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Research Article

A Robust Numerical Method for a Singularly Perturbed Semilinear Problem with Integral Boundary Conditions

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Abstract: In the present study, we provide an efficient numerical approach for solving singularly perturbed nonlinear ordinary differential equations with two integral boundary conditions. We specifically propose a numerical approach for the solution of a nonlinear singular perturbed problem with integral boundary conditions. To solve the nonlinear singularly perturbed issue, we also apply finite difference methods. It explores how a specific derivative and a problem-solving approach behave. Finally, a numerical technique that employs a finite difference scheme is built using a nonuniform mesh.

***Keywords*:** singularly perturbed problem, numerical solution, boundary layer, uniform convergence

MSC: 65N12, 65N30, 65N06

Nomenclature

C_0, C_1, C	Positive constants
$\lVert v \rVert_1$	l_1-norm
$\lVert v \rVert_\infty$	Maximum norm
$u(t)$	Solution of the continuous problem
y_0, y_N	Boundary values of the difference scheme
ε	Small parameter
σ	Transition parameter used in Bakhvalov mesh
ω_N	Bakhvalov mesh
ψ_i	The basis functions
z_i	Error function
R_i, $R_{1,i}$, $R_{2,i}$, $R_{3,i}$	Remainder terms from the discrete scheme
r_1, r_2	Remaining terms of the boundary conditions
e_{N_ε}	Pointwise error formula
p_{N_ε}	Rate of convergence

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1. Introduction

Consider the singularly perturbed nonlinear reaction-diffusion boundary value problem.

$$Lu \equiv \varepsilon u''(x) + a(x)u'(x) - f(x, u(x)) = 0, \; x \in (0, l), \tag{1}$$

$$u(0) = \int_0^l g_1(x)u(x)dx + A, \tag{2}$$

$$u(l) = \int_0^l g_2(x)u(x)dx + B, \tag{3}$$

where A and B are certain constants and the parameter ε is positive. The functions $g_1(x)$, $g_2(x)$, and $a(x) \geq \alpha > 0$ are also expected to be sufficiently smooth on $[0, l]$. In the range $(x, u) \in [0, 1] \times R$, the function $f(x, u)$ is sufficiently smooth. Furthermore, $\left|\frac{\partial f}{\partial u}\right| \geq \beta > 0$. Under these conditions, there is only one solution to problems (1) - (3). For small values of ε, this problem generally has only one boundary layer near $x = 0$.

Numerous disciplines have used singular perturbation and related ideas, including financial modeling, ion transport across cells, pollutant dispersion in waters,and many others [1-8]. Boundary layers are typically visible in the solution of a singularly perturbed problem. Due to the existence of boundary layers, conventional numerical methods applied to any uniform mesh are unable to yield accurate results. To capture the layer phenomena, one practical method is to create the appropriate layer-adapted meshes [9].

One must overcome challenges brought on by the small parameter ε to numerically solve (1) - (3) by applying a finite difference method. The uniform mesh with an exponentially fitted scheme can be utilized if one is concerned with a numerical solution for the boundary layers, however, the traditional schemes can produce respectable results, as shown in [10]. Layers are produced utilizing uniform or graded meshes, such as Shishkin mesh and Bakhvalov mesh. Additionally, the Bakhvalov-Shishkin mesh is created by combining Shishkin and Bakhvalov meshes [11]. Thus, it is clear that a numerical solution is required inside the border layers. This suggests the usage of a mesh that is thick in the boundary layer sections and nonuniform overall. In [12], certain increasingly fitted techniques on a nonuniform mesh are provided, however, it is not explained how to configure the mesh points for various ε. The first time this was accomplished was in [13] for traditional finite difference schemes, and several modifications of this Bakhvalov concept were developed.

In 1969, Bakhvalov first introduced this remarkable mesh-building approach for one- and two-dimensional reaction-diffusion equations. The mesh is constructed so that it is dense in boundary layers and coarse without them. Essentially, a function that is analogous to the inverse of the boundary layer function creates the mesh points in the layers. Moreover, for the whole integration interval, the Bakhvalov mesh can be rebuilt by a differentiable function.

For singularly perturbed boundary value problems, many traditional numerical approaches are inappropriate. Thus, we need to come up with uniform convergent methods to solve such problems whose precision and accuracy are independent of the parameter's value. Some of the easiest and most useful ways to produce these types of methods consist of suitable mesh techniques, finite element techniques employing peculiar elements, which involve exponential elements, and appropriate finite difference techniques. In this case, the finite difference method mentioned may be used very easily for a uniform mesh. In previous years, numerous studies have been conducted on numerical methods to solve singular perturbation problems [14, 15]. Additionally, when using heat transfer problems, these types of problems with integral nonlocal boundary conditions can obtained [16]. These problems have been broadly studied in the literature (see [17]). Ahsan et al. [18] considered a Haar wavelet multi-resolution collocation method for singularly perturbed differential equations with integral boundary conditions. Cakir and Gunes [19] proposed an exponentially fitted difference scheme for singularly perturbed mixed integro-differential equations. Kumar et al. [20] considered high-order convergent methods for singularly perturbed quasilinear problems with integral boundary conditions. Munyakazi and Kehinde [21] suggest a new parameter-uniform discretization of semilinear singularly perturbed problems. The concepts of existence and uniqueness can be examined to find solutions for such problems [22, 23]. The problem, which includes two boundary layers has been tried, where a finite difference scheme on a uniform mesh has been offered to study a singularly perturbed problem with an integral nonlocal condition [24, 25]. As a special fine mesh, these problems are necessary compared to coarser mesh in the boundary layer region and elsewhere. Cakir and Arslan [26] proposed a new numerical approach for a singularly perturbed problem with two integral boundary conditions. Durmaz et al. [27] considered an efficient numerical method for a singularly perturbed Fredholm integro-differential equation with integral boundary condition. Sekar and Tamilselvan [28] proposed singularly perturbed delay differential equations of convection-diffusion type with integral boundary condition.

