [10], it was shown that there exists a nontrivial cohomology class $\tilde{\alpha}_s^n \in \mathrm{Ext}_{\mathcal{A}}^{s,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ for $n < p$ and $s \not\equiv 0, 1, \cdots, n-1 \bmod p$. We call this cohomology class the *n*-th Greek letter element. Especially, we write $\tilde{\alpha}_s^n$ as $\tilde{\alpha}_s, \tilde{\beta}_s, \tilde{\gamma}_s, \tilde{\delta}_s$ for n = 1, 2, 3, and 4, respectively. In [10], it was shown that $\tilde{\alpha}_s, \tilde{\beta}_s$, and $\tilde{\gamma}_s$ represent the periodic elements α_s, β_s, and γ_s, respectively. Since the existence of the fourth periodic element is still an open problem, we do not know whether or not $\tilde{\delta}_s$ represents a possibly existing fourth periodic element.

In what follows, we introduce the notion of a product element in the ASS. Suppose that we are given two nontrivial elements $x, y \in \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ in the ASS. Thus, multiplication on the ASS gives us an element *xy* in the ASS, which is called the product element. Generally, *xy* may not be a permanent cycle in the ASS. This drives us to find the condition under which *xy* is a permanent cycle in the ASS. Among these product elements, we are especially interested in the nontriviality of the product of one of $\tilde{\beta}_s, \tilde{\gamma}_s$, and $\tilde{\delta}_s$ with some other element in E_2-term of the ASS. Some progress has been made on this problem. For example, in [11], it was shown under the conditions $p \ge 7$, $n \ge 4$, and $0 \le s < p - 4$, the product element $\tilde{\gamma}_{s+3} h_0 b_{n-1} \in \mathrm{Ext}_{\mathcal{A}}^{s+6,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ is a nontrivial permanent cycle in the ASS. In [12], a nontrivial product involving *β*-family was detected in the ASS. Also in [13], it was shown that under the conditions $0 \le s < p - 4$ and $p \ge 11$, the product element $\tilde{\delta}_{s+4} h_0 b_0 \in \mathrm{Ext}_{\mathcal{A}}^{s+7,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ is a permanent cycle in the ASS. Recently, more nontrivial product elements have been detected [14-16]. The proofs of these results are based on the method of combinatorial analysis with some restrictions on the range of the filtration *s*. In order to extend the range of the filtration *s*, we plan to apply a new computation method in this paper to obtain a relatively more general result by showing that the product $\tilde{\delta}_{s+4} h_0 b_{n-1} \in \mathrm{Ext}_{\mathcal{A}}^{s+7,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ is a permanent cycle in the ASS under the conditions $p \ge 11$, $n \ge 1$ and $n \ne 5$, $0 \le s < p - 4$. We state our main result as follows:

Theorem 1.1. Let $p \ge 11$ and $n \ge 1$ with $n \ne 5$. If $0 \le s \le p - 5$, then the product $\tilde{\delta}_{s+4} h_0 b_{n-1}$ is a nontrivial permanent cycle in the ASS.

Remark 1.2. The case $n = 5$ is special due to the existence of a possible obstruction, which is related to one possible higher Adams differential. A detailed discussion will be given in Remark 4.3.

We organize this paper as follows: in Section 2, a new method is introduced to compute the generators of the E_1-

term of the May spectral sequence (MSS). In Section 3, we will apply this method to give an explicit computation in order to prove Theorem 1.1. In Section 4, the proof of Theorem 1.1 is given.

2. Computation method for the MSS

In this section, we will recall some preliminary knowledge on the MSS and introduce an effective method to compute the E_1-term of the MSS. According to [7], we know that the MSS is a triple-graded algebra with differential $d_r : E_r^{s,t,M} \to E_r^{s+1,t,M-r}$. It is well known that the MSS converges to $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{Z}/p,\mathbb{Z}/p)$. According to [7], we know that the E_1-term of the MSS has the structure as

$$E_1^{*,*,*} = E[h_{m,i} \mid m > 0, i \geq 0] \otimes P[b_{m,i} \mid m > 0, i \geq 0] \otimes P[a_n \mid n \geq 0],$$

where the notation $E[\ ]$ denotes the exterior algebra and $P[\ ]$ denotes the polynomial algebra. Among these generators, $h_{1,i} \in E_1^{1,p^i q,*}$ can converge nontrivially to a generator $h_i \in \mathrm{Ext}_{\mathcal{A}}^{1,p^i q}(\mathbb{Z}/p,\mathbb{Z}/p)$. It follows that $d_r(h_{1,i}) = 0$ for $r \geq 1$. The degrees of the above generators are listed as follows:

$$h_{m,i} \in E_1^{1,2(p^m-1)p^i,2m-1}, \quad b_{m,i} \in E_1^{2,2(p^m-1)p^{i+1},(2m-1)p}, \quad a_n \in E_1^{1,2p^n-1,2n+1}.$$

The r-th May differential $d_r : E_r^{s,t,M} \to E_r^{s+1,t,M-r}$ satisfies the Leibniz law

$$d_r(xy) = d_r(x)y + (-1)^{s+t} x d_r(y)$$

for $x \in E_r^{s,t,*}$ and $y \in E_r^{s',t',*}$. If we are given $\{x, y\} \subset \{h_{m,i}, b_{m,i}, a_n\}$, then we have the graded commutativity $xy = (-1)^{(s+t)(s'+t')} yx$. For each generator, the first May differential $d_1 : E_1^{s,t,M} \to E_1^{s+1,t,M-1}$ can be given explicitly as

$$d_1(h_{i,j}) = \sum_{0<k<i} h_{i-k,k+j} h_{k,j}, \ d_1(a_i) = \sum_{0 \leq k < i} h_{i-k,k} a_k, \ d_1(b_{i,j}) = 0.$$

For a given $x \in E_1^{s,t,M}$, its three degrees are expressed as $\dim(x) = s, \deg(x) = t,$ and $M(x) = M$, respectively. For the above generators of $E_1^{*,*,*}$, we have

$$\begin{aligned}
&\dim(h_{i,j}) = \dim(a_i) = 1, \dim(b_{i,j}) = 2, \\
&M(h_{i,j}) = M(a_{i-1}) = 2i-1, M(b_{i,j}) = (2i-1)p, \\
&\deg(h_{i,j}) = 2(p^i - 1)p^j = (p^j + \cdots + p^{i+j-1})q, \\
&\deg(b_{i,j}) = 2(p^i - 1)p^{j+1} = (p^{j+1} + \cdots + p^{i+j})q, \\
&\deg(a_i) = 2p^i - 1 = (1 + \cdots + p^{i-1})q + 1, \\
&\deg(a_0) = 1
\end{aligned}$$

We use the letters x, y, and z to denote the generator $a_i, h_{i,j}$, and $b_{i,j}$, respectively. According to the graded commutativity, one generator $h \in E_1^{*,*,*}$ can be written as the form

$$h = (x_1 \cdots x_u)(y_1 \cdots y_v)(z_1 \cdots z_l) \in E_1^{s,t+b,*}.$$

For each degree of h, we have $s < b + q$ with $0 < b < q$, and $t = (\overline{c}_0 + \overline{c}_1 p + \cdots + \overline{c}_n p^n)q$ with $0 \leq \overline{c}_i, < p$ for $0 \leq i < n, \overline{c}_n > 0$. It is claimed that $u = b$. Otherwise, according to the above expressions of $\deg(a_i)$, $\deg(h_{i,j})$, $\deg(b_{i,j})$, and t, we see that there exists some positive integer w such that $u = b + wq$. It follows $\dim(h) \geq b + wq > s = \dim(h)$. This is clearly a contradiction. Thus, we have

$$h = (x_1 \cdots x_b)(y_1 \cdots y_v)(z_1 \cdots z_l) \in E_1^{b+v+2l,t+b,*}.$$

It should be noted that $\deg(x_i), \deg(y_i)$, and $\deg(z_i)$ are uniquely p-adic expressed as follows:

$$\deg(x_i) = (x_{i,0} + x_{i,1}p + \cdots + x_{i,n}p^n)q + 1,$$

$$\deg(y_i) = (y_{i,0} + y_{i,1}p + \cdots + y_{i,n}p^n)q,$$

$$\deg(z_i) = (0 + z_{i,1}p + \cdots + z_{i,n}p^n)q.$$

It is easily checked that among the above expressions the sequence $(x_{i,0}, x_{i,1}, \cdots, x_{i,n})$ has the form $(1, \cdots, 1, 0, \cdots, 0)$, but the sequences $(y_{i,0}, y_{i,1}, \cdots, y_{i,n})$ and $(0, z_{i,1}, \cdots, z_{i,n})$ both have form $(0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0)$. By the graded commutativity on $E_1^{*,*,*}$, we can rearrange $h = (x_1 \cdots x_b)(y_1 \cdots y_v)(z_1 \cdots z_l) \in E_1^{b+v+2l,t+b,*}$ as follows:

(a) if $i > j$, place a_i on the left side of a_j;
(b) if $j < k$, place $h_{i,j}$ on the left side of $h_{w,k}$;
(c) if $i > w$, place $h_{i,j}$ on the left side of $h_{w,j}$;
(d) use the same rules (b) and (c) for $b_{i,j}$.

After we finish the above operation, $x_{i,j}$, $y_{i,j}$, and $z_{i,j}$ in the above sequences satisfy the following conditions (α):

(i) $x_{1,j} \geq x_{2,j} \geq \cdots \geq x_{b,j}, x_{i,0} \geq x_{i,1} \geq \cdots \geq x_{i,n}$ for $i \leq b$ and $j \leq n$;
(ii) if $y_{i,j-1} = 0$ and $y_{i,j} = 1$, then $y_{i,k} = 0$ for $k < j$;
(iii) if $y_{i,j} = 1$ and $y_{i,j+1} = 0$, then $y_{i,k} = 0$ for $k > j$;
(iv) $y_{1,0} \geq y_{2,0} \geq \cdots \geq y_{v,0}$;
(v) if $y_{i,0} = y_{i+1,0}, y_{i,1} = y_{i+1,1}, \cdots, y_{i,j} = y_{i+1,j}$, then there is $y_{i,j+1} \geq y_{i+1,j+1}$;
(vi) use the same rules (ii)~(iv) for $z_{i,j}$.

According to the properties of p-adic numbers, for the p-adic expression of $\deg(x_i)$, $\deg(y_i)$, and $\deg(z_i)$, we have the following group of linear equations (β)

$$\begin{cases} x_{1,0} + \cdots + x_{b,0} + y_{1,0} + \cdots + y_{v,0} = \overline{c}_0 + k_1 p = c_0 \\ x_{1,1} + \cdots + x_{b,1} + y_{1,1} + \cdots + y_{v,1} + z_{1,1} + \cdots + z_{l,1} = \overline{c}_1 - k_1 + k_2 p = c_1 \\ \cdots \qquad \cdots \qquad \cdots \\ x_{1,n-1} + \cdots + x_{b,n-1} + y_{1,n-1} + \cdots + y_{v,n-1} + z_{1,n-1} + \cdots + z_{l,n-1} = \overline{c}_{n-1} - k_{n-1} + k_n p = c_{n-1} \\ x_{1,n} + \cdots + x_{b,n} + y_{1,n} + \cdots + y_{v,n} + z_{1,n} + \cdots + z_{l,n} = \overline{c}_n - k_n = c_n. \end{cases}$$

There are two integer sequences

$$K = (k_1, \cdots, k_n) \text{ and } S = (c_0, \cdots, c_n)$$

where S is determined by $(\overline{c}_0, \cdots, \overline{c}_n)$. The group of linear equations (β) is said to have a solution if its solution satisfies the conditions (α).

