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Research Article

Topological Indices and Structural Properties of Cubic Power Graph of Dihedral Group

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Abstract: The cubic power graph of finite group G with identity element e, is an undirected finite, simple graph in which a pair of distinct vertices x, y have an edge iff $xy = z^3$ or $yx = z^3$ for any $z \in D_n$ with $z^3 \neq e$. In this paper, we have studied the structural representation of the cubic power graph of the dihedral group and various structural properties such as clique, girth, vertex degree, chromatic number, independent number, matching number, perfect matching, dominating number, etc. We have also calculated various topological indices such as the Harary index, the first and second Zagreb indices, the Wiener and hyper-Wiener indices, the Schultz index, the harmonic index, the general Randic index, the eccentric connectivity index, the Gutman index, the atomic-bond connectivity index, and the geometric-arithmetic index of the cubic power graph of dihedral group D_n when $\gcd(n, 3) = 1$.

Keywords: dihedral group, cubic power graph, degree of vertex, chromatic number, matching number, topological indices

MSC: 05C07, 05C09, 05C25, 05C69, 05C70

1. Introduction

In recent years, various graphs associated with algebraic structures have been studied such as various properties of power graphs in [1-5], square element graphs in [6], square power graphs in [7], kth power graphs in [8], etc. A power graph of a finite group G is an undirected finite simple graph having a vertex set group G in which two distinct element vertices have an edge iff one is the power of the other. Various structural properties and topological indices of co-prime-order graphs of finite abelian p-groups are studied in [9]. A co-prime order graph of a finite group G is a graph with a vertex set G in which two distinct vertices a, b have an edge iff $\gcd(o(a), o(b)) = 1$ or a prime number. Various properties of equal-square graphs associated with finite groups are studied in [10]. An equal-square graph of a finite group is a simple graph with two distinct vertices x, y adjacent iff $x^2 = y^2$.

Graph invariants are important elements of graph theory. They are crucial in describing the structural features of networks and graphs. The computer networks, the World Wide Web, social networks, chemical reaction networks, chemical structures of molecules, etc. can be described by graph invariants. Topological indices are the common name for these measurements that are employed in the investigation of networks' structural properties. Chemistry, mathematics, and pharmacy engineering all make extensive use of topological indices [11-13]. In addition, topological indices are used for networks, internet routing, transport network flow, and people trafficking [14-18].

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Prathap and Chelvam [19] introduced and studied the cubic power graph of finite abelian groups. The cubic power graph of the finite abelian group G is an undirected simple graph with G as the vertex set and two distinct vertices x, y having edges iff $x + y = 3t$ for some $t \in G$ with $3t \neq 0$. In this paper, we have extended the concept of the cubic power graph of finite abelian groups to the cubic power graph of finite groups. The cubic power graph of the finite group G with identity element e is simple undirected graph with vertex set G having distinct vertices $x, y \in G$ adjacent iff $xy = z^3$ or $yx = z^3$ for some $z \in G$ with $z^3 \neq e$. It is denoted as $\Gamma_{cpg}(G)$. The cubic power graph of a finite abelian group is connected only when $\gcd(3, n) = 1$, where n is the order of the group, but the cubic power graph of a dihedral group is always a connected graph with a special type of regular subgraph when $\gcd(3, n) = 3$.

In Section 2, we have defined some symbols used throughout this paper and given the basic terminologies of graph theory and group theory. In Section 3, we have given a structural representation of the cubic power graph of the dihedral group. Structural properties such as vertex degree, connectedness, completeness, girth, clique number, chromatic number, weakly perfectness, independent number, matching number, perfect matching, and dominating number of the cubic power graph of the dihedral group are also studied in Section 3. In Section 4, topological indices such as the Harary index, the first and second Zagreb indices, the Wiener and hyper-Wiener indices, the Schultz index, the harmonic index, the general Randic index, the eccentric connectivity index, the Gutman index, the atomic-bond connectivity, and the geometric-arithmetic index of the cubic power graph of the dihedral group D_n are given for $\gcd(n, 3) = 1$.

2. Preliminaries

Let $\Gamma_{cpg}(D_n)$ be a cubic power graph of the dihedral group D_n of order $2n$ with identity element e, whose vertex set is D_n itself denoted as $V(\Gamma_{cpg}(D_n))$ and a set of edges as $E(\Gamma_{cpg}(D_n))$. The number of edges in $\Gamma_{cpg}(D_n)$ is denoted as $|E(\Gamma_{cpg}(D_n))|$. For any vertex $x \in V(\Gamma_{cpg}(D_n))$, $\deg_{\Gamma_{cpg}(D_n)}(x)$ is the number of vertices in $\Gamma_{cpg}(D_n)$ in which x is adjacent, known as degree of x. Distance between any pair of vertices $x, y \in V(\Gamma_{cpg}(D_n))$ is the shortest path x-y in $\Gamma_{cpg}(D_n)$, denoted as $d(x, y)$. For any $x \in V(\Gamma_{cpg}(D_n))$, the largest distance of x to any vertex in $\Gamma_{cpg}(D_n)$ is known as eccentricity of x and denoted as $ecc(x)$. A regular graph with n vertices and every vertex with vertex degree m is denoted as m - K_n. If all the vertices of graph G_1 are adjacent with all the vertices of graph G_2, then we have denoted it by using + as $G_1 + G_2$. If graph G have two disjoint components G_1 and G_2, then it is denoted as $G = G_1 \cup G_2$. $\overline{G}$ is used for complement of graph G.

Let D_n be the dihedral group of order $2n$ with identity element e, then $D_n = \{e, r, r^2, r^3, \cdots, r^{n-1}, s, rs, r^2s, r^3s, \cdots, r^{n-1}s : r^n = e, s^2 = e, r^is = sr^{n-i}\}$. Let $X = \{x : x \in D_n, x = x^{-1}\}$, $R = \{e, r, r^2, r^3, \cdots, r^{n-1}\}$, $R' = \{e, r^3, r^6, \cdots, r^{n-3}\}$ and $S = \{s, rs, r^2s, \cdots, r^{n-1}s\}$. Let $D_n' = \{x^3 : x \in D_n\}$ then we have $D_n' = D_n$ when $\gcd(3, n) = 1$ and $D_n' = \{e, r^3, r^6, \cdots, r^{n-3}, s, rs, r^2s, r^3s, \cdots, r^{n-1}s\} = R' \cup S$ when $\gcd(3, n) = 3$. It can be noted that any pair of distinct vertices x, y in $\Gamma_{cpg}(D_n)$ are adjacent iff $xy \in D_n' \setminus \{e\}$.

3. Structural properties of $\Gamma_{cpg}(D_n)$

Theorem 3.1. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and $\gcd(3, n) = 1$. Let $x, y \in V(\Gamma_{cpg}(D_n))$, then

(i) $$\Gamma_{cpg}(D_n) = \begin{cases} \overline{(n+1)K_1 \cup \left(\frac{n-1}{2}\right)K_2} & \text{when } n \text{ is an odd number,} \\ \overline{(n+2)K_1 \cup \left(\frac{n-2}{2}\right)K_2} & \text{when } n \text{ is an even number.} \end{cases}$$

(ii) $$\deg_{\Gamma_{cpg}}(x) = \begin{cases} 2n-2 & \text{when } x \in D_n \setminus X, \\ 2n-1 & \text{when } x \in X. \end{cases}$$

(iii) $$d(x, y) = \begin{cases} 1 & \text{if } x^{-1} \neq y, \\ 2 & \text{if } x^{-1} = y \neq x. \end{cases}$$

(iv) $\left|E\left(\Gamma_{cpg}\left(D_n\right)\right)\right| = \begin{cases} \dfrac{4n^2-3n+1}{2} & \text{if } n \text{ is an odd number,} \\ \dfrac{4n^2-3n+2}{2} & \text{if } n \text{ is an even number.} \end{cases}$

Proof. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n, dihedral group of order $2n$ and $\gcd(3, n) = 1$.

(i) Case 1. When n is an odd number.

In this case, we have $|X| = n+1$. For every $x \in X$, $y \in D_n \setminus \{x\}$, we get $xy \in D_n{}' \setminus \{e\}$ and for every $x \in D_n \setminus X$, $y \in D_n \setminus \{x, x^{-1}\}$, we have $xy \in D_n{}' \setminus \{e\}$. So, not self-inverse element vertex in $\Gamma_{cpg}(D_n)$ adjacent with all other vertices in $\Gamma_{cpg}(D_n)$ except its inverse and self-inverse element vertex in $\Gamma_{cpg}(D_n)$ adjacent with all other vertices in $\Gamma_{cpg}(D_n)$. Hence, $\Gamma_{cpg}(D_n) = \overline{(n+1)K_1 \cup \left(\dfrac{n-1}{2}\right)K_2}$ when n is an odd number.

Case 2. When n is an even number.

In this case, we have $|X| = n+2$. For every $x \in X$, $y \in D_n \setminus \{x\}$, we get $xy \in D_n{}' \setminus \{e\}$ and for every $x \in D_n \setminus X$, $y \in D_n \setminus \{x, x^{-1}\}$, we have $xy \in D_n{}' \setminus \{e\}$. So, we have not self-inverse element vertex in $\Gamma_{cpg}(D_n)$ adjacent with all other vertices in $\Gamma_{cpg}(D_n)$ except its inverse and self-inverse element vertex in $\Gamma_{cpg}(D_n)$ adjacent with all other vertices in $\Gamma_{cpg}(D_n)$. Hence, $\Gamma_{cpg} = \overline{(n+2)K_1 \cup \left(\dfrac{n-2}{2}\right)K_2}$ when n is an even number.

(ii) Case 1. When $x \in D_n \setminus X$.