The main result of the present study is to establish a uniform difference scheme for the boundary value problems (1) - (3) by applying the methods introduced. The difference schemes are obtained by applying exponential basis functions through integral identities and also by constructing integral-type quadrature forms that have weight and remainder terms [29, 30]. As far as we know, any paper which presents a numerical approach hasn't been published until now. This paper aims to close that gap. The finite difference approximation for nonlinear singularly perturbed equations with integral boundary conditions is thoroughly reviewed in this study. This paper's main contribution is the introduction of a novel numerical method and comprehensive uniform convergence result. Thanks to these results, we can also use our method for different types of differential equations. This work has special characteristics in that the singularly perturbed issue contains two integral boundary conditions and is nonlinear. We also took the Bakhvalov mesh into account and demonstrated that it had an ordered precision of one. In this way, a new innovation has been brought to the published articles. This method of approximation has the advantage that the schemes can also be effective in the case where the original problem is considered under certain singularities (presence of boundary layer, nonsmooth solutions, etc.)

The article has the following structure: In Section 2, we provide a few conclusions for the solution of the problem (1) - (3). In Section 3, we create a mesh and a finite difference scheme for the problems that have singularly perturbed. In Section 4, the stability analysis, uniform convergence, and error evaluations of the problem are proved by the finite difference method. Finally, the algorithm for this kind of problem is presented, and the numerical outcomes are displayed in graphs and tables.

2. Properties of exact solution

Now, we will present a result for the solution of problems (1) - (3). This result shall be necessary for the analysis of the suitable numerical solution.

Lemma 2.1. Assume $g_1(x), g_2(x), a(x), b(x), \dfrac{\partial f(x,u)}{\partial u} \geq \beta_* > 0$, and $f(x, u) \in C^1$ $[0, l]$. If function $u(x)$ is a solution of problems (1) - (3) and

$$\gamma = \int_0^l (|g_1(x)| + |g_2(x)|)dx < 1,$$

then

$$\|u\|_\infty \leq C_0, \, 0 \leq x \leq 1 \tag{4}$$

and

$$|u'(x)| \leq C\left\{1 + \frac{1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}}\right\}, \quad 0 \leq x \leq 1, \tag{5}$$

where

$$C_0 = (1-\gamma)^{-1}(|A| + |B| + \alpha^{-1}\|f\|_\infty). \tag{6}$$

Proof. For a nonlinear function $f(x,u)$ in (1), we may rewrite equation (1) as follows by utilizing the mean value theorem:

$$\varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = F(x),$$

where

$$F(x) = f(x,0),\, b(x) = \frac{\partial f}{\partial u}(x,\overline{u})$$

and

$$f(x,u) = f(x,0) + \frac{\partial f}{\partial u}(x,\overline{u})u,\, \overline{u} = \gamma u, 0 < \gamma < 1.$$

Here, the maximal principle will be applied to resolve the problems (1) - (3). If L is a differential operator in the problems (1) - (3) and $v \in C^2\,[0, l]$, and if $v\,(0) \geq 0$, $v(l) \geq 0$, and for $0 \leq x \leq 1$, $Lv \leq 0$, then for $0 \leq x \leq 1$, $v(x) \geq 0$. Then, we acquire

$$u(x) \leq |u(0)| + |u(l)| + \alpha^{-1}\|f\|_\infty,\, x \in [0,l]. \tag{7}$$

With the boundary conditions (2) and (3), we get

$$u(0) \leq |A| + \int_0^l |g_1(x)||u(x)|\,dx, \tag{8}$$

and

$$u(l) \leq |B| + \int_0^l |g_2(x)||u(x)|\,dx. \tag{9}$$

Substituting the inequalities (8) and (9) into (7), we obtain the following inequality:

$$\begin{aligned} u(x) &\leq |A| + |B| + \int_0^l |g_1(x)||u(x)|\,dx + \int_0^l |g_2(x)||u(x)|\,dx + \alpha^{-1}\|f\|_\infty \\ &\leq |A| + |B| + \max_{[0,l]}|u(x)|\int_0^l |g_1(x)|dx + \max_{[0,l]}|u(x)|\int_0^l |g_2(x)|dx + \alpha^{-1}\|f\|_\infty \\ &\leq |A| + |B| + \|u\|_\infty \int_0^l |g_1(x)|dx + \|u\|_\infty \int_0^l |g_2(x)|dx + \alpha^{-1}\|f\|_\infty \end{aligned}$$

and

$$\|u\|_\infty \leq \left(1 - \left(\int_0^l |g_1(x)|dx + \int_0^l |g_2(x)|dx\right)\right)^{-1}\left(|A| + |B| + \alpha^{-1}\|f\|_\infty\right) \leq (1-\gamma)^{-1}\left(|A| + |B| + \alpha^{-1}\|f\|_\infty\right). \tag{10}$$

Hence, the inequality (10) gives proof of the inequality (4). Now, let us give the proof of inequality (5). We can get as below:

$$\varepsilon u''(x) + a(x)u'(x) = \phi(x) \tag{11}$$

with $\phi(x) = f(x) + b(x)u(x)$.

Due to equation (11), for $u'(x)$ we will be able to use the following:

$$u'(x)=u'(0)e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)d\tau}+\frac{1}{\varepsilon}\int_0^x \phi(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau}d\xi. \tag{12}$$

On equality (12), we need to get an evaluation for $u'(0)$. Integrating the relation (12) over $[0, l]$, we see that

$$-\int_0^l g_1(x)u(x)+\int_0^l g_2(x)u(x)+B-A=u'(0)\int_0^l e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)d\tau}$$
$$+\frac{1}{\varepsilon}\int_0^l\int_0^x \phi(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau}d\xi dx. \tag{13}$$

From the relation (13), we have

$$u'(0)=\frac{B-A-\int_0^l g_1(x)u(x)dx+\int_0^l g_2(x)u(x)dx-\frac{1}{\varepsilon}\int_0^l\int_0^x \phi(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau}d\xi dx}{\int_0^l e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)d\tau}dx}. \tag{14}$$

Estimating the integral in the denominator of (14), we have

$$\begin{aligned}\int_0^l e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)d\tau}dx &\geq n\int_0^l e^{-\frac{1}{\varepsilon}\int_0^x \overline{a}d\tau}dx\\ &=\int_0^l e^{-\frac{\overline{a}x}{\varepsilon}}dx\\ &=-\frac{\varepsilon}{\overline{\alpha}}(e^{-\frac{\overline{a}l}{\varepsilon}}-1)=\varepsilon\overline{\alpha}^{-1}(1-e^{-\frac{\overline{\alpha}l}{\varepsilon}})\\ &=\delta\varepsilon\ (\delta\neq\delta\varepsilon>0,\ \overline{a}=\max_{[0,l]}|a(x)|).\end{aligned} \tag{15}$$