We express the above group of linear equations by the following matrix

$$\begin{array}{c} \qquad A \qquad\qquad\qquad B \qquad\qquad\qquad C \\ \left(\begin{array}{ccc|ccc|ccc} x_{1,0} & \cdots & x_{b,0} & y_{1,0} & \cdots & y_{m,0} & 0 & \cdots & 0 \\ x_{1,1} & \cdots & x_{b,1} & y_{1,1} & \cdots & y_{m,1} & z_{1,1} & \cdots & z_{l,1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{1,n} & \cdots & x_{b,n} & y_{1,n} & \cdots & y_{m,n} & z_{1,n} & \cdots & z_{l,n} \end{array}\right) \begin{array}{c} c_0 \\ c_1 \\ \vdots \\ c_n. \end{array} \end{array} \tag{1}$$

According to the conditions (α), we see that Section A is an echelon matrix as

$$\begin{pmatrix} 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 \\ 1 & \cdots & 1 & \cdots & \cdots & 1 & & \\ \vdots & \vdots & \vdots & \vdots & & & & \\ 1 & \cdots & 1 & & & & & \\ 1 & & & & & & & \end{pmatrix} \tag{2}$$

where the missing numbers are all zeroes. Section B has the form

$$\begin{pmatrix} 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 & & & & & & & & & \\ 1 & \cdots & 1 & \cdots & 1 & & & \ddots & & & & & & & & \\ \vdots & \vdots & \vdots & \vdots & & & & & \ddots & & & & & & & \\ 1 & \cdots & 1 & & & & & & & 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 \\ 1 & & & & & & & & & 1 & \cdots & 1 & \cdots & 1 & & \\ & & & & & & & & & \vdots & \vdots & \vdots & & & & \\ & & & & & & & & & 1 & \cdots & 1 & & & & \\ & & & & & & & & & 1 & & & & & & \end{pmatrix} \tag{3}$$

where the missing numbers are all zeroes. Section C has a similar form as Section B except that the first horizontal lines are all replaced by zeroes.

By observing the above matrices, we can easily find that each column in Section A determines some x_i, and each column in Section B or C determines some y_i or zi, respectively. It should be noticed that each column in the above matrix does not have the form $(\cdots,1,0,\cdots,0,1,\cdots)^T$, i.e., there are no zeroes between 1 and 1. Finally, the determination of $E_1^{s,t+b,*}$ can be reduced to the following steps:

Step 1. Express t/q by the p-adic number as the form $t = (\overline{c}_0 + \overline{c}_1 p + \cdots + \overline{c}_n p^n)q$.

Step 2. List all the possible of sequence K whose numbers lie in the corresponding of sequence S.

Step 3. For each sequence S, we can solve the group of linear equations (β) which satisfies the conditions (α).

Step 4. Do the replacement $h_{i,j+1} \to b_{i,j}$ and resolution $h_{i,j} \to h_{i-k,j+k}h_{k,j}$ or $a_i \to a_{i-j}h_{j,i-j}$.

3. Computation of E_1-term of the MSS

In order to prove our main theorem, we will apply the method introduced in Section 2 to determine the generators of $E_1^{s-r+7,tq+(s-r+1),*}$ where $1 \le r \le s+7$ and $t = s+2+(s+2)p+(s+3)p^2+(s+4)p^3+p^n$. In what follows, we use M_i to denote the May filtration for each $i \ge 1$.

Theorem 3.1. For each $0 \le s \le p-5$, we let $t = s+2+(s+2)p+(s+3)p^2+(s+4)p^3+p^n$. If $1 \le r \le s+7$ and $n \ge 1$, then we have the generators of $E_1^{s-r+7,tq+(s-r+1),*}$ as follows:

(1) when $2 \le r \le s+7$, there is no generator;

(2) when $r = 1$, the generators are stated as follows:

(a) if $n = 1$, there is one generator $a_4^s h_{4,0}h_{1,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}$ $(M_1 = 8s+18)$;

(b) if $n = 2$, there is one generator $a_4^s h_{4,0}h_{1,0}h_{3,1}h_{1,2}h_{2,2}h_{1,3}$ $(M_1 = 8s+18)$;

(c) if $n = 3$, there is no generator;

(d) if $n = 4$, there are 54 generators

$$
\left\{
\begin{array}{l}
a_4^s h_{5,0} h_{4,0} b_{2,1} b_{1,2} \quad (M_2 = 9s + 4p + 16) \\
\left.\begin{array}{ll}
a_4^{s-1} a_1 h_{5,0} h_{4,0} h_{2,2} h_{1,3} b_{3,0}, & a_4^s h_{2,0} h_{4,0} h_{2,2} h_{1,3} b_{3,1} \\
a_4^s h_{5,0} h_{1,0} h_{2,2} h_{1,3} b_{3,0}, & a_5 a_4^{s-1} h_{4,0} h_{1,0} h_{2,2} h_{1,3} b_{3,0} \\
a_4^s h_{4,0} h_{1,0} h_{3,2} h_{1,3} b_{3,0}, & a_4^s h_{4,0} h_{1,0} h_{3,1} h_{1,3} b_{3,1} \\
a_4^s h_{4,0} h_{2,0} h_{2,2} h_{1,3} b_{3,2}, & a_4^s h_{4,0} h_{1,0} h_{2,2} h_{2,3} b_{3,0}
\end{array}\right\} (M_3 = 9s + 5p + 14) \\
a_4^s h_{1,0} h_{4,0} h_{2,2} h_{1,3} b_{4,0} \quad (M_4 = 9s + 7p + 12) \\
\left.\begin{array}{ll}
a_4^{s-1} a_1 h_{3,1} h_{5,0} h_{4,0} h_{1,3} b_{2,1}, & a_4^{s-1} a_2 h_{5,0} h_{4,0} h_{2,2} h_{1,3} b_{2,1} \\
a_4^s h_{1,0} h_{4,1} h_{4,0} h_{1,3} b_{2,1}, & a_4^s h_{3,0} h_{4,0} h_{2,2} h_{1,3} b_{2,2} \\
a_4^s h_{5,0} h_{1,0} h_{3,1} h_{1,3} b_{2,1}, & a_4^s h_{5,0} h_{2,0} h_{2,2} h_{1,3} b_{2,1} \\
a_5 a_4^{s-1} h_{4,0} h_{1,0} h_{3,1} h_{1,3} b_{2,1}, & a_5 a_4^{s-1} h_{4,0} h_{2,0} h_{2,2} h_{1,3} b_{2,1} \\
a_4^s h_{4,0} h_{2,0} h_{3,2} h_{1,3} b_{2,1}, & a_4^s h_{4,0} h_{1,0} h_{3,1} h_{2,3} b_{2,1} \\
a_4^s h_{4,0} h_{1,0} h_{3,1} h_{2,2} b_{2,2}, & a_4^s h_{4,0} h_{2,0} h_{2,2} h_{2,3} b_{2,1} \\
a_4^s h_{4,0} h_{3,0} h_{1,3} h_{2,3} b_{2,1}, & a_4^s h_{4,0} h_{3,0} h_{1,3} h_{2,2} b_{2,2}
\end{array}\right\} (M_5 = 9s + 3p + 16) \\
a_4^s h_{2,0} h_{4,1} h_{4,0} h_{1,3} b_{2,1} \quad (M_6 = 9s + 3p + 18) \\
\left.\begin{array}{ll}
a_4^{s-1} a_1 h_{3,1} h_{5,0} h_{4,0} h_{2,2} b_{1,2}, & a_4^{s-1} a_3 h_{5,0} h_{4,0} h_{2,2} h_{1,3} b_{1,2} \\
a_4^s h_{3,0} h_{4,1} h_{4,0} h_{2,2} b_{1,2}, & a_4^s h_{5,0} h_{1,0} h_{3,1} h_{2,2} b_{1,2} \\
a_4^s h_{5,0} h_{3,0} h_{1,3} h_{2,2} b_{1,2}, & a_4^s h_{5,0} h_{4,0} h_{1,2} h_{1,3} b_{1,2} \\
a_5 a_4^{s-1} h_{4,0} h_{3,0} h_{1,3} h_{2,2} b_{1,2}, & a_4^s h_{4,0} h_{1,0} h_{3,1} h_{3,2} b_{1,2} \\
a_4^s h_{4,0} h_{3,0} h_{1,3} h_{3,2} b_{1,2}, & a_4^s h_{4,0} h_{3,0} h_{2,2} h_{2,3} b_{1,2}
\end{array}\right\} (M_7 = 9s + p + 18) \\
a_4^s h_{2,0} h_{4,1} h_{4,0} h_{2,2} b_{1,2}, \quad a_4^s h_{4,0} h_{2,0} h_{3,1} h_{3,2} b_{1,2} \quad (M_8 = 9s + p + 20) \\
a_4^s h_{3,0} h_{4,1} h_{4,0} h_{2,2} b_{1,2} \quad (M_9 = 9s + p + 22)
\end{array}
\right.
$$

$$
\left\{
\begin{array}{l}
a_4^s h_{1,0} h_{1,1} h_{2,3} h_{4,0} h_{2,2} h_{1,3} \quad (M_{10} = 9s + 16) \\
\left.\begin{array}{ll}
a_4^{s-1} a_1 h_{3,1} h_{1,0} h_{4,1} h_{4,0} h_{2,2} h_{1,3}, & a_4^{s-1} a_1 h_{3,1} h_{2,0} h_{3,2} h_{4,0} h_{2,2} h_{1,3} \\
a_4^{s-1} a_1 h_{3,1} h_{3,0} h_{2,3} h_{4,0} h_{2,2} h_{1,3}, & a_4^s h_{1,0} h_{3,1} h_{1,4} h_{4,0} h_{2,2} h_{1,3} \\
a_4^s h_{1,0} h_{1,2} h_{3,2} h_{4,0} h_{2,2} h_{1,3}, & a_4^s h_{2,0} h_{3,2} h_{1,0} h_{3,1} h_{2,2} h_{1,3} \\
a_4^s h_{1,0} h_{2,1} h_{2,3} h_{4,0} h_{2,2} h_{1,3}, & a_4^s h_{2,0} h_{1,2} h_{2,3} h_{4,0} h_{2,2} h_{1,3} \\
a_4^s h_{3,0} h_{2,3} h_{1,0} h_{3,1} h_{2,2} h_{1,3}, & a_4^{s-1} a_2 h_{4,0} h_{1,0} h_{3,1} h_{3,2} h_{2,2} h_{1,3} \\
a_4^{s-1} a_3 h_{4,0} h_{1,0} h_{3,1} h_{2,3} h_{2,2} h_{1,3}, & a_4^{s-1} a_0 h_{5,0} h_{4,0} h_{1,0} h_{3,1} h_{2,2} h_{1,3}, \\
a_4^s h_{4,0} h_{1,0} h_{3,1} h_{1,2} h_{2,3} h_{1,3}, & a_4^s h_{4,0} h_{1,0} h_{1,1} h_{2,2} h_{3,2} h_{1,3}
\end{array}\right\} (M_{11} = 9s + 18) \\
a_4^s h_{1,0} h_{2,1} h_{3,2} h_{4,0} h_{2,2} h_{1,3} \quad (M_{12} = 9s + 20)
\end{array}
\right.
$$