As discussed in Theorem 3.1(i), $x \in D_n \setminus X$ is adjacent with all $y \in D_n \setminus \{x, x^{-1}\}$. Thus, $\deg_{\Gamma_{cpg}}(x) = 2n-2$ when $x \in D_n \setminus X$.

Case 2. When $x \in X$.

As discussed in Theorem 3.1(i), $x \in X$ is adjacent with all $y \in D_n \setminus \{x\}$. Thus, $\deg_{\Gamma_{cpg}}(x) = 2n-1$ when $x \in X$.

(iii) For every $x \in V(\Gamma_{cpg}(D_n))$, $y \in V(\Gamma_{cpg}(D_n)) \setminus \{x, x^{-1}\}$, we get $xy \in D_n{}' \setminus \{e\}$. Now, when $x^{-1} \neq y$, we have $xy \in D_n{}' \setminus \{e\}$. So, $d(x, y) = 1$ when $x^{-1} \neq y$.

When $x^{-1} = y \neq x$, we have $xy = e \notin D_n{}'$, and so $d(x, y) \neq 1$. In such case, we always have $z \in D_n \setminus \{x, y\}$ such that $xz \in D_n{}' \setminus \{e\}$ and $yz \in D_n{}' \setminus \{e\}$. So, we have path $x - z - y$ between x and y vertices. Thus, $d(x, y) = 2$ when $x^{-1} = y \neq x$.

(iv) Case 1. When n is an odd number.

In any simple graph with $2n$ vertices, maximum possible number of vertices are $n(2n - 1)$. We obtain $\Gamma_{cpg}(D_n)$ by deleting $\dfrac{(n-1)}{2} x - x^{-1}$ type edges from K_n. Hence, $|E(\Gamma_{cpg}(D_n))| = n(2n-1) - \dfrac{(n-1)}{2} = \dfrac{4n^2-3n+1}{2}$.

Case 2. When n is an even number.

In any simple graph with $2n$ vertices, maximum possible number of vertices are $n(2n - 1)$. We obtain $\Gamma_{cpg}(D_n)$ by deleting $\dfrac{(n-2)}{2} x - x^{-1}$ type edges from K_n. Hence, $|E(\Gamma_{cpg}(D_n))| = n(2n-1) - \dfrac{(n-2)}{2} = \dfrac{4n^2-3n+2}{2}$.

Theorem 3.2. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and $\gcd(3, n) = 3$. Then,

(i) $$\Gamma_{cpg}(D_n) = \begin{cases} K_{3,3} \text{ when } n = 3 \\ [3K_2] + [3K_2] \text{ when } n = 6 \\ \left[\left[\overline{K_1 \cup \left(\dfrac{n-3}{6}\right)K_2}\right] \cup \left(\dfrac{n-3}{3}\right) - K_{\frac{2n}{3}}\right] + 3K_{\frac{n}{3}} \text{ when } n \text{ is an odd number other than 3,} \\ \left[\left[\overline{2K_1 \cup \left(\dfrac{n-6}{6}\right)K_2}\right] \cup \left(\dfrac{n-3}{3}\right) - K_{\frac{2n}{3}}\right] + 3K_{\frac{n}{3}} \text{ when } n \text{ is an even number other than 6.} \end{cases}$$

(ii) $\deg_{\Gamma_{cpg}}(x) = \begin{cases} \left(\dfrac{4n-3}{3}\right) \text{ when } x \in X \cup R \setminus R', \\ \left(\dfrac{4n-6}{3}\right) \text{ when } x \in R' \setminus X. \end{cases}$

Proof. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and gcd(3, n) = 3.

(i) Case 1. When $n = 3$.

In this case, we have all three reflection element vertices adjacent to all three rotation elements, and no pair of rotation elements or reflection elements is adjacent, as shown in Figure 1. So, $\Gamma_{cpg}(D_3) = K_{3,3}$.

Case 2. When $n = 6$.

In this case, we have all six reflection element vertices adjacent to all six rotation elements, and three pairs of rotation elements and three pairs of reflection elements are adjacent, as shown in Figure 1. So, $\Gamma_{cpg}(D_6) = [3K_2] + [3K_2]$.

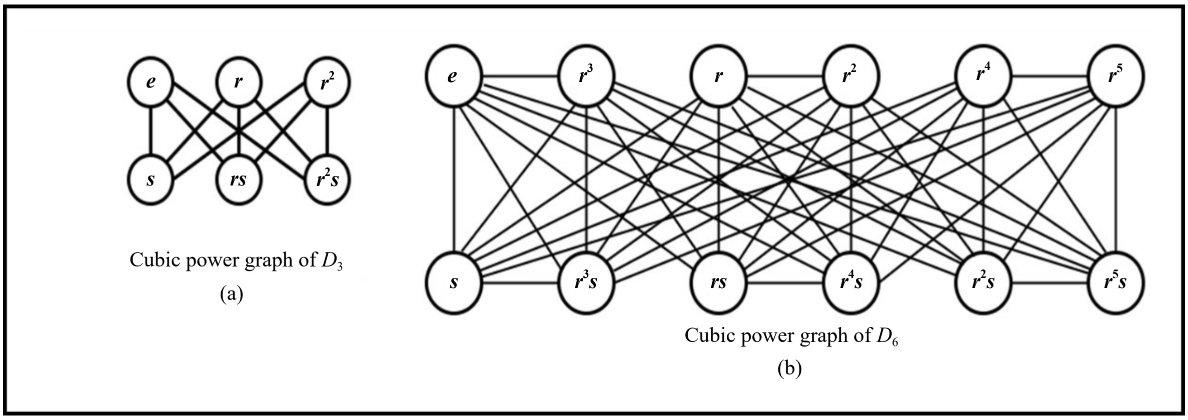


Figure 1. (a) $\Gamma_{cpg}(D_3)$; **(b)** $\Gamma_{cpg}(D_6)$

Case 3. When n is an odd number other than 3.

For every rotation element vertex $x \in R \subset V(\Gamma_{cpg}(D_n))$ and every reflection element vertex $y \in S \subset V(\Gamma_{cpg}(D_n))$, we have $xy \in D_n' \setminus \{e\}$. So, we have every rotation element vertex adjacent with every reflection element vertex.

For every pair of rotation element $x, y \in R'$, we have $xy \in D_n' \setminus \{e\}$ whenever $x^{-1} \neq y$. We have one self-inverse element in R' and $\dfrac{n-3}{6}$ pairs of x, x^{-1} in R', when n is an odd number. For $x \in R'$ and $y \in R \setminus R'$ we have $xy \notin D_n' \setminus \{e\}$.

So, forming $\overline{\left[K_1 \cup \left(\dfrac{n-3}{6}\right) K_2\right]}$ subgraph of $\Gamma_{cpg}(D_n)$ with R' as vertex set.

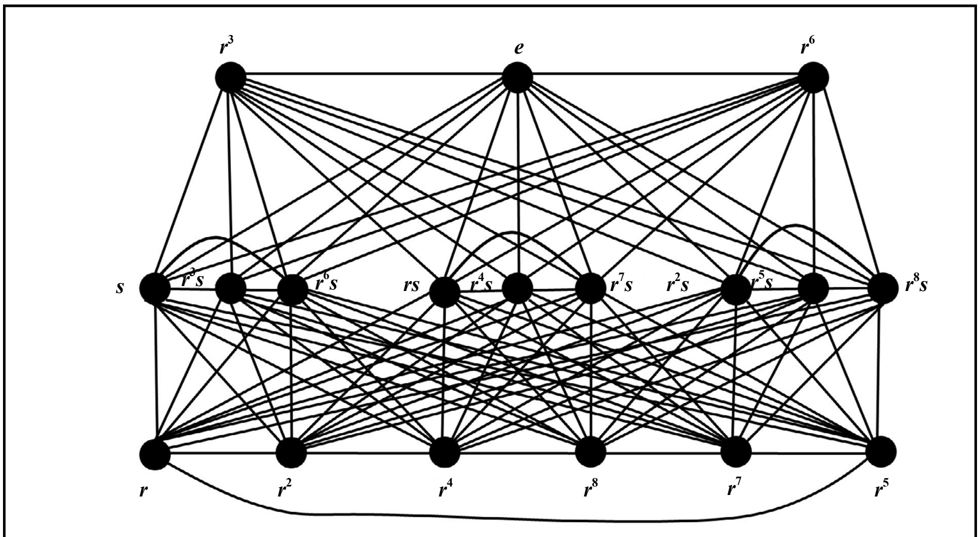
For $x, y \in R \setminus R'$, we have $xy \in D_n' \setminus \{e\}$ iff $xy \in R'$. We have every rotation element in R' adjacent with other $\dfrac{n-3}{3}$ rotation vertices of R' forming the connected $\left(\dfrac{n-3}{3}\right)$ – regular subgraph of $\Gamma_{cpg}(D_n)$ with $R \setminus R'$ as vertex set.

Let $S = S_1 \cup S_2 \cup S_3$ such that $S_1 = \{s, r^3 s, r^6 s, \cdots, r^{n-3} s\}$, $S_2 = \{rs, r^4 s, r^7 s, \cdots, r^{n-2} s\}$, and $S_3 = \{r^2 s, r^5 s, r^8 s, ..., r^{n-1} s\}$. For every pair of vertices $x, y \in S_1$, we have $xy \in D_n' \setminus \{e\}$. For $x \in S_1$ and $y \in S \setminus S_1$, we have $xy \notin D_n' \setminus \{e\}$. Thus, we have a complete subgraph of $\Gamma_{cpg}(D_n)$ with S_1 as a vertex set.