Here, the integral in (14), when subjected to the mean value theorem, yields the following conclusion:

$$\begin{aligned}&\left|\frac{1}{\varepsilon}\int_0^l\int_0^x \phi(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau}d\xi dx\right|\\ &\leq\frac{1}{\varepsilon}\int_0^l\left[\int_0^x|\phi(\xi)|e^{-\frac{1}{\varepsilon}\int_\xi^x \alpha(\tau)d\tau}d\xi\right]dx\\ &\leq\frac{\|\phi\|_\infty}{\varepsilon}\int_0^l\int_0^x e^{-\frac{1}{\varepsilon}\int_\xi^x \alpha d\tau}d\xi dx\\ &\leq\frac{\|\phi\|_\infty}{\varepsilon}\int_0^l\int_0^x e^{-\frac{\alpha}{\varepsilon}(x-\xi)}d\xi dx\\ &\leq\alpha^{-1}\|\phi\|_\infty\int_0^l\left[1-e^{-\frac{\alpha x}{\varepsilon}}\right]dx\ \leq\alpha^{-1}l\|\phi\|_\infty\leq C_1.\end{aligned} \tag{16}$$

And also, if the same applications are done for the other integral, the following inequality is obtained:

$$
\begin{aligned}
&\left|\int_0^l g_1(x)u(x)dx+\int_0^l g_2(x)u(x)dx\right| \\
&\leq \int_0^l |g_1(x)||u(x)|dx+\int_0^l |g_2(x)||u(x)|dx \\
&\leq \max_{[0,l]}|u(x)|\left[\int_0^l |g_1(x)|dx+\int_0^l |g_2(x)|dx\right] \\
&\leq \|u\|_\infty\left(\|g_1\|_1+\|g_2\|_1\right) \\
&\leq \left(\|g_1\|_1+\|g_2\|_1\right)C_0. \qquad (17)
\end{aligned}
$$

Substituting the inequalities (16) and (17) into the relation (14), we can write as follows:

$$
|u'(0)| \leq \frac{|A|+|B|+\alpha^{-1}l\|\phi\|_\infty+\left(\|g_1\|_1+\|g_2\|_1\right)C_0}{\delta\varepsilon} \leq \frac{C}{\varepsilon}. \qquad (18)
$$

By taking into account (18) in (12), we deduce that

$$
\begin{aligned}
|u'(x)| &= |u'(0)|e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)d\tau}+\frac{1}{\varepsilon}\int_0^x |\phi(\xi)|e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau}d\xi \\
&\leq \frac{C}{\varepsilon}e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)d\tau}dx+\frac{1}{\varepsilon}\|\phi\|_\infty\int_0^x e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau}d\xi \\
&= \frac{C}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}}+\frac{1}{\varepsilon}\|\phi\|_\infty\int_0^x e^{-\frac{\alpha}{\varepsilon}(x-\xi)}d\xi \\
&= \frac{C}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}}+\frac{\|\phi\|_\infty}{\varepsilon}(1-e^{-\frac{\alpha x}{\varepsilon}})\varepsilon\alpha^{-1} \\
&= \frac{C}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}}+C. \qquad (19)
\end{aligned}
$$

Thus, from (19) we arrive at (5).

3. Difference scheme generating using finite difference method

Let us define a nonuniform mesh in [0, l] as follows:

$$
\omega_N = \{\, 0 < x_1 < x_2 < \ldots < x_{N-1} < 1,\ h_i = x_i - x_{i-1},\ i = 1, 2, \ldots, N-1 \,\},
$$

and

$$
\bar{\omega}_N = \omega_N \cup \{x = l\}.
$$

On a Bakhvalov mesh, the approximate solution $u(x)$ to problems (1) - (3) will be calculated. The interval [0, l] is divided into two subintervals [0, σ] and [σ, l] for a positive integer N that may be split in two. $\sigma << l$ is typically used as a transition point in practice and is shown as follows:

$$
\sigma = \min\left\{\frac{l}{2},\ \alpha^{-1}\varepsilon|\ln\varepsilon|\right\}.
$$

We define the mesh points as

$$x_i \in [0,\sigma]: x_i = -\alpha^{-1}\varepsilon \ln\left[1-(1-\varepsilon)\frac{2i}{N}\right], i = 0,..., \frac{N}{2},$$

if $\sigma < \frac{l}{2}$, then

$$x_i = -\alpha^{-1}\varepsilon \ln\left[1-(1-e^{-\frac{\alpha l}{2\varepsilon}})\frac{2i}{N}\right],\ i = 0,..., \frac{N}{2},$$

and if $\sigma = \frac{l}{2}$, then

$$x_i \in (\sigma, l]: x_i = \sigma + (i-\frac{N}{2})h,\ i = \frac{N}{2}+1,..., N$$
$$h = 2(l-\sigma)/N.$$

We show specific instances of the mesh functions before outlining this numerical approach. Each nonuniform mesh function v_i described in ϖ_N is taken into consideration:

$$v_{\bar{x},i} = \frac{v_i - v_{i-1}}{h_i}, \qquad v_{x,i} = \frac{v_{i+1} - v_i}{h_{i+1}}, \qquad v_{\overset{o}{x},i} = \frac{v_{x,i} - v_{\bar{x},i}}{2}$$

and

$$v_{\hat{x},i} = \frac{v_{i+1} - v_i}{\hbar_i}, \qquad v_{\bar{x}\hat{x},i} = \frac{v_{x,i} + v_{\bar{x},i}}{\hbar_i}, \qquad \hbar_i = \frac{h_i + h_{i+1}}{2}.$$