(e) for $n = 5$, there are two generators:

$$
\left\{
\begin{array}{l}
a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} h_{1,5} \quad (M_{13} = 9s + 18) \\
a_5^{p-5} h_{5,0} h_{4,1} h_{3,2} h_{2,3} h_{1,0} h_{1,4} \quad (M_{14} = 11p - 29, s = p - 5)
\end{array}
\right.
$$

(f) for $n > 5$ and $s = p - 5$, there are 15 families of generators

$$\begin{cases} \mathrm{h}_1 = a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} h_{1,n} \ (M_{15} = 9s + 18) \\ \mathrm{h}_2 = a_n^{p-5} h_{n,0} h_{5,0} h_{n-2,2} h_{n-3,3} b_{n-4,3} \ \ (M_{16} = 4np - 4n - 8p - 9) \\ \mathrm{h}_3 = a_n^{p-5} h_{n,0} h_{5,0} h_{n-2,2} h_{n-4,4} b_{n-3,2} \ \ (M_{17} = 4np - 4n - 6p - 11) \\ \mathrm{h}_4 = a_n^{p-5} h_{n,0} h_{5,0} h_{n-3,3} h_{n-4,4} b_{n-2,1} \ \ (M_{18} = 4np - 4n - 4p - 13) \\ \left. \begin{array}{r} \mathrm{h}_5 = a_n^{p-5} h_{n,0} h_{1,0} h_{n-1,1} h_{n-2,2} h_{n-3,3} h_{1,4} \\ \mathrm{h}_6^{(\mathrm{i})} = a_n^{p-5} h_{n,0} h_{i,0} h_{n-i,i} h_{n-2,2} h_{n-3,3} h_{1,4} (4 \le i \le n-1) \\ \mathrm{h}_7^{(\mathrm{i})} = a_n^{p-5} h_{n,0} h_{i,0} h_{n-i,i} h_{n-2,2} h_{2,3} h_{n-4,4} (1 \le i \le n-1, i \ne 2, 4) \\ \mathrm{h}_8^{(\mathrm{i})} = a_n^{p-5} h_{n,0} h_{i,0} h_{n-i,i} h_{3,2} h_{n-3,3} h_{n-4,4} (1 \le i \le n-1, i \ne 3, 4) \\ \mathrm{h}_9 = a_n^{p-6} a_1 h_{n-1,1} h_{n,0} h_{5,0} h_{n-2,2} h_{n-3,3} h_{n-4,4} \\ \mathrm{h}_{10} = a_n^{p-5} h_{1,0} h_{n-1,1} h_{5,0} h_{n-2,2} h_{n-3,3} h_{n-4,4} \\ \mathrm{h}_{11}^{(\mathrm{i})} = a_n^{p-5} h_{i,0} h_{n-i,i} h_{5,0} h_{n-2,2} h_{n-3,3} h_{n-4,4} (6 \le i \le n-1) \\ \mathrm{h}_{12}^{(\mathrm{i})} = a_n^{p-5} h_{n,0} h_{i,0} h_{5-i,i} h_{n-2,2} h_{n-3,3} h_{n-4,4} (1 \le i \le 4) \\ \mathrm{h}_{13}^{(\mathrm{i})} = a_n^{p-5} h_{n,0} h_{5,0} h_{i,2} h_{n-i-2,i+2} h_{n-3,3} h_{n-4,4} (3 \le i \le n-3) \\ \mathrm{h}_{14}^{(\mathrm{i})} = a_n^{p-5} h_{n,0} h_{5,0} h_{n-2,2} h_{i,3} h_{n-i-3,i+3} h_{n-4,4} (2 \le i \le n-4) \\ \mathrm{h}_{15}^{(\mathrm{i})} = a_n^{p-5} h_{n,0} h_{5,0} h_{n-2,2} h_{n-3,3} h_{i,4} h_{n-i-4,i+4} (1 \le i \le n-5) \end{array} \right\} (M_{19}) \end{cases}$$

where $M_{19} = 2_{np} - 2_n +_p - 19$.

In what follows, we give the proof of Theorem 3.1 case-by-case when r and n take different values.

Proof. For $t = s + 2 + (s + 2)p + (s + 3)p^2 + (s + 4)p^3 + p^n$, we see that

when $n = 1$, there exists the sequence $\overline{S}_1 = (s + 2, s + 3, s + 3, s + 4)$;

when $n = 2$, there exists the sequence $\overline{S}_2 = (s + 2, s + 2, s + 4, s + 4)$;

when $n = 3$, there exists the sequence $\overline{S}_3 = (s + 2, s + 2, s + 3, s + 5)$;

when $n \geq 4$, there exists the sequence with form

$$\overline{S} = (\overline{c}_0, \cdots, \overline{c}_n) = (s + 2, s + 2, s + 3, s + 4, 0, \cdots, 0, 1).$$

For a given generator $h \in E_1^{s-r+7, tq+(s-r+1),*}$ with $1 \leq r \leq s + 7$, we have

$$\dim(h) = s - r + 7 \text{ and } \deg(h) = tq + (s - r + 1).$$

If $s + 1 < r \leq s + 7$, then there is $s - r + 1 < 0$. This implies that the number of a_i in h is $s - r + 1 + q$. This is impossible due to the reason of dimension. Thus, we can now assume $2 \leq r \leq s + 1$ which follows $s - r + 1 \geq 0$.

Since $s - r + 7 < s - r + 1 + q$, we see that the number of x_i in h is $s - r + 1$. According to the reason of dimension, we list all the possibilities of h as

$$\begin{cases} x_1 \cdots x_{s-r+1} z_1 z_2 z_3 \\ x_1 \cdots x_{s-r+1} y_1 y_2 z_1 z_2 \\ x_1 \cdots x_{s-r+1} y_1 y_2 y_3 y_4 z_1 \\ x_1 \cdots x_{s-r+1} y_1 y_2 y_3 y_4 y_5 y_6. \end{cases}$$

When $1 \leq n \leq 3$, the sequence $K = (k_1, \cdots, k_n)$ becomes $(0, \cdots, 0)$ and thus the corresponding $(c_0, \cdots, c_n)$ equals to $\overline{S}_n = (\overline{c}_0, \cdots, \overline{c}_n)$. When $n = 4$, we have $(k_1, \cdots, k_4) = (0, \cdots, 0)$ and then

$$(c_0, \cdots, c_n) = \overline{S} = (s + 2, s + 2, s + 3, s + 4, 1)$$

When $n \geq 5$, all the possibilities of $K = (k_1, \cdots, k_n)$ are listed as

$$\begin{cases} K_1 = (0, \cdots, 0) \\ K_i = (0, 0, 0, 0, \cdots, 0, 1^{(i)}, \cdots, 1) \quad (5 \leq i \leq n) \end{cases}$$

where $1^{(i)}$ means that the i-th term of K_i is 1. For each K_i, we get the corresponding $S = (c_0, \cdots, c_n)$ as

$$\begin{cases} S_1 = (s+2, s+2, s+3, s+4, 0\cdots, 0, 1) \\ S_i = (s+2, s+2, s+3, s+4, 0\cdots, 0, p^{(i)}, p-1, \cdots, p-1, 0) \quad (5 \le i \le n). \end{cases}$$

where $p^{(i)}$ means that the i-th term of S_i is p.

Case 1. $h = x_1 \cdots x_{s-r+1} z_1 z_2 z_3$.

In this case, we see that $s \le p - 5$ and $r \ge 1$, which follows the inequality

$$\sum_{i=1}^{s-r+1} x_{i,0} \le s - r + 1 \le s - 1 + 1 < \overline{c}_0 = s + 2.$$

This implies that the first equation of (β) has no solution. Thus, such h cannot exist.

Case 2. $h = x_1 \cdots x_{s-r+1} y_1 y_2 z_1 z_2$.

Subcase 2.1. $s \le p - 5$ and $r \ge 2$.

Since

$$\sum_{i=1}^{s-r+1} x_{i,3} + y_{1,3} + y_{2,3} + z_{1,3} + z_{2,3} \le s + 3 < \overline{c}_3 = s + 4,$$

it follows that the fourth equation of (β) has no solution. Such h cannot exist.

Subcase 2.2. $s \le p - 5$, $r = 1$, and $n = 1, 2$.

For $\overline{S}_1$ and $\overline{S}_2$, solving the corresponding group of equations (β) by virtue of (α) gives generators $h = a_4^s h_{4,0}^2 b_{3,0} b_{1,2}$ and $a_4^s h_{4,0}^2 b_{2,1}^2$, which are both zeroes due to $h_{4,0}^2 = 0$.

Subcase 2.3. $s \le p - 5$, $r = 1$, and $n = 3$.

Since

$$\sum_{i=1}^{s-r+1} x_{i,3} + y_{1,3} + y_{2,3} + z_{1,3} + z_{2,3} \le s + 4 < \overline{c}_3 = s + 5,$$

the fourth equation of (β) has no solution. It follows that such h cannot exist.

Subcase 2.4. $s \le p - 5$, $r = 1$, and $n = 4$.

For $\overline{S} = (s+2, s+2, s+3, s+4, 1)$, solving the corresponding group of equations (β) by virtue of (α) gives one generator

$$a_4^s h_{5,0} h_{4,0} b_{2,1} b_{1,2}. \tag{4}$$

Subcase 2.5. $s \le p - 5$, $r = 1$, and $n \ge 5$.

For S_1, the corresponding group of equations (β) by virtue of (α) has no solution since there is a solution with the form $(0, \cdots, 0, 1, \cdots, 1, 0 \cdots, 0, 1)$.

For S_i $(5 \le i \le n)$, the inequality

$$\sum_{j=1}^{s-r+1} x_{j,i-1} + y_{1,i-1} + y_{2,i-1} + z_{1,i-1} + z_{2,i-1} \le s + 4 < \overline{c}_{i-1} = p$$

implies that there is no solution for the i-th equation of (β). Thus, such h cannot exist.

Case 3. $h = x_1 \cdots x_{s-r+1} y_1 y_2 y_3 y_4 z_1$.

Subcase 3.1. $s \le p - 5$ and $r > 2$.

The inequality

$$\sum_{i=1}^{s-r+1} x_{i,3} + \sum_{i=1}^{4} y_{i,3} + z_{1,3} < s + 4 \le \overline{c}_3$$

implies that there is no solution for the fourth equation of (β). Thus, such h cannot exist.