For every pair of vertices $x, y \in S_2$, we have $xy \in D_n' \setminus \{e\}$. For $x \in S_2$ and $y \in S \setminus S_2$, we have $xy \notin D_n' \setminus \{e\}$. Thus, we have a complete subgraph of $\Gamma_{cpg}(D_n)$ with S_2 as a vertex set.,

For every pair of vertices $x, y \in S_3$, we have $xy \in D_n' \setminus \{e\}$. For $x \in S_3$ and $y \in S \setminus S_3$, we have $xy \notin D_n' \setminus \{e\}$. Thus, we have a complete subgraph of $\Gamma_{cpg}(D_n)$ with S_3 as a vertex set.

Hence, we have $\Gamma_{cpg}(D_n) = \left[\overline{\left[K_1 \cup \left(\frac{n-3}{6}\right)K_2\right]} \cup \left(\frac{n-3}{3}\right) - K_{\frac{2n}{3}}\right] + 3K_{\frac{n}{3}}$ when n is an odd number other than 3 and gcd(3, n) = 3. Cubic power graph of D_9, $\Gamma_{cpg}(D_9)$ is shown in Figure 2.



Cubic power graph of D_9

Figure 2. $\Gamma_{cpg}(D_9)$

Case 4. When n is an even number other than 6.

For every rotation element vertex $x \in R \subset V(\Gamma_{cpg}(D_n))$ and every reflection element vertex $y \in S \subset V(\Gamma_{cpg}(D_n))$, we have $xy \in D_n' \setminus \{e\}$. So, we have every rotation element vertex adjacent with every reflection element vertex.

For every pair of rotation element $x, y \in R'$, we have $xy \in D_n' \setminus \{e\}$ whenever $x^{-1} \neq y$. We have two self-inverse element in R' and $\frac{n-6}{6}$ pairs of x, x^{-1} in R', when n is an odd number. For $x \in R'$ and $y \in R \setminus R'$, we have $xy \notin D_n' \setminus \{e\}$. So, forming $\overline{\left[2K_1 \cup \left(\frac{n-6}{6}\right)K_2\right]}$ subgraph of $\Gamma_{cpg}(D_n)$ with R' as a vertex set.

For $x, y \in R \setminus R'$, we have $xy \in D_n' \setminus \{e\}$ if $xy \in R'$. We have every rotation element in R' adjacent with other $\left(\frac{n-3}{3}\right)$ rotation vertices of R' forming the connected $\left(\frac{n-3}{3}\right)$ – regular subgraph of $\Gamma_{cpg}(D_n)$ with $R \setminus R'$ as vertex set.

Let $S = S_1 \cup S_2 \cup S_3$ such that $S_1 = \{s, r^3s, r^6s, \cdots, r^{n-3}s\}$, $S_2 = \{rs, r^4s, r^7s, \cdots, r^{n-2}s\}$, and $S_3 = \{r^2s, r^5s, r^8s, \cdots, r^{n-1}s\}$. For every pair of vertices $x, y \in S_1$, we have $xy \in D_n' \setminus \{e\}$. For $x \in S_1$ and $y \in S \setminus S_1$, we have $xy \notin D_n' \setminus \{e\}$. Thus, we have a complete subgraph of $\Gamma_{cpg}(D_n)$ with S_1 as a vertex set.

For every pair of vertices $x, y \in S_2$, we have $xy \in D_n' \setminus \{e\}$. For $x \in S_2$ and $y \in S \setminus S_2$, we have $xy \notin D_n' \setminus \{e\}$. Thus, we have a complete subgraph of $\Gamma_{cpg}(D_n)$ with S_2 as a vertex set.

For every pair of vertices $x, y \in S_3$, we have $xy \in D_n' \setminus \{e\}$. For $x \in S_3$ and $y \in S \setminus S_3$, we have $xy \notin D_n' \setminus \{e\}$. Thus, we have a complete subgraph of $\Gamma_{cpg}(D_n)$ with S_3 as a vertex set.

Thus, we have $\Gamma_{cpg}(D_n) = \left[\overline{\left[2K_1 \cup \left(\frac{n-6}{6}\right)K_2\right]} \cup \left(\frac{n-3}{3}\right) - K_{\frac{2n}{3}}\right] + 3K_{\frac{n}{3}}$ when n is an even number other than 6 and gcd(3, n) = 3.

(ii) Case 1. For $n \in \{3,6\}$, we have $X \cup R \setminus R' = D_n$, and so using Theorem 3.1(i), $\deg_{\Gamma_{cpg}}(x) = \frac{4n-3}{3}$. When $x \in X \cup R \setminus R'$.

We have $X = S \cup (X \cap R')$. For $x \in S$, we have x vertex part of $K_{\frac{n}{3}}$ type subgraph of $\Gamma_{cpg}(D_n)$ as discussed in Theorem 3.2(i). So, we have $\deg_{\Gamma_{cpg}}(x) = n + \frac{n}{3} - 1 = \frac{4n-3}{3}$.

We have $X \cap R' = \{e\}$ for odd n other than 3, and $X \quad R = \{e, r^{\bar{2}}\}$ for even n other than 6. So, we have vertex $x \in X \cap R'$ self-inverse element vertex part of $\overline{\left[K_1 \cup \frac{n-3}{6} K_2\right]}$ type subgraph when n is an odd other than 3, and of $\overline{\left[2K_1 \cup \frac{n-6}{6} K_2\right]}$ type subgraph when n is an even other than 6. Thus, $\deg_{\Gamma_{cpg}}(x) = \frac{n-3}{3} + n = \frac{4n-3}{3}$.

If $x \in R \setminus R'$, then using Theorem 3.2(i), we have x-vertex part of $\left(\frac{n-3}{3}\right) - K_{\frac{2n}{3}}$ type subgraph of $\Gamma_{cpg}(D_n)$. Thus, $\deg_{\Gamma_{cpg}}(x) = \frac{n-3}{3} + n = \frac{4n-3}{3}$. Hence, $\deg_{\Gamma_{cpg}}(x) = \frac{n-3}{3} + n = \frac{4n-3}{3}$ when $x \in X \cup R \setminus R'$.

Case 2. When $x \in R' \setminus X$.

For $n \in \{3,6\}$ we have $R' \setminus X = \phi$. Using Theorem 3.2(i), we have $x \in R' \setminus X$ is the not self-inverse element vertex of $\overline{\left[K_1 \cup \frac{n-3}{6} K_2\right]}$ subgraph and of $\overline{\left[2K_1 \cup \frac{n-6}{6} K_2\right]}$ for odd number n other than 3, and even number n other than 6, respectively. Thus, $\deg_{\Gamma_{cpg}(D_n)} = 1 + \frac{n-3}{3} - 2 + n = \frac{4n-6}{3}$ for odd number n other than 3, and $\deg_{\Gamma_{cpg}(D_n)} = 2 + \frac{n-6}{3} - 2 + n = \frac{4n-6}{3}$ for even number n other than 6.

Thus, the required result.

Theorem 3.3. Let $\Gamma_{cpg}(D_n)$ be a cubic power graph of D_n, then

(i) $\Gamma_{cpg}(D_n)$ is always a connected graph.

(ii) $\Gamma_{cpg}(D_n)$ is a complete graph iff $n \in \{1,2\}$.

(iii) $\Gamma_{cpg}(D_n)$ is a bipartite graph iff $n \in \{1,3\}$.

Proof. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n with identity element e.

(i) Case 1. When gcd(3, n) = 1.

Using Theorem 3.1(i), we have for every vertex $x \in V(\Gamma_{cpg}(D_n)) \setminus \{e\}$, $xe \in D_n' \setminus \{e\}$. Thus, e is adjacent with every vertex $x \in V(\Gamma_{cpg}(D_n)) \setminus \{e\}$. Thus, $\Gamma_{cpg}(D_n)$ is a connected graph when gcd(3, n) = 1.

Case 2. When gcd(3, n) $\neq$ 1.

Using Theorem 3.2(i), for every rotation element vertex x and reflection element vertex y, we have $xy \in D_n' \setminus \{e\}$. Thus, every rotation element vertex adjacent with every reflection element vertex. Thus, $\Gamma_{cpg}(D_n)$ is connected when gcd(3, n) $\neq$ 1.

Hence, $\Gamma_{cpg}(D_n)$ is a connected graph.

(ii) Case 1. When $n \in \{1,2\}$.

Using Theorem 3.1(i), we have $\Gamma_{cpg}(D_1) = \overline{[2K_1]} = K_2$ and $\Gamma_{cpg}(D_2) = \overline{[4K_1]} = K_4$. Thus, $\Gamma_{cpg}(D_n)$ is a complete graph when $n \in \{1,2\}$.

Case 2. When $n \in N \setminus \{1,2\}$.

In this case, we always have a pair of rotation elements r^i, r^{n-i} such that $r^i \neq r^{n-i}$ and $r^i r^{n-i} = r^{n-i} r^i = e$. Thus, we always have at least a pair of vertices, which are not adjacent in $\Gamma_{cpg}(D_n)$. Thus, $\Gamma_{cpg}(D_n)$ is not a complete graph when $n \in N \setminus \{1,2\}$.

Hence, the required result.

(iii) Case 1. When $n \in \{1,3\}$.

For $n = 1$, we have $\Gamma_{cpg}(D_1) = K_2$ by using Theorem 3.1(i) and $\Gamma_{cpg}(D_3) = K_{3,3}$ when $n = 3$ by using Theorem 3.2(i). Thus, $\Gamma_{cpg}(D_n)$ is a bipartite graph for $n \in \{1,3\}$.

Case 2. When $n \in N \setminus \{1,3\}$.