We will build the difference scheme from the following identity, which we will integrate into problem (1) in (x_{i-1}, x_{i+1})

$$\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} Lu(t)\psi_i(t)dt = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} f(t)\psi_i(t)dt,\ i = 1,2,..., N-1, \tag{20}$$

where $\psi_i(x)$ are the basis functions, is determined by

$$\psi_i(x) = \begin{cases} \psi_i^{(1)}(x) \equiv \dfrac{e^{\frac{a_i(x-x_{i-1})}{\varepsilon}}-1}{e^{\frac{a_i h_i}{\varepsilon}}-1}, & x_{i-1} < x < x_i \\ \psi_i^{(2)}(x) \equiv \dfrac{1-e^{-\frac{a_i(x_{i+1}-x)}{\varepsilon}}}{1-e^{\frac{a_i h_{i+1}}{\varepsilon}}}, & x_i < x < x_{i+1} \\ 0 & , x \notin (x_{i-1}, x_{i+1}) \end{cases}$$

It can easily be seen that the basis functions $\psi_i^{(1)}(x)$ and $\psi_i^{(2)}(x)$ are the solutions to the following problems, respectively:

$$\varepsilon\psi_i'' + a_i\psi_i' = 0,\ x_{i-1} < x < x_i,$$
$$\psi_i(x_{i-1}) = 0,\ \psi_i(x_i) = 1$$

and

$$\varepsilon\psi_i{}'' + a_i\psi_i' = 0,\ \ x_i < x < x_{i+1},$$
$$\psi_i(x_i) = 1,\ \ \psi_i(x_{i+1}) = 0.$$

Also, we have

$$\hbar_i^{-1}\chi_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \psi_i^{(1)}(x)dx + \hbar_i^{-1}\chi_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \psi_i^{(2)}(x)dx = 1.$$

If the necessary arrangements are made and partial integration is applied in the equation (20), we obtain

$$\hbar_i^{-1}\chi_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \varepsilon u''(x)\psi_i(x)dx + \hbar_i^{-1}\chi_i^{-1}\int_{x_{i-1}}^{x_{i+1}} a(x)u'(x)\psi_i(x)dx$$
$$-\hbar_i^{-1}\chi_i^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x,u(x))\psi_i(x)dx = 0,\ i = 1,2,...,N-1,$$

and

$$\varepsilon\hbar_i^{-1}\chi_i^{-1}\int_{x_{i-1}}^{x_i} u'(x)\psi_i^{(1)'}(x)dx + \varepsilon\hbar_i^{-1}\chi_i^{-1}\int_{x_i}^{x_{i+1}} u'(x)\psi_i^{(2)^1}(x)dx$$
$$+a_i\hbar_i^{-1}\chi_i^{-1}\int_{x_{i-1}}^{x_i} u'(x)\psi_i^{(1)}(x)dx + a_i\hbar_i^{-1}\chi_i^{-1}\int_{x_i}^{x_{i+1}} u'(x)\psi_i^{(2)}(x)dx$$
$$-f(x_i,u_i) - R_i = 0,\ i = 1,2,...,N-1, \tag{21}$$

where the remainder term

$$R_i = \varepsilon\hbar_i^{-1}x_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \left[a(x) - a_i\right] u'(x)\psi_i(x)dx$$
$$-\hbar_i^{-1}x_i^{-1}\int_{x_{i-1}}^{x_{i+1}} dx\psi_i(x)\int_{x_{i-1}}^{x_{i+1}} \frac{d}{dx} f(\xi,u(\xi))K_0^*(x,\xi)d\xi,$$
$$K_{0,i}^* = T_0(x-\xi) - T_0(x_i - \xi),\quad 1 \le i \le N-1,$$
$$T_s(\lambda) = \begin{cases} \dfrac{\lambda^s}{s!}, & \lambda \ge 0,\ \text{for } s = 0, \\ 0, & \lambda < 0. \end{cases} \tag{22}$$

The remainder terms R_i is defined in the following manner:

$$R_i = \varepsilon\hbar_i^{-1}x_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \left[a(x) - a_i\right] u'(x)\psi_i(x)dx$$
$$-\hbar_i^{-1}x_i^{-1}\int_{x_{i-1}}^{x_{i+1}} dx\psi_i(x)\int_{x_{i-1}}^{x_{i+1}} \frac{d}{dx} f(\xi,u(\xi))K_0^*(x,\xi)d\xi. \tag{23}$$

If the quadrature formulas in (2.1) and (2.2) from [31] are applied to the equation (21) on each of the subintervals (x_{i-1}, x_i) and (x_i, x_{i+1}), we find the following difference approximation for (1):

$$\varepsilon\theta_i u_{\bar{x}\hat{x},i} + a_i u_{\overset{\circ}{x},i} - f(x_i,u_i) - R_i = 0,\ \ i = \overline{1,N-1}, \tag{24}$$

where

$$\theta_i = \begin{cases} \dfrac{a_i \hbar_i}{2\varepsilon} \left(\dfrac{h_i + 1\left(e\dfrac{a_i h_i}{\varepsilon} - 1 \right) + h_i \left(1 - e - \dfrac{a_i h_i + 1}{\varepsilon} \right)}{h_i + 1\left(e\dfrac{a_i h_i}{\varepsilon} - 1 \right) + h_i \left(1 - e - \dfrac{a_i h_i + 1}{\varepsilon} \right)} \right). \\ 1, \end{cases} \tag{25}$$

We use the right-sided rectangle rule to obtain an appropriate approximation to the boundary conditions (2) and (3). Then, we have

$$u_0 = \sum_{i=1}^{N} h_i g_{1,i} u_i - r_1 + A \tag{26}$$

and

$$u_N = \sum_{i=1}^{N} h_i g_{2,i} u_i - r_2 + B, \tag{27}$$

where

$$r_1 = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (x - x_{i-1}) \frac{d}{dx} (g_1(x) u(x)) dx \tag{28}$$

and

$$r_2 = \sum_{i\ 1}^{N} \int_{x_{i\ 1}}^{x_i} (x - x_{i\ 1}) \frac{d}{dx} (g_2(x) u(x)) dx. \tag{29}$$

Neglecting the remainder terms R_i, r_1, and r_2 in (24), (26), and (27), respectively, we suggest the following difference scheme for approximating the problems (1) - (3):

$$ly_i \equiv \varepsilon \theta_i y_{\bar{x}\hat{x},i} + a_i y_{\overset{\circ}{x},i} - f(x_i, y_i) = 0, \quad i = 1, 2, ..., N-1, \tag{30}$$

$$y_0 = \sum_{i=1}^{N} h_i g_{1,i} y_i + A, \tag{31}$$

$$y_N = \sum_{i=1}^{N} h_i g_{2,i} y_i + B, \tag{32}$$

where θ_i is defined by (25).