Subcase 3.2. $s \le p - 5$, $r = 2$, and $n = 1, 2$.

For $\overline{S}_1$ and $\overline{S}_2$, solving the corresponding group of equations (β) by virtue of (α) gives $h = a_4^{s-1} h_{4,0}^3 h_{3,1} b_{1,2}$, $a_4^{s-1} h_{4,0}^3 h_{1,4} b_{3,0}$, and $a_4^{s-1} h_{4,0}^3 h_{2,2} b_{2,1}$, respectively. They are all zeroes since $h_{4,0}^3 = 0$.

Subcase 3.3. $s \le p - 5$, $r = 2$, and $n = 3$.

For $\overline{S}_3$, the inequality

$$\sum_{i=1}^{s-r+1} x_{i,3} + \sum_{i=1}^{4} y_{i,3} + z_{1,3} \le s + 4 < \overline{c}_3 = s + 5$$

implies that there is no solution for the fourth equation of (β). Thus, such h cannot exist.

Subcase 3.4. $s \le p - 5$, $r = 2$, and $n = 4$.

For $\overline{S} = (s+2, s+2, s+3, s+4, 1)$, solving the corresponding group of equations (β) by virtue of (α) gives the following generators:

$$\begin{array}{llll} a_4^{s-2} a_5 h_{4,0}^3 h_{2,2} b_{1,2} & a_4^{s-1} h_{5,0} h_{4,0}^2 h_{2,2} b_{1,2} & a_4^{s-1} h_{4,0}^3 h_{3,2} b_{1,2} & a_4^{s-1} h_{4,0}^3 h_{2,2} b_{2,2} \\ a_4^{s-2} a_5 h_{4,0}^3 h_{1,3} b_{2,1} & a_4^{s-1} h_{5,0} h_{4,0}^2 h_{1,3} b_{2,1} & a_4^{s-1} h_{4,0}^3 h_{2,3} b_{2,1} & a_4^{s-1} h_{4,0}^3 h_{1,3} b_{3,1}. \end{array}$$

They are all zeroes since they all contain $h_{4,0}^2 = 0$.

Subcase 3.5. $s \le p - 5$, $r = 1$, and $1 \le n \le 3$.

For $\overline{S}_1$, solving the corresponding group of equations (β) by virtue of (α) gives the following generators:

$$a_4^{s-1} h_{4,0}^2 h_{2,2} h_{1,3} b_{1,0},\ a_4^{s-1} h_{4,0}^2 h_{1,3}^2 b_{2,0} \text{ and } a_4^{s-1} h_{4,0}^2 h_{2,1} h_{1,3} b_{1,2}.$$

They are all zeroes since they contain $h_{4,0}^2 = 0$.

For $\overline{S}_2$, solving the corresponding group of equations (β) by virtue of (α) gives the generators

$$a_4^{s-1} h_{4,0}^2 h_{2,2} h_{1,3} b_{1,1} \text{ and } a_4^{s-1} h_{4,0}^2 h_{2,2} h_{1,2} b_{1,2},$$

which are both zeroes since they contain $h_{4,0}^2 = 0$.

For $\overline{S}_3$, solving the corresponding group of equations (β) by virtue of (α) gives the generators

$$a_4^{s-1} h_{4,0}^2 h_{2,2} h_{1,3} b_{1,2} \text{ and } a_4^{s-1} h_{4,0}^2 h_{1,3}^2 b_{2,1},$$

which are both zeroes since they contain $h_{4,0}^2 = 0$.

Subcase 3.6. $s \le p - 5$, $r = 1$, and $n = 4$.

This case is a little difficult. Our strategy of computation is stated as follows: for the unique $\overline{S} = (s+2, s+2, s+3, s+4, 1)$, it can be seen that the maximal number in S is $s + 4$. This drives us to firstly compute out all the generators (which may be zero) of $E_1^{s+4,tq+s,*}$ with the form $x_1 \cdots x_s y_1 y_2 y_3 y_4$.

For some a_i or $h_{i,j}$ in $x_1 \cdots x_s y_1 y_2 y_3 y_4$, we first do the resolution

$$h_{i,j} \to h_{k,j} h_{i-k,j+k} \text{ or } a_i \to a_j h_{i-j,j},$$

and then do the replacement

$$h_{i,j} \to b_{i,j-1} \text{ for } (j \ge 1).$$

If what we got is all nonzero, then all the desired generators of $E_1^{s+7,tq+s,*}$ are obtained and also have the form $x_1 \cdots x_s y_1 y_2 y_3 y_4 z_1$. We do the explicit computation by the following two steps:

Step 1. For $\overline{S} = (s+2, s+2, s+3, s+4, 1)$, solving the corresponding group of equations (β) by virtue of (α) gives a set of generators of $E_1^{s+4,tq+s,*}$ as

$$a_4^s h_{5,0} h_{4,0} h_{2,2} h_{1,3}, \qquad a_5 a_4^{s-1} \underline{h_{4,0} h_{4,0}} h_{2,2} h_{1,3}, \qquad a_4^s \underline{h_{4,0} h_{4,0}} h_{3,2} h_{1,3}, \qquad a_4^s \underline{h_{4,0} h_{4,0}} h_{2,2} h_{2,3}.$$

Step 2. Resolve a_i or $h_{i,j}$, and then replace $h_{i',j'}$ by $b_{i',j'-1}$ for each $j' \geq 1$.

In the following table, the elements at the rightmost column are the obtained generators. For simplicity, we only write out the nonzero ones.

$$a_5 a_4^{s-1} \underline{h_{4,0} h_{4,0}} h_{2,2} h_{1,3} \xrightarrow{h_{4,0}} \begin{cases} a_5 a_4^{s-1} h_{4,0} \underline{h_{1,0} h_{3,1}} h_{2,2} h_{1,3} \xrightarrow{h_{3,1}, h_{2,2}, h_{1,3}} \begin{cases} a_5 a_4^{s-1} h_{4,0} h_{1,0} h_{2,2} h_{1,3} b_{3,0} \\ a_5 a_4^{s-1} h_{4,0} h_{1,0} h_{3,1} h_{1,3} b_{2,1} \\ a_5 a_4^{s-1} h_{4,0} h_{1,0} h_{3,1} h_{2,2} b_{1,2} \end{cases} \\ a_5 a_4^{s-1} h_{4,0} \underline{h_{2,0} h_{2,2}} h_{2,2} h_{1,3} \xrightarrow{h_{2,2}} a_5 a_4^{s-1} h_{4,0} h_{2,0} h_{2,2} h_{1,3} b_{2,1} \\ a_5 a_4^{s-1} h_{4,0} \underline{h_{3,0} h_{1,3}} h_{2,2} h_{1,3} \xrightarrow{h_{1,3}} a_5 a_4^{s-1} h_{4,0} h_{3,0} h_{1,3} h_{2,2} b_{1,2} \end{cases} \tag{5}$$

$$a_4^s \underline{h_{4,0} h_{4,0}} h_{3,2} h_{1,3} h_{4,0} \begin{cases} a_4^s h_{4,0} \underline{h_{1,0} h_{3,1}} h_{3,2} h_{1,3} \xrightarrow{h_{3,1}, h_{3,2}, h_{1,3}} \begin{cases} a_4^s h_{4,0} h_{1,0} h_{3,2} h_{1,3} b_{3,0} \\ a_4^s h_{4,0} h_{1,0} h_{3,1} h_{1,3} b_{3,1} \\ a_4^s h_{4,0} h_{1,0} h_{3,1} h_{3,2} b_{1,2} \end{cases} \\ a_4^s h_{4,0} \underline{h_{2,0} h_{2,2}} h_{3,2} h_{1,3} \xrightarrow{h_{2,2}, h_{3,2}, h_{1,3}} \begin{cases} a_4^s h_{4,0} h_{2,0} h_{3,2} h_{1,3} b_{2,1} \\ a_4^s h_{4,0} h_{2,0} h_{2,2} h_{1,3} b_{3,2} \\ a_4^s h_{4,0} h_{2,0} h_{3,1} h_{3,2} b_{1,2} \end{cases} \\ a_4^s h_{4,0} \underline{h_{3,0} h_{1,3}} h_{3,2} h_{1,3} \xrightarrow{h_{1,3}} a_4^s h_{4,0} h_{3,0} h_{1,3} h_{3,2} b_{1,2} \end{cases} \tag{6}$$

$$
a_4^s h_{5,0} h_{4,0} h_{2,2} h_{1,3} \overset{\text{one by one}}{\longrightarrow}
\begin{cases}
a_4^{s-1} \underline{a_1 h_{3,1}} h_{5,0} h_{4,0} h_{2,2} h_{1,3} \xrightarrow{h_{3,1}, h_{2,2}, h_{1,3}} \begin{cases} a_4^{s-1} a_1 h_{5,0} h_{4,0} h_{2,2} h_{1,3} b_{3,0} \\ a_4^{s-1} a_1 h_{3,1} h_{5,0} h_{4,0} h_{1,3} b_{2,1} \\ a_4^{s-1} a_1 h_{3,1} h_{5,0} h_{4,0} h_{2,2} b_{1,2} \end{cases} \\
a_4^{s-1} \underline{a_2 h_{2,2}} h_{5,0} h_{4,0} h_{2,2} h_{1,3} \xrightarrow{h_{2,2}} a_4^{s-1} a_2 h_{5,0} h_{4,0} h_{2,2} h_{1,3} b_{2,1} \\
a_4^{s-1} \underline{a_3 h_{1,3}} h_{5,0} h_{4,0} h_{2,2} h_{1,3} \xrightarrow{h_{1,3}} a_4^{s-1} a_3 h_{5,0} h_{4,0} h_{2,2} h_{1,3} b_{1,2} \\
a_4^s \underline{h_{1,0} h_{4,1}} h_{4,0} h_{2,2} h_{1,3} \xrightarrow{h_{4,1}, h_{2,2}, h_{1,3}} \begin{cases} a_4^s h_{1,0} h_{4,0} h_{2,2} h_{1,3} b_{4,0} \\ a_4^s h_{1,0} h_{4,1} h_{4,0} h_{1,3} b_{2,1} \\ a_4^s h_{1,0} h_{4,1} h_{4,0} h_{2,2} b_{1,2} \end{cases} \\
a_4^s \underline{h_{2,0} h_{3,2}} h_{4,0} h_{2,2} h_{1,3} \xrightarrow{h_{3,2}, h_{2,2}, h_{1,3}} \begin{cases} a_4^s h_{2,0} h_{4,0} h_{2,2} h_{1,3} b_{3,1} \\ a_4^s h_{2,0} h_{4,1} h_{4,0} h_{1,3} b_{2,1} \\ a_4^s h_{2,0} h_{4,1} h_{4,0} h_{2,2} b_{1,2} \end{cases} \\
a_4^s \underline{h_{3,0} h_{2,3}} h_{4,0} h_{2,2} h_{1,3} \xrightarrow{h_{2,3}, h_{2,2}, h_{1,3}} \begin{cases} a_4^s h_{3,0} h_{4,0} h_{2,2} h_{1,3} b_{2,2} \\ a_4^s h_{3,0} h_{4,1} h_{4,0} h_{1,3} b_{2,1} \\ a_4^s h_{3,0} h_{4,1} h_{4,0} h_{2,2} b_{1,2} \end{cases} \\
a_4^s h_{5,0} \underline{h_{1,0} h_{3,1}} h_{2,2} h_{1,3} \xrightarrow{h_{3,1}, h_{2,2}, h_{1,3}} \begin{cases} a_4^s h_{5,0} h_{1,0} h_{2,2} h_{1,3} b_{3,0} \\ a_4^s h_{5,0} h_{1,0} h_{3,1} h_{1,3} b_{2,1} \\ a_4^s h_{5,0} h_{1,0} h_{3,1} h_{2,2} b_{1,2} \end{cases} \\
a_4^s h_{5,0} \underline{h_{2,0} h_{2,2}} h_{2,2} h_{1,3} h_{2,2} a_4^s h_{5,0} h_{2,0} h_{2,2} h_{1,3} b_{2,1} \\
a_4^s h_{5,0} \underline{h_{3,0} h_{1,3}} h_{2,2} h_{1,3} h_{1,3} a_4^s h_{5,0} h_{3,0} h_{1,3} h_{2,2} b_{1,2} \\
a_4^s h_{5,0} h_{4,0} \underline{h_{1,2} h_{1,3}} h_{1,3} h_{1,3} a_4^s h_{5,0} h_{4,0} h_{1,2} h_{1,3} b_{1,2}
\end{cases}
\tag{7}
$$