Using Theorem 3.1(i) and Theorem 3.2(ii), we always have the cycle of length 3 in $\Gamma_{cpg}(D_n)$ for $n \in N \setminus \{1, 3\}$. Thus, $\Gamma_{cpg}(D_n)$ contains odd cycle for $n \in N \setminus \{1, 3\}$. Hence, $\Gamma_{cpg}(D_n)$ not a bipartite graph for $n \in N \setminus \{1, 3\}$.

Hence, the required result.

Theorem 3.4. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n, then girth

$$gr(\Gamma_{cpg}(D_n)) = \begin{cases} \infty \text{ if } n = 1, \\ 4 \text{ if } n = 3, \\ 3 \text{ if } n \in \mathbb{N} \setminus \{1,3\}. \end{cases}$$

Proof. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n.

Case 1. When $n = 1$.

Using Theorem 3.1(i), we have $\Gamma_{cpg}(D_1) = K_2$. So, there is no cycle in $\Gamma_{cpg}(D_1)$. Thus, $gr(\Gamma_{cpg}(D_n)) = \infty$ if $n = 1$.

Case 2. When $n = 3$.

In this case, we have $\Gamma_{cpg}(D_3) = K_{3,3}$ using Theorem 3.2(i). Thus, $gr(\Gamma_{cpg}(D_3)) = 4$.

Case 3. When $n \in N \setminus \{1, 3\}$.

In this case, we always have at least a pair of reflection vertices x, y such that both are adjacent with each other and adjacent with identity element e vertex using Theorem 3.1 and Theorem 3.2. Thus, $gr(\Gamma_{cpg}(D_n)) = 3$ when $n \in N \setminus \{1, 3\}$.

Theorem 3.5. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and $\gcd(3, n) = 1$, then clique number

$$\omega(\Gamma_{cpg}(D_n)) = \begin{cases} \dfrac{3n+1}{2} \text{ if } n \text{ is an odd number,} \\ \dfrac{3n+2}{2} \text{ if } n \text{ is an even number.} \end{cases}$$

Proof. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and $\gcd(3, n) = 1$.

Case 1. When n is an odd number.

Using Theorem 3.1, we have $\Gamma_{cpg}(D_n) = \overline{\left[(n+1)K_1 \cup \left(\dfrac{n-1}{2}\right)K_2\right]}$. So, we have $(n + 1)$ self-inverse element vertices and one vertex from each $\left(\dfrac{n-1}{2}\right)$ pairs of not-self inverse vertices (r^i, r^{n-i}) forms the complete subgraph of $\Gamma_{cpg}(D_n)$ with maximum number of possible vertices. Thus, $\omega(\Gamma_{cpg}(D_n)) = (n+1) + \left(\dfrac{n-1}{2}\right) = \dfrac{3n+1}{2}$.

Case 2. When n is an even number.

Using Theorem 3.1, we have $\Gamma_{cpg}(D_n) = \overline{\left[(n+2)K_1 \cup \left(\dfrac{n-2}{2}\right)K_2\right]}$. So, we have $(n + 2)$ self-inverse element vertices and one vertex from each $\left(\dfrac{n-2}{2}\right)$ pairs of not-self inverse vertices (r^i, r^{n-i}) forms the complete subgraph of $\Gamma_{cpg}(D_n)$ with maximum number of possible vertices. Thus, $\omega(\Gamma_{cpg}(D_n)) = (n+2) + \left(\dfrac{n-2}{2}\right) = \dfrac{3n+2}{2}$.

Theorem 3.6. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and $\gcd(3, n) = 1$, then chromatic number

$$\chi(\Gamma_{cpg}(D_n)) = \begin{cases} \dfrac{3n+1}{2} & \text{if } n \text{ is an odd number,} \\ \dfrac{3n+2}{2} & \text{if } n \text{ is an even number} \end{cases}$$

Proof. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and $\gcd(3, n) = 1$.

Case 1. When n is an odd number.

Using Theorem 3.1, we have $n+1$ self-inverse element vertices adjacent with all vertices other than itself and remaining not self-inverse element vertices adjacent with all vertices other than itself and its inverse in $\Gamma_{cpg}(D_n)$. Thus, we need at least $(n + 1)$ different colors for self-inverse element vertices and $\left(\dfrac{n-1}{2}\right)$ different colors for each pair of not self-inverse element vertices for proper coloring of $\Gamma_{cpg}(D_n)$. Thus, $\chi(\Gamma_{cpg}(D_n)) = (n+1) + \left(\dfrac{n-1}{2}\right) = \dfrac{3n+1}{2}$.

Case 2. When n is an even number.

Using Theorem 3.1, we have $n+2$ self-inverse element vertices adjacent with all vertices other than itself and remaining not self-inverse element vertices adjacent with all vertices other than itself and its inverse in $\Gamma_{cpg}(D_n)$. Thus, we need at least $(n+2)$ different colors for self-inverse element vertices and $\left(\dfrac{n-2}{2}\right)$ different colors for each pair of not self-inverse element vertices for proper coloring of $\Gamma_{cpg}(D_n)$. Thus, $\chi(\Gamma_{cpg}(D_n)) = (n+2) + \left(\dfrac{n-2}{2}\right) = \dfrac{3n+2}{2}$.

Corollary 3.1. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and $\gcd(3, n) = 1$. Then, $\Gamma_{cpg}(D_n)$ is weakly perfect.

Proof. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and $\gcd(3, n) = 1$. Then, by using Theorem 3.5 and Theorem 3.6, we have $\omega(\Gamma_{cpg}(D_n)) = \chi(\Gamma_{cpg}(D_n))$. Thus, $\Gamma_{cpg}(D_n)$ is weakly perfect.

Theorem 3.7. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and $\gcd(3, n) = 1$, then independent number

$$\beta(\Gamma_{cpg}(D_n)) = \begin{cases} 1 \text{ if } n \in \{1, 2\}, \\ 2 \text{ otherwise.} \end{cases}$$

Proof. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and $\gcd(3, n) = 1$.

Case 1. When $n \in \{1, 2\}$.

Using Theorem 3.1, we have $\Gamma_{cpg}(D_1) = K_2$ and $\Gamma_{cpg}(D_2) = K_4$. Thus, we have only one vertex in maximum independent set of vertices. Hence, $\beta(\Gamma_{cpg}(D_n)) = 1$ if $n \in \{1, 2\}$.

Case 2. When $n \in N \setminus \{1, 2\}$.

Using Theorem 3.1, we have only vertices, which are not adjacent with each other are rotation element vertices r^i, r^{n-i} for which $r^i \neq r^{n-i}$. So, any pair of rotation element vertices r^i, r^{n-i} for which $r^i \neq r^{n-i}$ forms the maximum independent set of vertices. Hence, $\beta(\Gamma_{cpg}(D_n)) = 2$ if $n \in N \setminus \{1, 2\}$ and $\gcd(3, n) = 1$.

Theorem 3.8. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n, then matching number, $\mu(\Gamma_{cpg}(D_n)) = n$.

Proof. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n. So, the maximum possible matching number of $\Gamma_{cpg}(D_n)$ is n.

Case 1. When n is an odd number and $\gcd(3, n) = 1$.

Using Theorem 3.1(i), we have $\Gamma_{cpg}(D_n) = \overline{\left[(n+1)K_1 \cup \left(\dfrac{n-1}{2}\right)K_2\right]}$. Thus, we have n edges from $\Gamma_{cpg}(D_n)$ forms maximum set of edges such that no pair of edges have common vertex. For example, one of the maximum set of edges such that no pair of edges have common vertex is set containing $n - 1$ edges with one end vertex self-inverse element vertex, and another end vertex not-self inverse element vertex and one edge with both end self-inverse vertices (which are not used in initial $n - 1$ edges). Thus, $\mu(\Gamma_{cpg}(D_n)) = n$ when n is an odd number and $\gcd(3, n) = 1$.

Case 2. When n is an even number and $\gcd(3, n) = 1$.

Using Theorem 3.1(i), we have $\Gamma_{cpg}\left(D_n\right) = \overline{\left[\left(n+2\right)K_1 \cup \left(\frac{n-2}{2}\right)K_2\right]}$. Thus, we have n edges from Γ_{cpg} (D_n) forms maximum set of edges such that no pair of edges have common vertex. For example, one of the maximum set of edges such that no pair of edges have common vertex is set containing $n - 2$ edges with one end vertex self-inverse element vertex, and another end vertex not-self inverse element vertex and two edges with both end self-inverse vertices (which are not used in initial $n - 2$ edges). Thus, μ $(\Gamma_{cpg}$ $(D_n)) = n$ when n is an even number and gcd(3, n) = 1.

Hence, the required result.

Case 3. When gcd(3, n) $\neq$ 1.

Using Theorem 3.2(i), we have all the n rotation element vertices adjacent with all n reflection element vertices in Γ_{cpg} (D_n). So, there are n^2 edges having one end reflection element vertex, and another end with rotation element vertex. So, we have $K_{n,n}$ as a subgraph of Γ_{cpg} (D_n). We have $\mu(\Gamma_{cpg}$ $K_{n,n})) = n$. As maximum possible matching number of Γ_{cpg} (D_n) is n, and Γ_{cpg} (D_n) has a subgraph with matching number n. Hence, $\mu(\Gamma_{cpg}$ $(D_n)) = n$ when gcd(3, n) $\neq$ 1.

Corollary 3.2. Let Γ_{cpg} (D_n) be a cubic power graph of D_n, then Γ_{cpg} (D_n) has perfect matching.

Proof. Let Γ_{cpg} (D_n) be the cubic power graph of D_n. Then, by using Theorem 3.8, we have $\mu(\Gamma_{cpg}$ $(D_n)) = n = \frac{|D_n|}{2}$. Hence, Γ_{cpg} (D_n) has perfect matching.