4. Stability bound and uniform convergence

First, we demonstrate our method's uniform convergence. The following discrete problem is solved by the error function $z_i = y_i - u_i$ for each $i \in [0, N]$.

$$\varepsilon\theta_i z_{\bar{x}\hat{x},i} + a_i z_{\overset{o}{x},i} - \left[f(x_i, y_i) - f(x_i, u_i) \right] = R_i,\ \ i = 1, 2, \ldots, N-1, \tag{33}$$

where

$$z_0 = \sum_{i=1}^{N} h_i g_{1,i} z_i - r_1 \tag{34}$$

and

$$z_N = \sum_{i=1}^{N} h_i g_{2,i} z_i - r_2. \tag{35}$$

Here, the remainder terms R_i, r_1 and r_2 are given in (23), (28), and (29), respectively.

Lemma 4.1. Under the condition of $a(x)$, $b(x)$, and $f(x,u) \in C^1\,[[0, l] \times R]$, the remaining terms R_i, r_1, and r_2 satisfy the following inequalities:

$$\|R\|_{\infty,\omega_N} \le CN^{-1}, \tag{36}$$

$$|r_1| \le CN^{-1}, \tag{37}$$

and

$$|r_2| \le CN^{-1}. \tag{38}$$

Proof. On any type of mesh, we defined the equation (23) for R_i, we have

$$R_i = \varepsilon\hbar_i^{-1} x_i^{-1} \int_{x_{i-1}}^{x_{i+1}} \left[a(x) - a_i \right] u'(x)\psi_i(x)dx - \hbar_i^{-1} x_i^{-1} \int_{x_{i-1}}^{x_{i+1}} dx\psi_i(x) \int_{x_{i-1}}^{x_{i+1}} \frac{d}{dx} f(\xi, u(\xi)) K_0^*(x, \xi) d\xi, \tag{39}$$

and

$$|R_i| \le C\left\{ \varepsilon \max_{[x_{i-1}, x_{i+1}]} |x_i - x| + \int_{x_{i-1}}^{x_{i+1}} \left| \frac{\partial f(\xi, u(\xi))}{\partial \xi} + \frac{\partial f}{\partial u}\frac{du(\xi)}{d\xi} \right| d\xi \right\}$$

$$\le C\left\{ h_i + h_{i+1} + \int_{x_{i-1}}^{x_{i+1}} (1 + |u'(\xi)|) d\xi \right\}, \qquad 1 \le i \le N.$$

Together with (5), this inequality allows us to dictate

$$|R_i| \le C\left\{ h_i + h_{i+1} + \frac{1}{\varepsilon} \int_{x_{i-1}}^{x_{i+1}} e^{-\frac{ax}{\varepsilon}} dx \right\}. \tag{40}$$

From equation 40, we get

$$\left|R_i\right| \le Ch_i. \tag{41}$$

For the first situation $x_i \in [0, \sigma]$:

(1) If $\sigma < \dfrac{1}{2}$, then

$$x_{i-1} = \alpha^{-1}\varepsilon \ln\left[1-(1-\varepsilon)\frac{2(i-1)}{N}\right], \tag{42}$$

and

$$h_i = \alpha^{-1}\varepsilon \ln\left[1-(1-\varepsilon)\frac{2i}{N}\right] - \alpha^{-1}\varepsilon \ln\left[1-(1-\varepsilon)\frac{2(i-1)}{N}\right]. \tag{43}$$

Applying the mean value theorem in (43), we can see that

$$h_i = \alpha^{-1}\varepsilon \frac{2(1-\varepsilon)N^{-1}}{1-2i(1-\varepsilon)N^{-1}} \le CN^{-1}. \tag{44}$$

Consequently, from (41) and (44), the following can be written:

$$\begin{aligned}
\left|R_i\right| &\le C\left\{h + \frac{1}{\varepsilon}\int_{x_{i-1}}^{x_{i+1}} e^{-\frac{\alpha x}{\varepsilon}} dx\right\}, \; i = \overline{0, \frac{N}{2}} \\
&\le C\left\{h + \alpha^{-1}\left(e^{\frac{\alpha x_{i-1}}{\varepsilon}} - e^{-\frac{\alpha x_{i+1}}{\varepsilon}}\right)\right\} \\
&\le C\left\{h + \alpha^{-1}N^{-1}\right\} \\
&\le CN^{-1}.
\end{aligned}$$

(2) If $\sigma = \dfrac{1}{2}$, then

$$x_{i-1} = \alpha^{-1}\varepsilon \ln\left[1-(1-e^{\frac{\alpha}{2\varepsilon}})\frac{2(i-1)}{N}\right] \tag{45}$$

and

$$h_i = \alpha^{-1}\varepsilon \ln\left[1-(1-e^{\frac{\alpha}{2\varepsilon}})\frac{2i}{N}\right] - \alpha^{-1}\varepsilon \ln\left[1-(1-e^{\frac{\alpha}{2\varepsilon}})\frac{2(i-1)}{N}\right]. \tag{46}$$

Applying the mean value theorem in (46), it is concluded that

$$h_i = \alpha^{-1}\varepsilon \frac{2(1-\varepsilon)N^{-1}}{1-2i(1-\varepsilon)N^{-1}} \le CN^{-1}. \tag{47}$$

As a result of (41) and (47), we can reveal that:

$$\left|R_i\right| \le C\left\{h + \frac{1}{\varepsilon}\int_{x_{i-1}}^{x_{i+1}} e^{-\frac{\alpha x}{\varepsilon}}dx\right\}, i = \overline{0, \frac{N}{2}}$$

$$\le C\left\{h + \alpha^{-1}(e^{\frac{\alpha x_{i-1}}{\varepsilon}} - e^{-\frac{\alpha x_{i+1}}{\varepsilon}})\right\}$$

$$\le C\left\{h + \alpha^{-1}N^{-1}\right\}$$

$$\le CN^{-1}.$$

For the first situation, $x_i \in [\sigma, l]$:

$$x_i = \sigma + (i - \frac{N}{2})h, i = \overline{\frac{N}{2} + 1, N}, \tag{48}$$

here

$$h = \frac{2(1-\sigma)}{N}.$$

Using equations (41) and (48), we achieve

$$\left|R_i\right| \le Ch \le CN^{-1}.$$

Taking into account the above circumstances, we conclude that

$$\left|R_i\right| \le CN^{-1}.$$

Thus, we arrive at (36).