$$
a_4^s \underline{h_{4,0} h_{4,0}} h_{2,2} h_{2,3} h_{4,0}
\begin{cases}
a_4^s h_{4,0} \underline{h_{1,0} h_{3,1}} h_{2,2} h_{2,3} \xrightarrow{h_{3,1}, h_{2,2}, h_{2,3}} \begin{cases} a_4^s h_{4,0} h_{1,0} h_{2,2} h_{2,3} b_{3,0} \\ a_4^s h_{4,0} h_{1,0} h_{3,1} h_{2,3} b_{2,1} \\ a_4^s h_{4,0} h_{1,0} h_{3,1} h_{2,2} b_{2,2} \end{cases} \\
a_4^s h_{4,0} \underline{h_{2,0} h_{2,2}} h_{2,2} h_{2,3} \xrightarrow{h_{2,2}} a_4^s h_{4,0} h_{2,0} h_{2,2} h_{2,3} b_{2,1} \\
a_4^s h_{4,0} \underline{h_{3,0} h_{1,3}} h_{2,2} h_{2,3} \xrightarrow{h_{1,3}, h_{2,2}, h_{2,3}} \begin{cases} a_4^s h_{4,0} h_{3,0} h_{2,2} h_{2,3} b_{1,2} \\ a_4^s h_{4,0} h_{3,0} h_{1,3} h_{2,3} b_{2,1} \\ a_4^s h_{4,0} h_{3,0} h_{1,3} h_{2,2} b_{2,2} \end{cases}
\end{cases}
\tag{8}
$$

In the above tables, the elements over the left arrows denote the resolutions, and the elements over the right arrows mean their replacements.

Subcase 3.7. $s \leq p - 5$, $r = 1$, and $n \geq 5$.

For S_1, solving the corresponding group of equations (β) by virtue of (α) gives generators as

$$
a_4^s h_{4,0}^2 h_{2,2} h_{1,3} b_{1,n-1} \qquad a_4^s h_{4,0}^2 h_{1,n} h_{1,3} b_{2,1} \qquad a_4^s h_{4,0}^2 h_{2,2} h_{1,n} b_{1,2},
$$

which are all zeroes since $h_{4,0}^2 = 0$.

For $S_i (5 \leq i \leq n)$, if $s < p - 5$ and $n \geq 5$, then the inequality

$$
\sum_{j=1}^{s} x_{j,i-1} + y_{1,i-1} + y_{2,i-1} + y_{3,i-1} + y_{4,i-1} + z_{1,i-1} \leq s + 5 < \overline{c}_{i-1} = p
$$

implies that the i-th equation of (β) has no solution. Thus, such h does not exist.

For S_5, if $s = p - 5$ and $n = 5$, then solving the corresponding group of equations (β) by virtue of (α) gives generators

$$a_5^s h_{5,0}^2 h_{3,2} h_{2,3} b_{1,3} \qquad a_5^s h_{5,0}^2 h_{3,2} h_{1,4} b_{2,2} \qquad a_5^s h_{5,0}^2 h_{2,3} h_{1,4} b_{3,1}.$$

They are all zeroes since $h_{5,0}^2 = 0$.

For S_5, if $s = p - 5$ and $n > 5$, solving the corresponding group of equations (β) by virtue of (α) gives nonzero generators

$$a_n^{p-5} h_{n,0} h_{5,0} h_{n-3,3} h_{n-4,4} b_{n-2,1} \quad a_n^{p-5} h_{n,0} h_{5,0} h_{n-2,2} h_{n-4,4} b_{n-3,2} \quad a_n^{p-5} h_{n,0} h_{5,0} h_{n-2,2} h_{n-3,3} b_{n-4,3}. \tag{9}$$

For S_i ($6 \le i \le n$), if $s = p - 5$ and $n > 5$, then the corresponding group of equations (β) by virtue of (α) has no solution since there is a solution of the form $(0,\cdots,0,1,\cdots,1,0\cdots,0,1)$. Thus, h does not exist.

Case 4. $h = x_1 \cdots x_{s-r+1} y_1 y_2 y_3 y_4 y_5 y_6$.

Subcase 4.1 $s \le p - 5$ and $r > 3$.

The inequality

$$\sum_{i=1}^{s-r+1} x_{i,3} + \sum_{i=1}^{6} y_{i,3} < s + 4 \le \overline{c}_3,$$

implies that the fourth equation of (β) has no solution. Thus, such h does not exist.

Subcase 4.2 $s \le p - 5$, $r = 3$, and $1 \le n \le 4$.

Solving the corresponding group of equations (β) by virtue of (α) gives all zero generators since they all contain $h_{4,0}^2 = 0$. Thus, such nonzero h does not exist.

Subcase 4.3. $s \le p - 5$, $r = 3$ and $n \ge 5$.

There is no solution for the corresponding group of equations (β) by virtue of (α). Thus such h does not exist.

Subcase 4.4. $s \le p - 5$, $r = 2$, and $1 \le n \le 4$.

Solving the corresponding group of equations (β) by virtue of (α) gives all zero generators since they all contain $h_{4,0}^2 = 0$. Thus, such nonzero h does not exist.

Subcase 4.5. $s \le p - 5$, $r = 2$, and $n \ge 5$.

For S_1, solving the corresponding group of equations (β) by virtue of (α) gives one generator $a_4^{s-1} h_{4,0}^3 h_{2,2} h_{1,3} h_{1,n}$, which is zero due to $h_{4,0}^3 = 0$.

For S_5, solving the corresponding group of equations (β) by virtue of (α) gives all zero generators since they all contain $h_{n,0}^2 = 0$. Thus, such nonzero h does not exist.

For S_i with ($6 \le i \le n$), there is no solution for the corresponding group of equations (β) by virtue of (α) since there will be a solution of the form $(0,\cdots,0,1,\cdots,1,0\cdots,0,1)$.

Subcase 4.6. $s \le p - 5$, $r = 1$, and $1 \le n \le 3$.

For $\overline{S}_1$, solving the corresponding group of equations (β) by virtue of (α) gives one nonzero generator

$$a_4^s h_{4,0} h_{1,0} h_{3,1} h_{1,1} h_{2,2} h_{1,3}. \tag{10}$$

For $\overline{S}_2$, solving the corresponding group of equations (β) by virtue of (α) gives one nonzero generator

$$a_4^s h_{4,0} h_{1,0} h_{3,1} h_{1,2} h_{2,2} h_{1,3}. \tag{11}$$

For $\overline{S}_3$, it is easy to see that the obtained generators are all zeroes since they all contain $h_{1,3}^2 = 0$.

Subcase 4.7. $s \le p - 5$, $r = 1$, and $n = 4$.

We deal with this case by a similar method as Subcase 3.6. We see that the maximal number of the unique sequence is $s + 4$. We compute out all the generators (which may be zero) of the form $x_1 \cdots x_s y_1 y_2 y_3 y_4$. Firstly, we resolve a_i or $h_{i,j}$ in $x_1 \cdots x_s y_1 y_2 y_3 y_4$, and then we repeat the first step for the obtained generators of $E_1^{s+5,tq+s,*}$. If the obtained generators are nonzero, then they are our desired generators of $E_1^{s+6,tq+s,*}$ of the form $x_1 \cdots x_s y_1 y_2 y_3 y_4 y_5 y_6$.

As for Subcase 3.6, we have already obtained a set of generators (which may be zero) of $E_1^{s+4,tq+s,*}$ as

$$a_4^s h_{5,0} h_{4,0} h_{2,2} h_{1,3} \qquad a_5 a_4^{s-1} \underline{h_{4,0} h_{4,0}} h_{2,2} h_{1,3} \qquad a_4^s \underline{h_{4,0} h_{4,0}} h_{3,2} h_{1,3} \qquad a_4^s \underline{h_{4,0} h_{4,0}} h_{2,2} h_{2,3}.$$

Then, as stated before, we will do resolution twice as shown in the following table, where the elements in the rightmost column are the obtained generators. For simplicity, we only write out the nonzero ones. Since there will be some identical generators coming from different resolutions, we will just write out only one of them.