Theorem 3.9. Let Γ_{cpg} (D_n) be the cubic power graph of D_n. Then, the dominating number,

$$\gamma\left(D_n\right) = \begin{cases} 1 \text{ if } \gcd(3, n) = 1, \\ 2 \text{ if } \gcd(3, n) = 3. \end{cases}$$

Proof. Let Γ_{cpg} (D_n) be the cubic power graph of D_n.

Case 1. When gcd(3, n) = 1.

Using Theorem 3.1, we have e identity element vertex adjacent with all other vertices in Γ_{cpg} (D_n). Thus, we have a minimal dominating set with cardinality 1. Hence, γ $(D_n) = 1$.

Case 2. When gcd(3, n) = 3.

Using Theorem 3.2, we have every rotation element vertex adjacent with every reflection element vertex. So, a minimum dominating set contains two vertices one rotation element vertex and another reflection element vertex. Hence, γ $(D_n) = 2$.

4. Topological indices of Γ_{cpg} (D_n)

Theorem 4.1. Let Γ_{cpg} (D_n) be the cubic power graph of D_n and gcd(3, n) = 1. Then, Wiener index,

$$W(\Gamma_{cpg}(D_n)) = \begin{cases} \dfrac{4n^2 - n - 1}{2} \text{when } n \text{ is an odd number,} \\ \dfrac{4n^2 - n - 2}{2} \text{when } n \text{ is an even number.} \end{cases}$$

Proof. Let Γ_{cpg} (D_n) be the cubic power graph of D_n and gcd(3, n) = 1.

The Wiener index [30] of Γ_{cpg} (D_n) is $W(\Gamma_{cpg}(D_n)) = \sum_{\{x,y\} \subset V(\Gamma_{cpg}(D_n))}, d(x, y)$. Using Theorem 3.1(iii), we have $d(x, y) = 1$ when $x^{-1} \neq y$ and $d(x, y) = 2$ when $x^{-1} = y$.

Case 1. When n is an odd number.

We have $\frac{n \quad 1}{2}$ number of vertices x, y pairs in this case such that $x^{-1} = y$ and so number of vertices x, y pairs with $x^{-1} \neq y$ are $n\left(2n-1\right) - \left(\frac{n-1}{2}\right) = \frac{4n^2 - 3n + 1}{2}$. Thus, $W(\Gamma_{cpg}(D_n)) = 1 \times \left(\frac{4n^2 - 3n + 1}{2}\right) + 2 \times \left(\frac{n-1}{2}\right) = \frac{4n^2 - n - 1}{2}$.

Case 2. When n is an even number.

We have $\frac{n-2}{2}$ number of vertices x, y pairs in this case such that $x^{-1} = y$ and so number of vertices x, y pairs such that $x^{-1} \neq y$ are $n(2n-1) - \left(\frac{n-2}{2}\right) = \frac{4n^2-3n+2}{2}$. Thus, $W\left(\Gamma_{cpg}\left(D_n\right)\right) = 1 \times \left(\frac{4n^2-3n+2}{2}\right) + 2 \times \left(\frac{n-2}{2}\right) = \frac{4n^2-n-2}{2}$.

Hence, the required result.

Theorem 4.2. Let Γ_{cpg} (D_n) be the cubic power graph of D_n and gcd(3, n) = 1.Then, the hyper-Wiener index,

$$WW\left(\Gamma_{cpg}\left(D_n\right)\right) = \begin{cases} 2n^2 - 1 \text{ if } n \text{ is an odd number,} \\ 2n^2 - 2 \text{ if } n \text{ is an even number.} \end{cases}$$

Proof. Let Γ_{cpg} (D_n) be the cubic power graph of D_n and gcd(3, n) = 1. The hyper-Wiener index [31] of Γ_{cpg} (D_n) is

$$WW(\Gamma_{cpg}(D_n)) = \frac{1}{2}W(\Gamma_{cpg}(D_n)) + \frac{1}{2}\sum_{\{x,y\}\}\subset V(\Gamma_{cpg}(D_n))} d^2(x,y)$$

Case 1. When n is an odd number.
Using Theorem 3.1 and Theorem 4.1, we have

$$WW\left(\Gamma_{cpg}\left(D_n\right)\right) = \frac{1}{2}W\left(\Gamma_{cpg}\left(D_n\right)\right) + \frac{1}{2}\sum_{\{x,y\}\subset V\left(\Gamma_{cpg}\left(D_n\right)\right)}$$
$$d^2\left(x,y\right) = \frac{1}{2} \times \frac{4n^2-n-1}{2} + \frac{1}{2} \times \frac{4n^2+n-3}{2} = 2n^2 - 1$$

Case 2. When n is an even number.
Using Theorem 3.1 and Theorem 4.1, we have

$$WW\left(\Gamma_{cpg}\left(D_n\right)\right) = \frac{1}{2}W\left(\Gamma_{cpg}\left(D_n\right)\right) + \frac{1}{2}\sum_{\{x,y\}\subset V\left(\Gamma_{cpg}\left(D_n\right)\right)}$$
$$d^2\left(x,y\right) = \frac{1}{2} \times \frac{4n^2-n-2}{2} + \frac{1}{2} \times \frac{4n^2+n-6}{2} = 2n^2 - 2$$

Theorem 4.3. Let Γ_{cpg} (D_n) be the cubic power graph of D_n and gcd(3, n) = 1. Then, the first Zagreb index,

$$M_1\left(\Gamma_{cpg}\left(D_n\right)\right) = \begin{cases} 8n^3 - 12n^2 + 9n - 3 \text{ when } n \text{ is an odd number,} \\ 8n^3 - 12n^2 + 13n - 6 \text{ when } n \text{ is an even number.} \end{cases}$$

Proof. Let Γ_{cpg} (D_n) be the cubic power graph of D_n and gcd(3, n) = 1.

$M_1\left(\Gamma_{cpg}\left(D_n\right)\right) = \sum_{x \in V(\Gamma_{cpg}(D_n))} (\deg(x))^2$ is the first Zagreb index [29] of Γ_{cpg} (D_n).

Case 1. When n is an odd number.

Using Theorem 3.1(ii), we have $n - 1$ vertices with degree $2n - 2$ and $n + 1$ vertices with degree $2n - 1$. Thus, $M_1(\Gamma_{cpg}(D_n)) = (n-1) \times (2n-2)^2 + \left(n+1\right)(2n-1)^2 = 8n^3 - 12n^2 + 9n - 3$.

Case 2. When n is an even number.

Using Theorem 3.1(ii), we have $n - 2$ vertices with degree $2n - 2$ and $n + 2$ vertices with degree $2n - 1$. Thus, $M_1(\Gamma_{cpg}(D_n)) = (n-2) \times (2n-2)^2 + (n+2)(2n-1)^2 = 8n^3 - 12n^2 + 13n - 6$.

Theorem 4.4. Let Γ_{cpg} (D_n) be the cubic power graph of D_n and gcd(3, n) = 1. Then, the second Zagreb index,

$$M_2(\Gamma_{cpg}(D_n)) = \begin{cases} \dfrac{16n^4 - 36n^3 + 41n^2 - 27n + 8}{2} & \text{when } n \text{ is an odd number,} \\ \dfrac{16n^4 - 36n^3 + 53n^2 - 45n + 18}{2} & \text{when } n \text{ is an even number.} \end{cases}$$

Proof. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and gcd(3, n) = 1.

$M_2(\Gamma_{cpg}(D_n)) = \sum_{xy \in E(\Gamma_{cpg}(D_n))} [\deg(x) \times \deg(y)]$ is the second Zagreb index [29] of $\Gamma_{cpg}(D_n)$.

Case 1. When n is an odd number.

Using Theorem 3.1, we have $\dfrac{(n-1)(n-3)}{2}$ edges with vertices on both ends having degree $(2n-2)$, $\dfrac{n^2+n}{2}$ edges with vertices on both ends having degree $(2n-1)$ and (n^2-1) edges with vertex on one end with degree $(2n-1)$, and vertex on another end with $(2n-2)$ degree. Thus,

$$\begin{aligned} M_2(\Gamma_{cpg}(D_n)) &= (2n-2)(2n-2)\left(\frac{(n-1)(n-3)}{2}\right) + (2n-1)(2n-1)\left(\frac{n^2+n}{2}\right) + (2n-1)(2n-2)(n^2-1) \\ &= \frac{16n^4 - 36n^3 + 41n^2 - 27n + 8}{2} \end{aligned}$$

Case 2. When n is an even number.

Using Theorem 3.1, we have $\left(\dfrac{(n-2)(n-4)}{2}\right)$ edges with vertices on both ends having degree $(2n-2)$, $\left(\dfrac{(n+1)(n+2)}{2}\right)$ edges with vertices on both ends having degree $(2n-1)$ and (n^2-4) edges with vertex on one end with degree $(2n-1)$, and vertex on another end with $(2n-2)$ degree. Thus,

$$\begin{aligned} M_2(\Gamma_{cpg}(D_n)) &= (2n-2)^2\left(\frac{(n-2)(n-4)}{2}\right) + (2n-1)^2\left(\frac{(n+2)(n+1)}{2}\right) + (2n-1)(2n-2)(n^2-4) \\ &= \frac{16n^4 - 36n^3 + 53n^2 - 45n + 18}{2} \end{aligned}$$

Theorem 4.5. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and gcd(3, n) = 1. Then, the Harary index,

$$\mathcal{H}\left(\Gamma_{cpg}(D_n)\right) = \begin{cases} \dfrac{8n^2 - 5n + 1}{4} & \text{when } n \text{ is an odd number,} \\ \dfrac{8n^2 - 5n + 2}{4} & \text{when } n \text{ is an even number.} \end{cases}$$

Proof. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and gcd(3, n) = 1.