Next, we estimate (37) and (38). From the written expression of (28) and (29), we obtain

$$\left|r_1\right| \le \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}(x - x_{i-1})\frac{d}{dx}\big(g_1(x)u(x)\big)dx, i = 1, 2, ..., N$$

and

$$\left|r_2\right| \le \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}(x - x_{i-1})\frac{d}{dx}\big(g_2(x)u(x)\big)dx, i = 1, 2, ..., N.$$

With the inequality of (5), we can write the following expression:

$$\left|r_1\right| \le \left\|g_1\right\|_\infty \sum_{i=1}^{N} h_i \int_{x_{i-1}}^{x_i}\left(1 + \frac{1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}}\right)dx, i = 1, 2, ..., N. \tag{49}$$

From inequalities (28) and (49), we obtain

$$
\begin{aligned}
|r_1| &\le C\sum_{i=1}^{\frac{N}{2}} h_i\int_{x_{i-1}}^{x_i}(1+\frac{1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}})dx + C\sum_{i=\frac{N}{2}+1}^{N} h_i\int_{x_{i-1}}^{x_i}(1+\frac{1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}})dx \\
&\le C\left\{N^{-1}+\sum_{i=1}^{N}h_i\int_{x_{i1}}^{x_i}\left(1+\frac{1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}}\right)dx\right\} \\
&\le C\left\{N^{-1}+\alpha^{-1}\sum_{i=1}^{N}h_i\left(e^{-\frac{\alpha x_{i-1}}{\varepsilon}}+e^{-\frac{\alpha x_i}{\varepsilon}}\right)\right\} \\
&\le C\left\{N^{-1}+\sum_{i=1}^{N}h_i\right\} \\
&\le CN^{-1}.
\end{aligned}
$$

With the inequality of (5), the following expression is possible to type:

$$
|r_2| \le \|g_2\|_\infty \sum_{i=1}^{N}h_i\int_{x_{i-1}}^{x_i}\left(1+\frac{1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}}\right)dx,\ i=1,2,...,N. \tag{50}
$$

From (29) and (50), we obtain

$$
\begin{aligned}
|r_2| &\le C\sum_{i=1}^{\frac{N}{2}} h_i\int_{x_{i-1}}^{x_i}(1+\frac{1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}})dx + C\sum_{i=\frac{N}{2}+1}^{N} h_i\int_{x_{i-1}}^{x_i}(1+\frac{1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}})dx \\
&\le C\left\{N^{-1}+\sum_{i=1}^{N}h_i\int_{x_{i-1}}^{x_i}\left(1+\frac{1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}}\right)dx\right\} \\
&\le C\left\{N^{-1}+\alpha^{-1}\sum_{i=1}^{N}h_i\left(e^{-\frac{\alpha x_{i-1}}{\varepsilon}}+e^{-\frac{\alpha x_i}{\varepsilon}}\right)\right\} \\
&\le C\left\{N^{-1}+\sum_{i=1}^{N}h_i\right\} \\
&\le CN^{-1}.
\end{aligned}
$$

Thus, we can express the convergence result of the problem (1) - (3). This shows the validity of (37) and (38).

Lemma 4.2. Assume for all $0 \le i \le N$, z_i are the solutions of (33) - (35) and

$$
\overline{\gamma} = \sum_{i=1}^{N-1}h_i\left(|g_{i,1}(x)|+|g_{i,2}(x)|\right) < 1.
$$

Afterward, the following inequality is achieved:

$$
\|z\|_{\infty,\overline{\omega}_h} \le C(\beta^{-1}\|R\|_{\infty,\omega_h}+|r_1|+|r_2|). \tag{51}
$$

Proof. We can write the difference problem for (33) - (35) as follows:

$$
lz_i := \varepsilon z_{\bar{x}\hat{x},i} + a_i z_{\overset{\circ}{x},i} = R_i,\quad i=1,2,...,N-1, \tag{52}
$$

$$l_1 z := z_0 - \sum_{i=1}^{N} h_i g_1 z_i = r_1, \tag{53}$$

$$l_2 z := z_N - \sum_{i=1} h_i g_2 z_i = r_2, \tag{54}$$

where

$$b_i = \frac{\partial f}{\partial u}(x_i, u_i).$$

In (52) - (54), let l, l_1, and l_2 denote the finite difference operators. If v is a mesh function constructed on $\overline{\omega}_h$, such that $v_0 \geq 0$, $v_N \geq 0$, and for each $i = 1, 2, ..., N - 1$, $lv_i \leq 0$, then $v_i \geq 0$ for each $i = 1, 2, ..., N$. Because of the maximum principle, we can write the following inequality:

$$\|z\|_{\infty,\overline{\omega}_N} \leq |z_0| + |z_N| + \alpha^{-1} \|R\|_{\infty,\overline{\omega}_N}. \tag{55}$$

Then, we take from boundary condition (34) - (35)

$$|z_0| \leq |r_1| + \sum_{i=1}^{N} \left(\int_{x_{i-1}}^{x_i} |g_1(x)| dx \right) |z_i|, \tag{56}$$

$$|z_N| \leq |r_2| + \sum_{i=1}^{N} \left(\int_{x_{i-1}}^{x_i} |g_2(x)| dx \right) |z_i|. \tag{57}$$

Substituting the inequality (56) and (57) in (55), we have

$$\begin{aligned} \|z\|_{\infty,\overline{\omega}_N} &\leq \alpha^{-1} \|R\|_{\infty,\omega_N} + |r_1| + |r_2| + \sum_{i=1}^{N} h_i \,|\, g_{1,i} \,\|\, z_i \,| + \sum_{i=1}^{N} h_i \,|\, g_{2,i} \,\|\, z_i \,| \\ &\leq \alpha^{-1} \|R\|_{\infty,\omega N} + |r_1| + |r_2| + \max_{1 \leq i \leq N} \|z_i\| \sum_{i=1}^{N} h_i \,|\, g_{1,i} + \max_{1 \leq i \leq N} |\, z_i \,| \sum_{i=1}^{N} h_i \,|\, g_{2,i} \,| \\ &\leq \alpha^{-1} \|R\|_{\infty,\omega N} + |r_1| + |r_2| + \|z\|_{\infty,\overline{\omega}_N} \sum_{i=1}^{N} h_i \,|\, g_{1,i} \,\|z\|_{\infty,\overline{\omega}_N} \sum_{i=1}^{N} h_i \,|\, g_{2,i} \,|. \end{aligned}$$