$$a_4^s h_{5,0} h_{4,0} h_{2,2} h_{1,3} \xrightarrow{a_4, h_{5,0}} \begin{cases} a_4^{s-1} \underline{a_1 h_{3,1}} h_{5,0} h_{4,0} h_{2,2} h_{1,3} \xrightarrow{h_{5,0}} \begin{cases} a_4^{s-1} a_1 h_{3,1} \underline{h_{1,0} h_{4,1}} h_{4,0} h_{2,2} h_{1,3} \\ a_4^{s-1} a_1 h_{3,1} \underline{h_{2,0} h_{3,2}} h_{4,0} h_{2,2} h_{1,3} \\ a_4^{s-1} a_1 h_{3,1} \underline{h_{3,0} h_{2,3}} h_{4,0} h_{2,2} h_{1,3} \end{cases} \\ a_4^s \underline{h_{1,0} h_{4,1}} h_{4,0} h_{2,2} h_{1,3} \xrightarrow{h_{4,1}} \begin{cases} a_4^s h_{1,0} \underline{h_{1,1} h_{2,3}} h_{4,0} h_{2,2} h_{1,3} \\ a_4^s h_{1,0} \underline{h_{2,1} h_{3,2}} h_{4,0} h_{2,2} h_{1,3} \\ a_4^s h_{1,0} \underline{h_{3,1} h_{1,4}} h_{4,0} h_{2,2} h_{1,3} \end{cases} \\ a_4^s \underline{h_{2,0} h_{3,2}} h_{4,0} h_{2,2} h_{1,3} \xrightarrow{h_{2,0}, h_{4,0}} \begin{cases} a_4^s \underline{h_{1,0} h_{1,2}} h_{3,2} h_{4,0} h_{2,2} h_{1,3} \\ a_4^s h_{2,0} h_{3,2} \underline{h_{1,0} h_{3,1}} h_{2,2} h_{1,3} \end{cases} \\ a_4^s \underline{h_{3,0} h_{2,3}} h_{4,0} h_{2,2} h_{1,3} \xrightarrow{h_{3,0}, h_{4,0}} \begin{cases} a_4^s \underline{h_{1,0} h_{2,1}} h_{2,3} h_{4,0} h_{2,2} h_{1,3} \\ a_4^s \underline{h_{2,0} h_{1,2}} h_{2,3} h_{4,0} h_{2,2} h_{1,3} \\ a_4^s h_{3,0} h_{2,3} \underline{h_{1,0} h_{3,1}} h_{2,2} h_{1,3} \end{cases} \end{cases} \tag{12}$$

$$a_5 a_4^{s-1} \underline{h_{4,0} h_{4,0}} h_{2,2} h_{1,3} \xrightarrow{a_5, h_{4,0}} \begin{cases} \underline{a_2 h_{3,2}} a_4^{s-1} \underline{h_{4,0} h_{4,0}} h_{2,2} h_{1,3} \xrightarrow{h_{4,0}} a_4^{s-1} a_2 h_{4,0} \underline{h_{1,0} h_{3,1}} h_{3,2} h_{2,2} h_{1,3} \\ \underline{a_3 h_{2,3}} a_4^{s-1} \underline{h_{4,0} h_{4,0}} h_{2,2} h_{1,3} \xrightarrow{h_{4,0}} a_4^{s-1} a_3 h_{4,0} \underline{h_{1,0} h_{3,1}} h_{2,3} h_{2,2} h_{1,3} \\ a_5 a_4^{s-1} h_{4,0} \underline{h_{1,0} h_{3,1}} h_{2,2} h_{1,3} \xrightarrow{a_5} a_4^{s-1} \underline{a_0 h_{5,0}} h_{4,0} h_{1,0} h_{3,1} h_{2,2} h_{1,3} \end{cases} \tag{13}$$

$$a_4^s \underline{h_{4,0} h_{4,0}} h_{3,2} h_{1,3} \xrightarrow{h_{4,0}} \begin{cases} a_4^s h_{4,0} \underline{h_{1,0} h_{3,1}} h_{3,2} h_{1,3} \xrightarrow{h_{3,2}} a_4^s h_{4,0} h_{1,0} h_{3,1} \underline{h_{1,2} h_{2,3}} h_{1,3} \\ a_4^s h_{4,0} \underline{h_{2,0} h_{2,2}} h_{3,2} h_{1,3} \xrightarrow{h_{2,0}} a_4^s h_{4,0} \underline{h_{1,0} h_{1,1}} h_{2,2} h_{3,2} h_{1,3} \end{cases} \tag{14}$$

In the above tables, the elements over the arrows denote their resolutions. We do not write out the resolutions of $a_4^s \underline{h_{4,0} h_{4,0}} h_{2,2} h_{2,3}$ as it will produce the same nonzero generators produced by the first three generators of

Subcase 4.8. $s \le p - 5$, $r = 1$, and $n \ge 5$.

For S_1, solving the corresponding group of equations (β) by virtue of (α) gives one nonzero generator

$$a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} h_{1,n}. \tag{15}$$

For S_5, if $s = p - 5$ and $n = 5$, solving the corresponding group of equations (β) by virtue of (α) gives one nonzero generator

$$a_5^{p-5} h_{5,0} h_{4,1} h_{3,2} h_{2,3} h_{1,0} h_{1,4}. \tag{16}$$

For S_5, if $s = p - 6$ and $n = 5$, solving the corresponding group of equations (β) by virtue of (α) gives all zero generators since they all contain h_5^2. Thus, such nonzero h does not exist.

For S_5, if $s < p - 6$ and $n = 5$, the inequality

$$\sum_{j=1}^{s} x_{j,i-1} + \sum_{j=1}^{6} y_{j,i-1} \leq s + 6 < \overline{c}_{i-1} = p,$$

implies that there is no solution for the i-th equation of (β). Thus, such h does not exist.

For S_i ($6 \leq i \leq n$), if $s \leq p - 5$ and $n > 5$, there is no solution for the corresponding group of equations (β) by virtue of (α), since there is a solution of the form $(0,\cdots,0,1,\cdots,1,0\cdots,0,1,\cdots,1)$.

For S_5, when $s = p - 5$ and $n > 5$, we will apply the method in Subcase 4.7. We see that the maximal term of the sequence $S_5 = (p - 3, p - 3, p - 2, p - 1, p, p - 1, \cdots, p - 1, 0)$ is p. Firstly, we compute out the generators (which may be zero) of which have form $x_1 \cdots x_p{-}_5 y_1 y_2 y_3 y_4 y_5$. Then, we resolve a_i or $h_{i,j}$ in $x_1 \cdots x_p{-}_5 y_1 y_2 y_3 y_4 y_5$. After finishing these two steps, if what we obtained is nonzero, then they are generators of $E_1^{p+1,tq+p-5,*}$, which have the form $x_1 \cdots x_p -$ $5 y_1 y_2 y_3 y_4 y_5 y_6$.

For $E_1^{p,tq+p-5,*}$, solving the corresponding group of equations (β) by virtue of (α) gives

$$a_n^{p-5}\underline{h_{n,0}h_{n,0}}h_{n-2,2}h_{n-3,3}h_{1,4} \qquad a_n^{p-5}\underline{h_{n,0}h_{n,0}}h_{n-2,2}h_{2,3}h_{n-4,4}.$$
$$a_n^{p-5}\underline{h_{n,0}h_{n,0}}h_{3,2}h_{n-3,3}h_{n-4,4} \qquad a_n^{p-5}\underline{h_{n,0}h_{5,0}}h_{n-2,2}h_{n-3,3}h_{n-4,4}.$$

Then, we do the following resolutions:

$$a_n^{p-5}\underline{h_{n,0}h_{n,0}}h_{n-2,2}h_{n-3,3}h_{1,4} \xrightarrow{h_{n,0}} \begin{cases} a_n^{p-5}h_{n,0}\underline{h_{1,0}h_{n-1,1}}h_{n-2,2}h_{n-3,3}h_{1,4} \\ a_n^{p-5}h_{n,0}\underline{h_{i,0}h_{n-i,i}}h_{n-2,2}h_{n-3,3}h_{1,4} \quad (4 \leq i \leq n-1) \end{cases} \tag{17}$$

$$a_n^{p-5}\underline{h_{n,0}h_{n,0}}h_{n-2,2}h_{2,3}h_{n-4,4} \xrightarrow{h_{n,0}} a_n^{p-5}h_{n,0}\underline{h_{i,0}h_{n-i,i}}h_{n-2,2}h_{2,3}h_{n-4,4} (1 \leq i \leq n-1, i \neq 2, 4) \tag{18}$$

$$a_n^{p-5}\underline{h_{n,0}h_{n,0}}h_{3,2}h_{n-3,3}h_{n-4,4} \xrightarrow{h_{n,0}} a_n^{p-5}h_{n,0}\underline{h_{i,0}h_{n-i,i}}h_{3,2}h_{n-3,3}h_{n-4,4} \quad (1 \leq i \leq n-1, i \neq 3, 4) \tag{19}$$

$$a_n^{p-5}h_{n,0}h_{5,0}h_{n-2,2}h_{n-3,3}h_{n-4,4} \xrightarrow{\text{one by one}} \begin{cases} a_n^{p-6}\underline{a_1h_{n-1,1}}h_{n,0}h_{5,0}h_{n-2,2}h_{n-3,3}h_{n-4,4} \\ a_n^{p-5}\underline{h_{1,0}h_{n-1,1}}h_{5,0}h_{n-2,2}h_{n-3,3}h_{n-4,4} \\ a_n^{p-5}\underline{h_{i,0}h_{n-i,i}}h_{5,0}h_{n-2,2}h_{n-3,3}h_{n-4,4} \quad (6 \leq i \leq n-1) \\ a_n^{p-5}h_{n,0}\underline{h_{i,0}h_{5-i,i}}h_{n-2,2}h_{n-3,3}h_{n-4,4} \quad (1 \leq i \leq 4) \\ a_n^{p-5}h_{n,0}h_{5,0}\underline{h_{i,2}h_{n-i-2,i+2}}h_{n-3,3}h_{n-4,4} \quad (3 \leq i \leq n-3) \\ a_n^{p-5}h_{n,0}h_{5,0}h_{n-2,2}\underline{h_{i,3}h_{n-i-3,i+3}}h_{n-4,4} \quad (2 \leq i \leq n-4) \\ a_n^{p-5}h_{n,0}h_{5,0}h_{n-2,2}h_{n-3,3}\underline{h_{i,4}h_{n-i-4,i+4}} \quad (1 \leq i \leq n-5) \end{cases} \tag{20}$$

In the above table, the rightmost elements are the obtained nonzero generators.

4. Proof of Theorem 1.1

We first give two lemmas for the sake of proof of Theorem 1.1.

Lemma 4.1. Suppose that $0 \leq s \leq p - 5$ and $p \geq 11$. We fix $t = s + 1 + (s + 2)p + (s + 3)p^2 + (s + 4)p^3$. Then, in the E_1-term of the MSS, the 4th Greek letter element $\tilde{\delta}_{s+4}$ is represented by $a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{s+4,tq+s,*}$.

Proof. Suppose we are given a generator $h \in E_1^{s+4,tq+s,*}$. Then, we have $\dim(h) = s+4$ and $\deg(h) = tq+s$. From the p-adic expression of t, we see that

$$(\overline{c}_0, \overline{c}_1, \overline{c}_2, \overline{c}_3) = (s+1, s+2, s+3, s+4).$$

Case 1. $h = x_1 \cdots x_s z_1 z_2$ or $x_1 \cdots x_s y_1 y_2 z_1$.

Both cases give the same sequence $S = (s + 1, s + 2, s + 3, s + 4)$. There is no solution for the corresponding group of equations (β) since the number of elements in h is at most $s + 3$ being less than $c_3 = s + 4$. Thus, such h does not exist.

Case 2. $h = x_1 \cdots x_s y_1 y_2 y_3 y_4$.

There is a sequence $S = (s + 1, s + 2, s + 3, s + 4)$ as Case 1. Solving the corresponding group of equations (β) by virtue of (α) gives

$$\begin{cases} x_{1,i} = \cdots = x_{s,i} = 1 \quad (0 \le i \le 3) \\ y_{1,0} = y_{1,1} = y_{1,2} = y_{1,3} = 1 \\ y_{2,0} = 0 \\ y_{2,1} = y_{2,2} = y_{2,3} = 1 \end{cases} \quad \text{and} \quad \begin{cases} y_{3,0} = y_{3,1} = 0 \\ y_{3,2} = y_{3,3} = 1 \\ y_{4,0} = y_{4,1} = y_{4,2} = 0. \\ y_{4,3} = 1. \end{cases}$$

It follows that $h = a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3}$, and then

$$E_1^{s+4,t+s,*} = \mathbb{Z}/p\{a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3}\}.$$

This shows the result.