$\mathcal{H}(\Gamma_{cpg}(D_n)) = \sum_{\{x,y\} \subset V(\Gamma_{cpg}(D_n))} \dfrac{1}{d(x,y)}$ is the Harary index [28] of $\Gamma_{cpg}(D_n)$.

Case 1. When n is an odd number.

Using Theorem 3.1, we have $\dfrac{n-1}{2}$ pairs x, y vertices having $d(x, y) = 2$, and $\dfrac{4n^2-3n+1}{2}$ x, y vertices pairs having $d(x, y) = 1$ in $\Gamma_{cpg}(D_n)$. Thus, $\mathcal{H}\left(\Gamma_{cpg}(D_n)\right) = \dfrac{1}{1} \times \left(\dfrac{4n^2-3n+1}{2}\right) + \dfrac{1}{2} \times \left(\dfrac{n-1}{2}\right) = \dfrac{8n^2-5n+1}{4}$.

Case 2. When n is an even number.

Using Theorem 3.1, we have $\frac{n-2}{2}$ pairs of x, y vertices having $d(x, y) = 2$ and $\frac{4n^2-3n+2}{2}$ x, y vertices pairs having $d(x, y) = 1$ in $\Gamma_{cpg}(D_n)$. Thus, $\mathcal{H}\left(\Gamma_{cpg}\left(D_n\right)\right) = \frac{1}{1}\times\left(\frac{4n^2-3n+2}{2}\right)+\frac{1}{2}\times\left(\frac{n-2}{2}\right)=\frac{8n^2-5n+2}{4}$.

Theorem 4.6. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and $\gcd(3, n) = 1$. Then, the Schultz index,

$$MTI\left(\Gamma_{cpg}\left(D_n\right)\right)=\begin{cases}8n^3-8n^2+n+1 \text{ when } n \text{ is an odd number,}\\ 8n^3-8n^2+n+2 \text{ when } n \text{ is an even number.}\end{cases}$$

Proof. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and $\gcd(3, n) = 1$.
$MTI(\Gamma_{cpg}(D_n)) = \sum_{\{x,y\}\subset V(\Gamma_{cpg}(D_n))} d(x, y)\left[\deg(x)+\deg(y)\right]$ is the Schultz index [27] of $\Gamma_{cpg}(D_n)$.

Case 1. When n is an odd number.

Using Theorem 3.1, we have $\frac{n(n+1)}{2}$ x, y vertices pairs having both vertices degree $(2n - 1)$ and $d(x, y) = 1$, $\frac{(n-1)(n-3)}{2}$ x, y vertices pairs having both vertices degree $(2n - 2)$ and $d(x, y) = 1$, $\frac{n-1}{2}$ pairs of vertices x, y with both vertices degree $(2n - 2)$, and $d(x, y) = 2$ and $(n^2 - 1)$ pairs of vertices with one vertex degree $(2n - 1)$, and another vertex $(2n - 2)$ with $d(x, y) = 1$ in $\Gamma_{cpg}(D_n)$. Thus,

$$\begin{aligned}MTI(\Gamma_{cpg}(D_n)) &= 1\times\left\{\left(2n-1\right)+\left(2n-1\right)\right\}\times\frac{n\left(n+1\right)}{2}+1\\ &\times\left\{\left(2n-2\right)+\left(2n-2\right)\right\}\times\frac{\left(n-1\right)\left(n-3\right)}{2}+2\times\left\{\left(2n-2\right)+\left(2n-2\right)\right\}\\ &\times\frac{\left(n-1\right)}{2}+1\times\left\{\left(2n-1\right)+\left(2n-2\right)\right\}\times\left(n^2-1\right)\\ &= 8n^3-8n^2+n+1\end{aligned}$$

Case 2. When n is an even number.

Using Theorem 3.1, we have $\frac{(n+1)(n+2)}{2}$ x, y vertices pairs having both vertices degree $(2n - 1)$ and $d(x, y) = 1$, $\frac{(n-2)(n-4)}{2}$ pairs of vertices x, y having both vertices with degree $(2n - 2)$, and $d(x, y) = 1$, $\frac{n-2}{2}$ pairs of vertices x, y with both vertices degree $(2n - 2)$, and $d(x, y) = 2$ and $(n^2 - 4)$ pairs of vertices x, y with one vertex degree $(2n - 1)$, and another vertex $(2n - 2)$ with $d(x, y) = 1$ in $\Gamma_{cpg}(D_n)$. Thus,

$$\begin{aligned}MTI\left(\Gamma_{cpg}\left(D_n\right)\right) &= 1\times\left\{\left(2n-1\right)+\left(2n-1\right)\right\}\times\frac{\left(n+1\right)\left(n+2\right)}{2}+1\\ &\times\left\{\left(2n-2\right)+\left(2n-2\right)\right\}\times\frac{\left(n-2\right)\left(n-4\right)}{2}+2\times\left\{\left(2n-2\right)+\left(2n-2\right)\right\}\\ &\times\frac{(n-2)}{2}+1\times\left\{(2n-1)+(2n-2)\right\}\times(n^2-4)\\ &= 8n^3-8n^2+n+2\end{aligned}$$

Theorem 4.7. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and $\gcd(3, n) = 1$. Then, the Gutman index,

$$Gut\left(\Gamma_{cpg}\left(D_n\right)\right)=\begin{cases}\dfrac{16n^4-28n^3+17n^2-3n}{2} & \text{when } n \text{ is an odd number,}\\ \dfrac{16n^4-28n^3+21n^2-5n+2}{2} & \text{when } n \text{ is an even number}\end{cases}$$

Proof. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and $\gcd(3, n) = 1$.

$Gut(\Gamma_{cpg}(D_n)) = \sum_{\{x,y\}\subset V(\Gamma_{cpg}(D_n))} d(x, y)[\deg(x)\times\deg(y)]$ is the Gutman index [22] of $\Gamma_{cpg}(D_n)$.

Case 1. When n is an odd number.

Using Theorem 3.1, we have $\frac{(n+1)\,n}{2}$ x, y vertices pair having both vertices degree $(2n - 1)$ and $d(x, y) = 1$, $\frac{(n-1)(n-3)}{2}$ pairs of vertices x, y with both vertices degree $(2n - 2)$ and $d(x, y) = 1$, $\frac{n-1}{2}$ pairs of vertices x, y with both vertices degree $(2n - 2)$, and $d(x, y) = 2$ and $(n^2 - 1)$ pairs of vertices x, y with one vertex having degree $(2n - 1)$, and another vertex having $(2n - 2)$ with $d(x, y) = 1$ in $\Gamma_{cpg}(D_n)$. Thus,

$$\begin{aligned}Gut(\Gamma_{cpg}(D_n)) &= 1\times\{(2n-1)(2n-1)\}\times\frac{n(n+1)}{2}+1\\ &\times\{(2n-2)(2n-2)\}\times\frac{(n-1)(n-3)}{2}+2\times\{(2n-2)(2n-2)\}\\ &\times\frac{(n-1)}{2}+1\times\{(2n-1)(2n-2)\}\times(n^2-1)\\ &=\frac{16n^4-28n^3+17n^2-3n}{2}\end{aligned}$$

Case 2. When n is an even number.

Through Theorem 3.1, we have $\frac{(n+1)(n+2)}{2}$ pairs of vertices x, y having both vertices degree $(2n - 1)$ and $d(x, y) = 1$, $\frac{(n-2)(n-4)}{2}$ pairs of vertices x, y having both vertices degree $(2n - 2)$ and $d(x, y) = 1$, $\frac{n-2}{2}$ pairs of vertices x, y having both vertices degree $(2n - 2)$ with $d(x, y) = 2$, and $(n^2 - 4)$ pairs of vertices x, y with one vertex having degree $(2n - 1)$, and another vertex having $(2n - 2)$ with $d(x, y) = 1$ in $\Gamma_{cpg}(D_n)$. Thus,

$$\begin{aligned}Gut(\Gamma_{cpg}(D_n)) &= 1\times\{(2n-1)(2n-1)\}\times\frac{(n+1)(n+2)}{2}+1\\ &\times\{(2n-2)(2n-2)\}\times\frac{(n-2)(n-4)}{2}+2\times\{(2n-2)(2n-2)\\ &\times\frac{(n-2)}{2}+1\times\{(2n-1)(2n-2)\}\times(n^2-4)\\ &=\frac{16n^4-28n^3+21n^2-5n+2}{2}\end{aligned}$$

Theorem 4.8. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and $\gcd(3, n) = 1$. Then, the harmonic index,

$$\mathcal{H}_r\left(\Gamma_{cpg}\left(D_n\right)\right)=\begin{cases}\dfrac{32n^3-40n^2+11n-1}{32n^2-40n+12} & \text{when } n \text{ is an odd number,}\\ \dfrac{32n^4-72n^3+51n^2-12n+4}{32n^3-72n^2+52n-12} & \text{when } n \text{ an is even number.}\end{cases}$$

Proof. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and gcd(3, n) = 1.

$\mathcal{H}_r(\Gamma_{cpg}(D_n)) = \sum_{xy\in E(\Gamma_{cpg}(D_n))} \frac{2}{\deg(x)+\deg(y)}$ is the harmonic index [25] of $\Gamma_{cpg}(D_n)$.

Case 1. When n is an odd number.