So, we have

$$\|z\|_{\infty\overline{\omega}_N} \leq \frac{\alpha^{-1} \|R\|_{\infty,\omega_N} + |r_1| + |r_2|}{1 - \sum_{i=1}^{N} h_i \left(|\, g_{1,i} \,| + |\, g_{2,i} \,| \right)}. \tag{58}$$

From (58), we include that

$$\|z\|_{\infty} \leq (1 - \overline{\gamma})^{-1} (\alpha^{-1} \|R\|_{\infty,\overline{\omega}_N} + |r_1| + |r_2|),$$

where

$$\overline{\gamma} = \sum_{i=1}^{N} \left(\int_{x_{i-}}^{x_i} \left(|g_1(x)| + |g_2(x)| \right) dx \right) < 1.$$

Thus, whenever $\overline{\gamma} < 1$, the evaluation (51) takes place.

Now, we can present the essential results of this paper.

Theorem 4.3. Assume u and y are the solutions of (1) - (3) and (30) - (32), respectively. Then, under conditions of previous lemmas, the ε-uniform estimate provides

$$\|y-u\|_{\infty,\overline{\omega}_h} \le CN^{-1}.$$

Proof. The claim of the theorem follows from the previous two propositions.

5. Test examples and numerical results

To confirm the theoretical conclusions, we give numerical results derived using the difference scheme (30) - (32).

Example 5.1. The following nonlinear singularly perturbed problem with boundary conditions is considered

$$\varepsilon u''(x)+(\frac{1+x}{2})u'(x)-\tan(x+u)+u^2-\varepsilon x=0,\ \ 0<x<1,$$

and considered the integral initial condition as

$$u(0)=\int_0^1 \sin(2\pi x)u(x)dx+1,$$

$$u(1)=\int_0^1 \cos(\pi x)u(x)dx+1.$$

Since the solution to the check problem is undetermined, we apply the double-mesh rule to evaluate the errors in the solutions and calculate the convergence rates. So, we cross-check the calculated solutions by placing the solutions on a mesh divided into two [32]. The error evaluations acquired in this way are as follows:

$$e_\varepsilon^N = \max_{\omega_N}\left|y_i^{\varepsilon,N}-\tilde{y}_i^{\varepsilon,2N}\right|,$$

where $\tilde{y}_i^{\varepsilon,2N}$ denotes the approximate solution of the relevant approach in $\overline{\omega}_{2N}=\left\{x_{i/2}: i=\overline{0,2N}\right\}$ and $x_{i+1/2}=\dfrac{x_i+x_{i+1}}{2}$, $i=\overline{1,N-1}$. Appropriate convergence rates are computed as follows:

$$p_\varepsilon^N=\frac{\ln(e_\varepsilon^N/e_\varepsilon^{2N})}{\ln 2}.$$

Example 5.2. Consider the nonlinear singular perturbation problem with the following boundary conditions.

$$\varepsilon u''(x)+(1+x)u'+\exp(-u)+\frac{u^2}{4}=0,\ \ \ 0<x<1,$$

$$u(0)=\int_0^1 \sin(\pi x)u(x)dx+1,$$

$$u(1)=\int_0^1 \tan\left(\frac{\pi x}{5}\right)u(x)dx+1.$$

Since the solution to the control problem is uncertain, we apply the double-mesh rule to evaluate the errors in the solutions and calculate the convergence rates. Therefore, we cross-check the calculated solutions with solutions on a bisected mesh. The error evaluations obtained in this way are as follows:

$$e_\varepsilon^N = \max_{\omega N} | y_i^{\varepsilon,N} - y_i^{\varepsilon,2N} |,$$

where $\tilde{y}_i^{\varepsilon,2N}$ denotes the approximate solution of the relevant approach in $\varpi_{2N} = \left\{ x_{i/2} : i + \overline{0,2N} \right\}$ and $x_{i+1/2} = \dfrac{x_i + x_{i+1}}{2}$, $i = \overline{1, N-1}.$ Appropriate convergence rates are computed as follows:

$$p_\varepsilon^N = \frac{\text{In}(e_\varepsilon^N / e_\varepsilon^2)}{\text{In } 2}.$$

We applied the present method above to the example. According to Tables 1 and 2 and Figures 1-4, when N takes increasing values, it is seen that the convergence rate of the smooth convergence speed p^N is first-order. In Figures 1, 2, 3, and 4, the errors are maximum in the boundary layer region because of the irregularity caused by the sudden and rapid change of solution around $x = 0$ for different values of ε. That is, numerical results show that the proposed scheme is working very effectively.

Table 1. Maximum pointwise errors e^N and order of convergence p^N on ω_N

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-2}	0.01782540 0.7855	0.01034136 0.8464	0.00575144 0.8971	0.00308820 0.9150	0.00165125
2^{-4}	0.07945299 0.8746	0.04333375 0.9050	0.02314232 0.9160	0.01226472 0.9307	0.00643389
2^{-6}	0.04670852 0.8300	0.02627446 0.8609	0.01446658 0.9182	0.00765524 0.9562	0.00394547
2^{-8}	0.01105469 0.8657	0.00606675 0.9011	0.00324859 0.9100	0.00172886 0.9612	0.00088798
2^{-10}	0.03906250 0.8719	0.02134485 0.8923	0.01449945 0.9155	0.00609663 0.9282	0.00320395
e^N	0.07945299	0.04333375	0.02314232	0.01226472	0.00643389
p^N	0.7855	0.8464	0.8831	0.9150	