Lemma 4.2. Suppose that $p \geq 11$ and $n \geq 1$ with n 6 = 5. We fix $t = s+2+(s+2)p+(s+3)p^2+(s+4)p^3+p^n$. Then, for $2 \le r \le s + 7$, we have

$$\mathrm{Ext}_{\mathcal{A}}^{s-r+7,tq+s-r+1}(\mathbb{Z}/p, \mathbb{Z}/p) = 0.$$

Proof. According to Theorem 3.1 (1), we have $E_1^{s-r+7,tq+s-r+1,*} = 0$ for $2 \le r \le s + 7$. Since

$$E_1^{s-r+7,tq+s-r+1,*} \to \mathrm{Ext}_{\mathcal{A}}^{s-r+7,tq+s-r+1}(\mathbb{Z}/p, \mathbb{Z}/p),$$

it implies that $\mathrm{Ext}_{\mathcal{A}}^{s-r+7,tq+s-r+1}(\mathbb{Z}/p, \mathbb{Z}/p) = 0$ for $2 \le r \le s + 7$.

Proof of Theorem 1.1. Since $h_0, b_{n-1} \in \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$, converge to $h_0, b_{n-1} \in \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$, respectively, by Lemma 4.1 we see that $\tilde{\delta}_{s+4} h_0 b_{n-1} \in \mathrm{Ext}_{\mathcal{A}}^{s+7,t+s}(\mathbb{Z}/p, \mathbb{Z}/p)$ is represented by

$$a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,n-1} \in E_1^{s+7,tq+s,9s+p+17}$$

in the MSS. For the sake of proving Theorem 1.1, we first need to show that the representative $a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,n-1}$ is not hit by any May differential $d_r : E_r^{s+6,tq+s,9s+p+17-r} \to E_r^{s+7,tq+s,9s+p+17}$ for $r \geq 1$. This implies that we only need to deal with the generators of $E_r^{s+6,tq+s,M}$ with $M > 9_s + p + 17$. Thus, according to Theorem 3.1, there is no need to consider those generators with May filtrations M_1, M_{10}, and M_i ($11 \le i \le 15$). For the remaining generators, we write out their first May differentials as follows (Table 1):

Table 1. May differentials

	n	$E_1^{s+6,tq+s,M_i}$	M_i	first May differential
1st	$n = 4$	$a_4^s h_{5,0} h_{4,0} b_{2,1} b_{1,2}$	M_2	$a_4^s \underline{h_{1,0} h_{4,1}} h_{4,0} b_{2,1} b_{1,2} + \cdots$
2nd	$n = 4$	$a_4^{s-1} a_1 h_{5,0} h_{4,0} h_{2,2} h_{1,3} b_{3,0}$	M_3	$a_4^{s-1} a_1 \underline{h_{1,0} h_{4,1}} h_{4,0} h_{2,2} h_{1,3} b_{3,0} + \cdots$
3rd	$n = 4$	$a_4^s h_{2,0} h_{4,0} h_{2,2} h_{1,3} b_{3,1}$	M_3	$a_4^s \underline{h_{1,0} h_{1,1}} h_{4,0} h_{2,2} h_{1,3} b_{3,1} + \cdots$
4th	$n = 4$	$a_4^s h_{5,0} h_{1,0} h_{2,2} h_{1,3} b_{3,0}$	M_3	$a_4^s \underline{h_{2,0} h_{3,2}} h_{1,0} h_{2,2} h_{1,3} b_{3,0} + \cdots$
5th	$n = 4$	$a_5 a_4^{s-1} h_{4,0} h_{1,0} h_{2,2} h_{1,3} b_{3,0}$	M_3	$\underline{a_0 h_{5,0}} a_4^{s-1} h_{4,0} h_{1,0} h_{2,2} h_{1,3} b_{3,0} + \cdots$
6th	$n = 4$	$a_4^s h_{4,0} h_{1,0} h_{3,2} h_{1,3} b_{3,0}$	M_3	$a_4^s \underline{h_{2,0} h_{2,2}} h_{1,0} h_{3,2} h_{1,3} b_{3,0} + \cdots$
7th	$n = 4$	$a_4^s h_{4,0} h_{1,0} h_{3,1} h_{1,3} b_{3,1}$	M_3	$a_4^s \underline{h_{1,0} h_{4,1}} h_{1,0} h_{3,1} h_{1,3} b_{3,1} + \cdots$
8th	$n = 4$	$a_4^s h_{4,0} h_{2,0} h_{2,2} h_{1,3} b_{3,2}$	M_3	$a_4^s h_{4,0} h_{2,0} h_{2,2} h_{1,3} b_{3,2} + \cdots$
9th	$n = 4$	$a_4^s h_{4,0} h_{1,0} h_{2,2} h_{2,3} b_{3,0}$	M_3	$a_4^s \underline{h_{3,0} h_{1,3}} h_{1,0} h_{2,2} h_{2,3} b_{3,0} + \cdots$
10th	$n = 4$	$a_4^s h_{1,0} h_{4,0} h_{2,2} h_{1,3} b_{4,0}$	M_4	$a_4^{s-1} \underline{a_1 h_{3,1}} h_{1,0} h_{4,0} h_{2,2} h_{1,3} b_{4,0} + \cdots$
11th	$n = 4$	$a_4^{s-1} a_1 h_{3,1} h_{5,0} h_{4,0} h_{1,3} b_{2,1}$	M_5	$a_4^{s-1} a_1 h_{3,1} \underline{h_{1,0} h_{4,1}} h_{4,0} h_{1,3} b_{2,1} + \cdots$
12th	$n = 4$	$a_4^{s-1} a_2 h_{5,0} h_{4,0} h_{2,2} h_{1,3} b_{2,1}$	M_5	$a_4^{s-1} a_2 \underline{h_{1,0} h_{4,1}} h_{4,0} h_{2,2} h_{1,3} b_{2,1} + \cdots$
13th	$n = 4$	$a_4^s h_{1,0} h_{4,1} h_{4,0} h_{1,3} b_{2,1}$	M_5	$a_4^s h_{1,0} \underline{h_{1,1} h_{3,2}} h_{4,0} h_{1,3} b_{2,1} + \cdots$
14th	$n = 4$	$a_4^s h_{3,0} h_{4,0} h_{2,2} h_{1,3} b_{2,2}$	M_5	$a_4^s h_{3,0} \underline{h_{1,0} h_{3,1}} h_{2,2} h_{1,3} b_{2,2} + \cdots$
15th	$n = 4$	$a_4^s h_{5,0} h_{1,0} h_{3,1} h_{1,3} b_{2,1}$	M_5	$a_4^s \underline{h_{2,0} h_{3,2}} h_{1,0} h_{3,1} h_{1,3} b_{2,1} + \cdots$
16th	$n = 4$	$a_4^s h_{5,0} h_{2,0} h_{2,2} h_{1,3} b_{2,1}$	M_5	$a_4^s \underline{h_{1,0} h_{4,1}} h_{2,0} h_{2,2} h_{1,3} b_{2,1} + \cdots$
17th	$n = 4$	$a_5 a_4^{s-1} h_{4,0} h_{1,0} h_{3,1} h_{1,3} b_{2,1}$	M_5	$a_5 a_4^{s-1} \underline{h_{2,0} h_{2,2}} h_{1,0} h_{3,1} h_{1,3} b_{2,1} + \cdots$
18th	$n = 4$	$a_5 a_4^{s-1} h_{4,0} h_{2,0} h_{2,2} h_{1,3} b_{2,1}$	M_5	$a_5 a_4^{s-1} h_{4,0} \underline{h_{1,0} h_{1,1}} h_{2,2} h_{1,3} b_{2,1} + \cdots$
19th	$n = 4$	$a_4^s h_{4,0} h_{2,0} h_{3,2} h_{1,3} b_{2,1} S$	M_5	$a_4^s h_{4,0} \underline{h_{1,0} h_{1,1}} h_{3,2} h_{1,3} b_{2,1} + \cdots$
20th	$n = 4$	$a_4^s h_{4,0} h_{1,0} h_{3,1} h_{2,3} b_{2,1}$	M_5	$a_4^s \underline{h_{2,0} h_{2,2}} h_{1,0} h_{3,1} h_{2,3} b_{2,1} + \cdots$
21st	$n = 4$	$a_4^s h_{4,0} h_{1,0} h_{3,1} h_{2,2} b_{2,2}$	M_5	$a_4^s \underline{h_{3,0} h_{1,3}} h_{1,0} h_{3,1} h_{2,2} b_{2,2} + \cdots$
22nd	$n = 4$	$a_4^s h_{4,0} h_{2,0} h_{2,2} h_{2,3} b_{2,1}$	M_5	$a_4^s \underline{h_{1,0} h_{3,1}} h_{2,0} h_{2,2} h_{2,3} b_{2,1} + \cdots$
23rd	$n = 4$	$a_4^s h_{4,0} h_{3,0} h_{1,3} h_{2,3} b_{2,1}$	M_5	$a_4^s \underline{h_{1,0} h_{3,1}} h_{3,0} h_{1,3} h_{2,3} b_{2,1} + \cdots$
24th	$n = 4$	$a_4^s h_{4,0} h_{3,0} h_{1,3} h_{2,2} b_{2,2}$	M_5	$a_4^s \underline{h_{1,0} h_{3,1}} h_{3,0} h_{1,3} h_{2,2} b_{2,2} + \cdots$
25th	$n = 4$	$a_4^s h_{2,0} h_{4,1} h_{4,0} h_{1,3} b_{2,1}$	M_5	$a_4^s \underline{h_{1,0} h_{1,1}} h_{4,1} h_{4,0} h_{1,3} b_{2,1} + \cdots$
26th	$n = 4$	$a_4^{s-1} a_1 h_{3,1} h_{5,0} h_{4,0} h_{2,2} b_{1,2}$	M_6	$a_4^{s-1} a_1 h_{3,1} \underline{h_{1,0} h_{4,1}} h_{4,0} h_{2,2} b_{1,2} + \cdots$
27th	$n = 4$	$a_4^{s-1} a_3 h_{5,0} h_{4,0} h_{2,2} h_{1,3} b_{1,2}$	M_7	$a_4^{s-1} a_3 \underline{h_{1,0} h_{4,1}} h_{4,0} h_{2,2} h_{1,3} b_{1,2} + \cdots$