Using Theorem 3.1, we have $\frac{n(n+1)}{2}$ edges having both end vertices degree $(2n-1)$, $\frac{(n-1)(n-3)}{2}$ edges having vertices on both ends with degree $(2n-2)$ and (n^2-1) edges having one end vertex with degree $(2n-1)$, and another end vertex having degree $(2n-2)$. Thus,

$$\begin{aligned}\mathcal{H}_r(\Gamma_{cpg}(D_n)) &= \frac{n(n+1)}{2}\times\frac{2}{(2n-1)+(2n-1)}+\frac{(n-1)(n-3)}{2}\\ &\times\frac{2}{(2n-2)+(2n-2)}+(n^2-1)\times\frac{2}{(2n-2)+(2n-1)}\\ &=\frac{32n^3-40n^2+11n-1}{32n^2-40n+12}\end{aligned}$$

Case 2. When n is an even number.

Using Theorem 3.1, we have $\frac{(n+1)(n+2)}{2}$ edges having both end vertices degree $(2n-1)$, $\frac{(n-2)(n-4)}{2}$ edges having both end vertices degree $(2n-2)$, and (n^2-4) edges having vertex on one end with degree $(2n-1)$, and vertex on another end with degree $(2n-2)$. Thus,

$$\begin{aligned}\mathcal{H}_r(\Gamma_{cpg}(D_n)) &= \frac{(n+1)(n+2)}{2}\times\frac{2}{(2n-1)+(2n-1)}+\frac{(n-2)(n-4)}{2}\\ &\times\frac{2}{(2n-2)+(2n-2)}+(n^2-4)\times\frac{2}{(2n-2)+(2n-1)}\\ &=\frac{32n^4-72n^3+51n^2-12n+4}{32n^3-72n^2+52n-12}\end{aligned}$$

Theorem 4.9. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and gcd(3, n) = 1. Then,

(i) general Randic index,

$$R_\alpha(\Gamma_{cpg}(D_n)) = \begin{cases}\frac{n(n+1)(2n-1)^{2\alpha}+(n-1)(n-3)(2n-2)^{2\alpha}+2(n^2-1)\{(2n-1)(2n-2)\}^{\alpha}}{2} & \text{when } n \text{ is an odd number,}\\ \frac{(n+1)(n+2)(2n-1)^{2\alpha}+(n-2)(n-4)(2n-2)^{2\alpha}+2(n^2-4)\{(2n-2)(2n-1)\}^{\alpha}}{2} & \text{when } n \text{ is an even number.}\end{cases}$$

(ii) Randic index,

$$R_{-\frac{1}{2}} = \begin{cases}\frac{n(n+1)}{2(2n-1)}+\frac{(n-1)(n-3)}{2(2n-2)}+\frac{n^2-1}{\sqrt{(2n-2)(2n-1)}} & \text{when } n \text{ is an odd number,}\\ \frac{(n+1)(n+2)}{2(2n-1)}+\frac{(n-2)(n-4)}{2(2n-2)}+\frac{(n^2-4)}{\sqrt{(2n-1)(2n-2)}} & \text{when } n \text{ is an even number.}\end{cases}$$

Proof. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and gcd(3, n) = 1.

$R_{\alpha(\Gamma_{cpg}(D_n))} = \sum_{xy\in E(\Gamma_{cpg}(D_n))}\{\deg(x)\deg(y)\}^{\alpha}$ is the general Randic index [23], and $R_{-\frac{1}{2}}(\Gamma_{cpg}(D_n)) = \sum_{xy\in E(\Gamma_{cpg}(D_n))}$ $\frac{1}{\sqrt{\deg(x)\deg(y)}}$ is the Randic index [24] of $\Gamma_{cpg}(D_n)$.

(i) Case 1. When n is an odd number.

Using Theorem 3.1, we have $\frac{(n+1)\,n}{2}$ edges having vertices on both ends with degree $(2n-1)$, $\frac{(n-1)(n-3)}{2}$ edges having vertices on both ends with degree $(2n-2)$, and (n^2-1) edges having vertex on one end with degree $(2n-1)$, and vertex on another end with degree $(2n-2)$. Thus,

$$\begin{aligned}R_\alpha(\Gamma_{cpg}(D_n)) &= \frac{n(n+1)}{2}\{(2n-1)(2n-1)\}^{\alpha} + \frac{(n-1)(n-3)}{2}\{(2n-2)(2n-2)\}^{\alpha} + (n^2-1)\{(2n-2)(2n-1)\}^{\alpha}\\ &= \frac{n(n+1)(2n-1)^{2\alpha} + (n-1)(n-3)(2n-2)^{2\alpha} + 2(n^2-1)\{(2n-1)(2n-2)\}^{\alpha}}{2}.\end{aligned}$$

Case 2. When n is an even number.

Using Theorem 3.1, we have $\frac{(n+1)(n+2)}{2}$ edges having vertices on both ends with degree $(2n-1)$, $\frac{(n-2)(n-4)}{2}$ edges having vertices on both ends with degree $(2n-2)$, and (n^2-4) edges having vertex on one end with degree $(2n-1)$, and vertex on another end with degree $(2n-2)$. Thus,

$$\begin{aligned}R_\alpha(\Gamma_{cpg}(D_n)) &= \frac{(n+1)(n+2)}{2}\{(2n-1)(2n-1)\}^{\alpha} + \frac{(n-2)(n-4)}{2}\{(2n-2)(2n-2)\}^{\alpha} + (n^2-4)\{(2n-2)(2n-1)\}^{\alpha}\\ &= \frac{(n+1)(n+2)(2n-1)^{2\alpha} + (n-2)(n-4)(2n-2)^{2\alpha} + 2(n^2-4)\{(2n-2)(2n-1)\}^{\alpha}}{2}.\end{aligned}$$

(ii) Putting $\alpha = -\frac{1}{2}$ in Theorem 4.9(i), we get $R_{-\frac{1}{2}} = \frac{n(n+1)}{2(2n-1)} + \frac{(n-1)(n-3)}{2(2n-2)} + \frac{n^2-1}{\sqrt{(2n-2)(2n-1)}}$ when n is an odd number and $R_{-\frac{1}{2}} = \frac{(n+1)(n+2)}{2(2n-1)} + \frac{(n-2)(n-4)}{2(2n-2)} + \frac{(n^2-4)}{\sqrt{(2n-1)(2n-2)}}$ when n is an even number.

Theorem 4.10. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and $\gcd(3, n) = 1$. Then, the atomic-bond connectivity index,

$$ABC(\Gamma_{cpg}(D_n)) = \begin{cases} \dfrac{n(n+1)\sqrt{(n-1)}}{2n-1} + \dfrac{(n-1)(n-3)\sqrt{4n-6}}{2(2n-2)} + \dfrac{(n^2-1)\sqrt{(4n-5)}}{\sqrt{(2n-2)(2n-1)}} & \text{when } n \text{ is an odd number,}\\[2ex] \dfrac{(n+2)(n+1)\sqrt{(n-1)}}{2n-1} + \dfrac{(n-2)(n-4)\sqrt{(4n-6)}}{2(2n-2)} + \dfrac{(n^2-4)\sqrt{(4n-5)}}{\sqrt{(2n-2)(2n-1)}} & \text{when } n \text{ is an even number.}\end{cases}$$

Proof. Let $\Gamma_{cpg}(D_n)$ be the cubic power graph of D_n and $\gcd(3, n) = 1$. The atomic-bond connectivity index [21] of $\Gamma_{cpg}(D_n)$ is $ABC(\Gamma_{cpg}(D_n)) = \sum_{xy\in E(\Gamma_{cpg}(D_n))}\sqrt{\frac{\deg(x)+\deg(y)-2}{\deg(x)\deg(y)}}$.

Case 1. When n is an odd number.

Using Theorem 3.1, we have $\frac{n(n+1)}{2}$ edges having vertices on both ends with degree $(2n-1)$, $\frac{(n-1)(n-3)}{2}$ edges having vertices on both ends with degree $(2n-2)$ and (n^2-1) edges having vertex on one end with degree $(2n-1)$, and vertex on another end with degree ($2n-2$). Thus,

$$ABC(\Gamma_{cpg}(D_n)) = \frac{n(n+1)}{2}\times\sqrt{\frac{4n-4}{(2n-1)(2n-1)}}+\frac{(n-1)(n-3)}{2}\times\sqrt{\frac{4n-6}{(2n-2)(2n-2)}}+(n^2-1)\sqrt{\frac{4n-5}{(2n-2)(2n-1)}}$$
$$= \frac{n(n+1)\sqrt{(n-1)}}{2n-1}+\frac{(n-1)(n-3)\sqrt{4n-6}}{2(2n-2)}+\frac{(n^2-1)\sqrt{(4n-5)}}{\sqrt{(2n-2)(2n-1)}}.$$

Case 2. When n is an even number.

Using Theorem 3.1, we have $\frac{(n+1)(n+2)}{2}$ edges with both end vertices degree ($2n-1$), $\frac{(n-2)(n-4)}{2}$ edges with both end vertices degree ($2n-2$) and (n^2-4) edges with one end vertex having degree ($2n-1$), and another end vertex having degree ($2n-2$). Thus,

$$ABC(\Gamma_{cpg}(D_n)) = \frac{(n+1)(n+2)}{2}\times\sqrt{\frac{4n-4}{(2n-1)(2n-1)}}+\frac{(n-2)(n-4)}{2}\sqrt{\frac{4n-6}{(2n-2)(2n-2)}}+(n^2-4)\sqrt{\frac{4n-5}{(2n-1)(2n-2)}}$$
$$= \frac{(n+1)(n+2)\sqrt{(n-1)}}{2n-1}+\frac{(n-2)(n-4)\sqrt{(4n-6)}}{2(2n-2)}+\frac{(n^2-4)\sqrt{(4n-5)}}{\sqrt{(2n-2)(2n-1)}}.$$

Theorem 4.11. Let Γ_{cpg} (D_n) be the cubic power graph of D_n and gcd(3, n) = 1. Then, the geometric-arithmetic index,

$$GA(\Gamma_{cpg}(D_n)) = \begin{cases} \frac{n(2n-1)(n+1)}{2(2n-1)}+\frac{(n-1)(n-3)}{2}+\frac{2(n^2-1)\sqrt{(2n-1)(2n-2)}}{(4n-3)} & \text{when } n \text{ is an odd number,} \\ \frac{(2n-1)(n+1)(n+2)}{2(2n-1)}+\frac{(n-2)(n-4)}{2}+\frac{2(n^2-4)\sqrt{(2n-1)(2n-2)}}{(4n-3)} & \text{when } n \text{ is an even number.} \end{cases}$$

Proof. Let Γ_{cpg} (D_n) be the cubic power graph of D_n and gcd(3, n) = 1.