Table 2. Maximum pointwise errors e^N and order of convergence p^N on ω_h

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-2}	0.0019308513 1.0165567016	0.0009544095 1.0083651078	0.0004744458 1.0042045093	0.0002365326 1.0021077773	0.0001180936
2^{-4}	0.0004827323 1.0166002110	0.0002386048 1.0083760262	0.0001186118 1.0042072441	0.0000591332 1.0021084617	0.0000295234
2^{-6}	0.0000678850 1.0166126744	0.0000335539 1.0083791538	0.0000166798 1.0042080276	0.0000083156 1.0021086577	0.0000041517
2^{-8}	0.0000075428 1.0166144873	0.0000037282 1.0083796087	0.0000018533 1.0042081414	0.0000009240 1.0021086866	0.0000004613
2^{-10}	0.0000017240 1.8910457518	0.0000004648 1.3607459016	0.0000001810 1.0042081549	0.0000000902 1.0021086885	0.0000000450
e^N	0.0019308513	0.0009544095	0.0004744458	0.0002365326	0.0001180936
p^N	1.0165567016	1.0083651078	1.0042045093	1.0021077773	

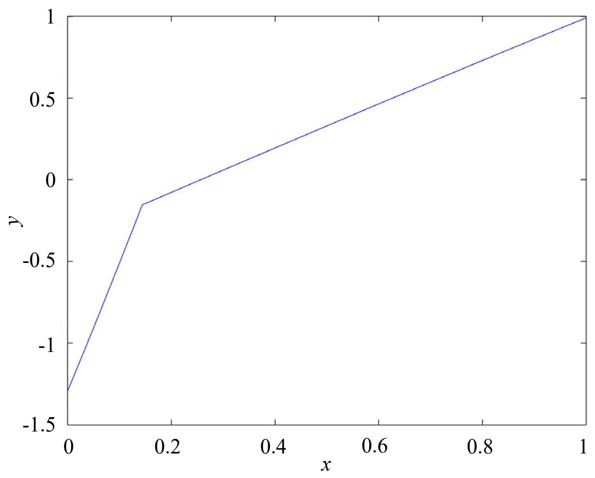

Figure 1. Behavior of the numerical solution for $\varepsilon = 2^{-4}$ and $N = 64$

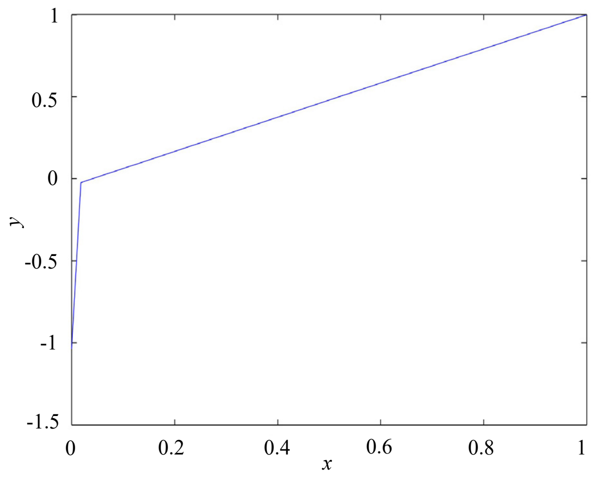

Figure 2. Behavior of the numerical solution for $\varepsilon = 2^{-8}$ and $N = 128$

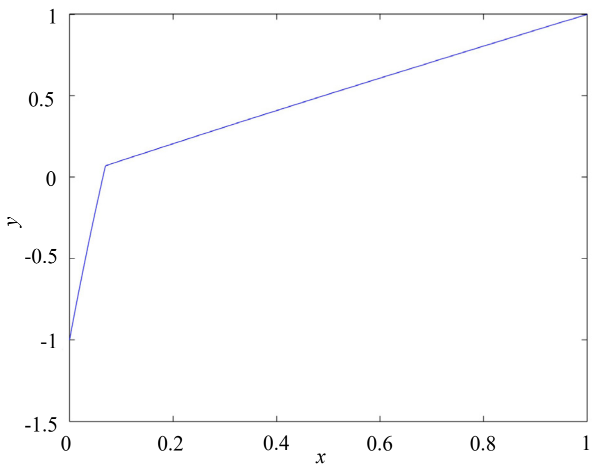

Figure 3. Behavior of the numerical solution for $\varepsilon = 2^{-4}$ and $N = 64$

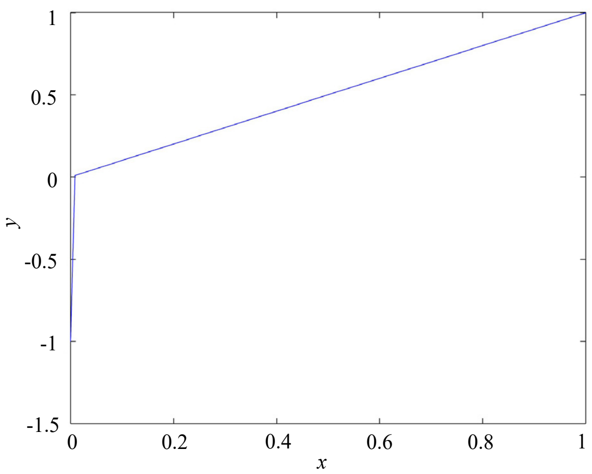

Figure 4. Behavior of the numerical solution for $\varepsilon = 2^{-8}$ and $N = 128$

6. Conclusion

The finite difference method is effectively applied to solve the singularly perturbed problem with two integral boundary states. The applied method supports the boundary layer using the asymptotic estimates technique, exponential basis functions, weight and remainder terms, and interpolating quadrature forms. In Table 1, the numerical results for various values of ε and varied numbers of mesh intervals N are also presented. We demonstrate that the finite difference approach has ε-uniform convergence with a maximum norm depending on the perturbation parameter. In particular, a suitable method is applied to the usual control problem. The biggest benefit of our method is that it provides suitable and good solutions. On test problems, we use the current method. We obtained the results we obtained using this method through the MATLAB (R2022b) program. Consequently, the numerical data show how accurate and reliable the analytical technique we suggest for singularly perturbed problems of the reaction-diffusion kind is. Numerical investigations can be sustained for more sophisticated types, such as mixed-type linear and nonlinear problems, delay forms, higher-dimensional problems, etc.

Conflict of interest

The authors declare that they have no conflict of interest.

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