Table 1. *Continued*

	n	$E_1^{s+6,tq+s,M_i}$	M_i	first May differential
28th	$n = 4$	$a_4^s h_{3,0} h_{4,1} h_{4,0} h_{2,2} b_{1,2}$	M_7	$a_4^s \underline{h_{1,0} h_{2,1}} h_{4,1} h_{4,0} h_{2,2} b_{1,2} + \cdots$
29th	$n = 4$	$a_4^s h_{5,0} h_{1,0} h_{3,1} h_{2,2} b_{1,2}$	M_7	$a_4^s \underline{h_{2,0} h_{3,2}} h_{1,0} h_{3,1} h_{2,2} b_{1,2} + \cdots$
30th	$n = 4$	$a_4^s h_{5,0} h_{3,0} h_{1,3} h_{2,2} b_{1,2}$	M_7	$a_4^s \underline{h_{1,0} h_{4,1}} h_{3,0} h_{1,3} h_{2,2} b_{1,2} + \cdots$
31st	$n = 4$	$a_4^s h_{5,0} h_{4,0} h_{1,2} h_{1,3} b_{1,2}$	M_7	$a_4^s \underline{h_{1,0} h_{4,1}} h_{4,0} h_{1,2} h_{1,3} b_{1,2} + \cdots$
32nd	$n = 4$	$a_5 a_4^{s-1} h_{4,0} h_{3,0} h_{1,3} h_{2,2} b_{1,2}$	M_7	$a_5 a_4^{s-1} h_{4,0} h_{3,0} h_{1,3} h_{2,2} b_{1,2} + \cdots$
33rd	$n = 4$	$a_4^s h_{4,0} h_{1,0} h_{3,1} h_{3,2} b_{1,2}$	M_7	$a_4^s \underline{h_{2,0} h_{2,2}} h_{1,0} h_{3,1} h_{3,2} b_{1,2} + \cdots$
34th	$n = 4$	$a_4^s h_{4,0} h_{3,0} h_{1,3} h_{3,2} b_{1,2}$	M_7	$a_4^s \underline{h_{1,0} h_{3,1}} h_{3,0} h_{1,3} h_{3,2} b_{1,2} + \cdots$
35th	$n = 4$	$a_4^s h_{4,0} h_{3,0} h_{2,2} h_{2,3} b_{1,2}$	M_7	$a_4^s \underline{h_{1,0} h_{3,1}} h_{3,0} h_{2,2} h_{2,3} b_{1,2} + \cdots$
36th	$n = 4$	$a_4^s h_{2,0} h_{4,1} h_{4,0} h_{2,2} b_{1,2}$	M_8	$a_4^s \underline{h_{1,0} h_{1,1}} h_{4,1} h_{4,0} h_{2,2} b_{1,2} + \cdots$
37th	$n = 4$	$a_4^s h_{4,0} h_{2,0} h_{3,1} h_{3,2} b_{1,2}$	M_8	$a_4^s h_{4,0} \underline{h_{1,0} h_{1,1}} h_{3,1} h_{3,2} b_{1,2} + \cdots$
38th	$n = 4$	$a_4^s h_{3,0} h_{4,1} h_{4,0} h_{2,2} b_{1,2}$	M_9	$a_4^s \underline{h_{1,0} h_{2,1}} h_{4,1} h_{4,0} h_{2,2} b_{1,2} + \cdots$
39th	$n > 5$	$\mathbf{h}_2$	M_{16}	$a_n^{p-5} h_{n,0} \underline{h_{1,0} h_{4,1}} h_{n-2,2} h_{n-3,3} b_{n-4,3}$
40th	$n > 5$	$\mathbf{h}_3$	M_{17}	$a_n^{p-5} h_{n,0} \underline{h_{1,0} h_{4,1}} h_{n-2,2} h_{n-4,4} b_{n-3,2}$
41st	$n > 5$	$\mathbf{h}_4$	M_{18}	$a_n^{p-5} h_{n,0} \underline{h_{1,0} h_{4,1}} h_{n-3,3} h_{n-4,4} b_{n-2,1}$
42nd	$n > 5$	$\mathbf{h}_5$	M_{19}	$a_n^{p-5} \underline{h_{4,0} h_{n-4,4}} h_{1,0} h_{n-1,1} h_{n-2,2} h_{n-3,3} h_{1,4} + \cdots$
43rd	$n > 5$	$\mathbf{h}_6^{(\mathbf{i})}$ $(4 \le i \le n-1)$	M_{19}	$a_n^{p-5} h_{n,0} \underline{h_{1,0} h_{i-1,1}} h_{n-i,i} h_{n-2,2} h_{n-3,3} h_{1,4} + \cdots$
44th	$n > 5$	$\mathbf{h}_7^{(\mathbf{i})}$ $(1 \le i \le n-1,\ i \neq 2,4)$	M_{19}	$a_n^{p-5} h_{n,0} \underline{h_{1,0} h_{i-1,1}} h_{n-i,i} h_{n-2,2} h_{2,3} h_{n-4,4}$
45th	$n > 5$	$\mathbf{h}_8^{(\mathbf{i})}$ $(1 \le i \le n-1,\ i \neq 3,4)$	M_{19}	$a_n^{p-5} h_{n,0} \underline{h_{1,0} h_{i-1,1}} h_{n-i,i} h_{3,2} h_{n-3,3} h_{n-4,4}$
46th	$n > 5$	$\mathbf{h}_9$	M_{19}	$a_n^{p-6} \underline{a_0 h_{1,0}} h_{n-1,1} h_{n,0} h_{5,0} h_{n-2,2} h_{n-3,3} h_{n-4,4} + \cdots$
47th	$n > 5$	$\mathbf{h}_{10}$	M_{19}	$a_n^{p-5} h_{1,0} h_{n-1,1} \underline{h_{2,0} h_{3,2}} h_{n-2,2} h_{n-3,3} h_{n-4,4} + \cdots$
48th	$n > 5$	$\mathbf{h}_{11}^{(\mathbf{i})}$ $(6 \le i \le n-1)$	M_{19}	$a_n^{p-5} h_{i,0} h_{n-i,i} \underline{h_{1,0} h_{4,1}} h_{n-2,2} h_{n-3,3} h_{n-4,4} + \cdots$
49th	$n > 5$	$\mathbf{h}_{12}^{(\mathbf{i})}$ $(1 \le i \le 4)$	M_{19}	$a_n^{p-5} h_{n,0} \underline{h_{1,0} h_{i-1,1}} h_{5-i,i} h_{n-2,2} h_{n-3,3} h_{n-4,4} + \cdots$
50th	$n > 5$	$\mathbf{h}_{13}^{(\mathbf{i})}$ $(3 \le i \le n-3)$	M_{19}	$a_n^{p-5} h_{n,0} \underline{h_{1,0} h_{4,1}} h_{i,2} h_{n-i-2,i+2} h_{n-3,3} h_{n-4,4} + \cdots$
51st	$n > 5$	$\mathbf{h}_{14}^{(\mathbf{i})}$ $(2 \le i \le n-4)$	M_{19}	$a_n^{p-5} h_{n,0} \underline{h_{1,0} h_{4,1}} h_{n-2,2} h_{i,3} h_{n-i-3,i+3} h_{n-4,4} + \cdots$
52nd	$n > 5$	$\mathbf{h}_{15}^{(\mathbf{i})}$ $(1 \le i \le n-5)$	M_{19}	$a_n^{p-5} h_{n,0} \underline{h_{1,0} h_{4,1}} h_{n-2,2} h_{n-3,3} h_{i,4} h_{n-i-4,i+4} + \cdots$

In Table 1, for the generator with May filtration M_2, since its first May differential is nonzero and does not contain $a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,n-1}$, this implies that $E_r^{s+6,tq+s,M_2} = 0$ for $r \geq 2$, and hence,

$$a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,n-1} \notin d_r(E_r^{s+6,tq+s,M_2}) \text{ for } r \geq 1.$$

For the generators with May filtration M_3, their first May differentials all contain at least one term, which does not lie in the first May differential of any other generators. This implies that all the first May differentials of the generators are linearly independent and thus $E_1^{s+6,tq+s,9s+p+16}$ is trivial. It follows that $E_r^{p+1,t+p-5,2np+p-2n-19} = 0$ for $r \geq 2$, and then

$$a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,n-1} \notin d_r(E_r^{s+6,tq+s,M_3}) \text{ for } r \geq 1.$$

The same argument shows that for the remaining generators, there is

$$a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,n-1} \notin d_r(E_r^{s+6,tq+s,M_i}).$$

for $r \geq 1$ and $4 \leq i \leq 19$.

From the above results, we see that $a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,n-1}$ is not hit by any May differential when $s \leq p - 5$, $n \geq 1$, and $n \neq 5$. It follows that $a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,n-1}$ is a permanent cycle in the MSS and then converges nontrivially to $\tilde{\delta}_{s+4} h_0 b_{n-1} \in \mathrm{Ext}_{\mathcal{A}}^{s+7,tq+s}(\mathbb{Z}/p, \mathbb{Z}/p)$.

Let us then consider the Adams differential $d_r : E_r^{s-r+7,tq+s-r+1} \to E_r^{s,tq+s}$ for $r \geq 2$, which possibly hit $\tilde{\delta}_{s+4} h_0 b_{n-1}$ in the ASS. According to Lemma 4.2, we know that $\mathrm{Ext}_{\mathcal{A}}^{s-r+7,t+s-r+1}(\mathbb{Z}/p, \mathbb{Z}/p) = 0$, which follows $E_r^{s-r+7,tq+s-r+1} = 0$. Hence, the corresponding Adams differential is trivial and then cannot hit $\tilde{\delta}_{s+4} h_0 b_{n-1}$ in the ASS. Thus, $\tilde{\delta}_{s+4} h_0 b_{n-1}$ is nontrivial in the ASS. This finishes our proof.

Remark 4.3. For $s = p - 5$, $r = 1$, and $n = 5$, by Theorem 3.1(e) there is one non-zero generator

$$E_1^{p+1,tq+p-5,11p-29} = \mathbb{Z}/p\{a_5^{p-5} h_{4,1} h_{3,2} h_{2,3} h_{1,4} h_{1,0}\}.$$

By applying the three-filtrated cobar construction, we can show the following differential holds:

$$d_{p-1}(a_5^{p-5} h_{4,1} h_{3,2} h_{2,3} h_{1,4} h_{1,0}) = a_4^{p-5} h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,4}.$$

It follows that $a_4^{p-5} h_{4,0} h_{3,1} h_{2,2} h_{1,3} h_{1,0} b_{1,4}$ vanishes in the E_p-term of the MSS. Thus, it cannot converge nontrivially to $\tilde{\delta}_{p-1} h_0 b_4$, and then follows $\tilde{\delta}_{p-1} h_0 b_4 = 0$.

5. Conclusion

This paper applies a new effective computation method to determine the convergence of a product element $\tilde{\delta}_{s+4} h_0 b_{n-1}$ in the classical ASS. The key point of this method is to construct a group of linear equations according to the triple degrees of the representative in the MSS of $\tilde{\delta}_{s+4} h_0 b_{n-1}$, and then compute out the related generators in the corresponding E_1-term of the MSS. What we compute helps us to show that the product element $\tilde{\delta}_{s+4} h_0 b_{n-1}$ is a permanent cycle without being bound in any term of the ASS. This new method admits a wide application range in determining nontrivial elements of the stable homotopy groups of spheres. In the future, we plan to use this method to detect more nontrivial elements in the stable homotopy groups of spheres.

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Conflict of interest

The authors declare that there is no personal or organizational conflict of interest in this work

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