$GA(\Gamma_{cpg}(D_n)) = \sum_{xy\in E(\Gamma_{cpg}(D_n))}\frac{2\sqrt{\deg(x)\times\deg(y)}}{\deg(x)+\deg(y)}$ is the geometric-arithmetic index [20] of Γ_{cpg} (D_n).

Case 1. When n is an odd number.

Using Theorem 3.1, we have $\frac{n(n+1)}{2}$ edges having vertices on both end with degree ($2n-1$), $\frac{(n-1)(n-3)}{2}$ edges having vertices on both end with degree ($2n-2$), and (n^2-1) edges having one end vertex with degree ($2n-1$), and vertex on another end having degree ($2n-2$). Thus,

$$GA(\Gamma_{cpg}(D_n)) = \frac{2\sqrt{(2n-1)(2n-1)}}{(2n-1)+(2n-1)}\times\frac{n(n+1)}{2}+\frac{2\sqrt{(2n-2)(2n-2)}}{(2n-2)+(2n-2)}\times\frac{(n-1)(n-3)}{2}+\frac{2\sqrt{(2n-1)(2n-2)}}{(2n-2)+(2n-1)}\times(n^2-1)$$
$$= \frac{n(2n-1)(n+1)}{2(2n-1)}+\frac{(n-1)(n-3)}{2}+\frac{2(n^2-1)\sqrt{(2n-1)(2n-2)}}{(4n-3)}.$$

Case 2. When n is an even number.

Using Theorem 3.1, we have $\frac{(n+1)(n+2)}{2}$ edges having vertices on both end with degree ($2n-1$), $\frac{(n-2)(n-4)}{2}$ edges having vertices on both end with degree ($2n-2$), and (n^2-4) edges having vertex on one end with degree ($2n-1$), and vertex on another end having degree ($2n-2$). Thus,

$$GA(\Gamma_{cpg}(D_n)) = \frac{2\sqrt{(2n-1)(2n-1)}}{(2n-1)+(2n-1)} \times \frac{(n+1)(n+2)}{2} + \frac{2\sqrt{(2n-2)(2n-2)}}{(2n-2)+(2n-2)} \times \frac{(n-2)(n-4)}{2} + \frac{2\sqrt{(2n-1)(2n-2)}}{(2n-2)+(2n-1)} \times (n^2-4)$$
$$= \frac{(2n-1)(n+1)(n+2)}{2(2n-1)} + \frac{(n-2)(n-4)}{2} + \frac{2(n^2-4)\sqrt{(2n-1)(2n-2)}}{(4n-3)}.$$

Theorem 4.12. Let Γ_{cpg} (D_n) be the cubic power graph of D_n and gcd(3, n) = 1. Then, the eccentric connectivity index,

$$\xi(\Gamma_{cpg}(D_n)) = \begin{cases} 6n^2 - 7n + 3 \text{ when } n \text{ is an odd number,} \\ 6n^2 - 9n + 6 \text{ when } n \text{ is an even number.} \end{cases}$$

Proof. Let Γ_{cpg} (D_n) be the cubic power graph of D_n and gcd(3, n) = 1. The eccentric connectivity index [26] of Γ_{cpg} (D_n), is $(\Gamma_{cpg}(D_n)) = \sum_{x \in V(\Gamma_{cpg}(D_n))} ecc(x)\deg(x)$.

Case 1. When n is an odd number.

Using Theorem 3.1, we have $n + 1$ vertices with degree $2n - 1$, and eccentricity 1 and $n - 1$ vertices with degree $2n - 2$ and eccentricity 2. Thus,

$$\xi(\Gamma_{cpg}(D_n)) = 1 \times (n+1) \times (2n-1) + 2 \times (n-1) \times (2n-2) = 6n^2 - 7n + 3.$$

Case 2. When n is an even number.

Using Theorem 3.1, we have $n + 2$ vertices with degree $2n - 1$, and eccentricity 1 and $n - 2$ vertices with degree $2n - 2$ and eccentricity 2. Thus,

$$\xi(\Gamma_{cpg}(D_n)) = 1 \times (n+2) \times (2n-1) + 2 \times (n-2) \times (2n-2) = 6n^2 - 9n + 6.$$

5. Conclusion

We conclude that the cubic power graph of the dihedral group is always a connected graph and complete only when the order of the dihedral group is two or four. We have also given structural representation, vertex degree, girth, clique number, chromatic number, independent number, matching and dominating number of cubic power graph of dihedral group, and studied its various topological indices.

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Conflict of interest

The authors declare no conflict of interest.

References

[1] Takshak N, Sehgal A, Malik A. Power graph of a finite group is always divisor graph. *Asian-European Journal of Mathematics*. 2023; 16(1): 2250236. Available from: https://doi.org/10.1142/S1793557122502369.

[2] Cameron PJ, Swathi VV, Sunitha MS. Matching in power graphs of finite groups. *Annals of Combinatorics*. 2022;

26(2): 379-391. Available from: https://doi.org/10.1007/s00026-022-00576-5.

[3] Sehgal A, Takshak N, Maan P, Malik A. Graceful labeling of power graph of group $Z_2^{k-1} \times Z_4$. *Asian-European Journal of Mathematics*. 2022; 15(7): 2250121. Available from: https://doi.org/10.1142/S1793557122501212.

[4] Sehgal A, Singh SN. The degree of a vertex in the power graph of a finite abelian group. *Southeast Asian Bulletin of Mathematics*. 2023; 47: 289-296.

[5] Vetrik T, Balachandran S. General multiplicative Zagreb indices of graphs with given clique number. *Opuscula Mathematica*. 2019; 39(3): 433-446. Available from: https://doi.org/10.7494/OpMath.2019.39.3.433.

[6] Gupta RS, Sen MK. The square element graph over a ring. *Southeast Asian Bulletin of Mathematics.* 2017; 41(5): 663-682.

[7] Siwach A, Rana P, Sehgal A, Bhatia V. The square power graph of Z_n and $Z_2^m \times Z_{2^n}$ group. *AIP Conference Proceedings*. 2023; 2782(1): 020099. Available from: https://doi.org/10.1063/5.0154191.

[8] Rana P, Siwach A, Sehgal A, Bhatia P. The degree of a vertex in the *k*th-power graph of a finite abelian group. *AIP Conference Proceedings*. 2023; 2782(1): 020078. Available from: https://doi.org/10.1063/5.0154183.

[9] Saini M, Khasraw SMS, Sehgal A, Singh D. On co-prime order graphs of finite abelian *p*-groups. *Journal of Mathematical and Computational Science*. 2021; 11(6): 7052-7061. Available from: https://doi.org/10.28919/jmcs/6103.

[10] Rehman SU, Farid G, Tariq T, Bonyah E. Equal-square graphs associated with finite groups. *Journal of Mathematics*. 2022; 2022: 9244325. Available from: https://doi.org/10.1155/2022/9244325.

[11] Gao W, Siddiqui MK, Imran M, Jamil MK, Farahani MR. Forgotten topological index of chemical structure in drugs. *Saudi Pharmaceutical Journal*. 2016; 24(3): 258-264. Available from: https://doi.org/10.1016/j.jsps.2016.04.012.

[12] Ashrafi AR, Shabania H, Diudeab MV. Balaban index of dendrimers. *MATCH Communications in Mathematical and in Computer Chemistry*. 2013; 69: 151-158.

[13] Gao W, Wang WF, Farahani MR. Topological indices study of molecular structure in anticancer drugs. *Journal of Chemistry*. 2016; 2016: 3216327. Available from: https://doi.org/10.1155/2016/3216327.

[14] Naeem T, Gumaei A, Jamil MK, Alsanad A, Ullah K. Connectivity indices of intuitionistic fuzzy graphs and their applications in internet routing and transport network flow. *Mathematical Problems in Engineering.* 2021; 2021: 4156879. Available from: https://doi.org/10.1155/2021/4156879.

[15] Kalathian S, Ramalingam S, Raman S, Srinivasan N. Some topological indices in fuzzy graphs. *Journal of Intelligent and Fuzzy Systems*. 2020; 39(5): 6033-6046. Available from: https://doi.org/10.3233/JIFS-189077.

[16] Binu M, Mathew S, Mordeson JN. Wiener index of a fuzzy graph and application to illegal immigration networks. *Fuzzy Sets and Systems*. 2020; 384: 132-147. Available from: https://doi.org/10.1016/j.fss.2019.01.022.

[17] Binu M, Mathew S, Mordeson JN. Connectivity index of a fuzzy graph and its application to human trafficking. *Fuzzy Sets and Systems*. 2019; 360: 117-136. Available from: https://doi.org/10.1016/j.fss.2018.06.007.

[18] Zeryouh M, Marraki ME, Essalih M. On the terminal Wiener index of networks. In: *5th International Conference on Multimedia Computing and Systems (ICMCS).* Marrakech, Morocco: IEEE; 2016. p.533-536. Available from: http://doi.org/10.1109/ICMCS.2016.7905645.

[19] Prathap RR, Chelvam TT. The cubic power graph of finite abelian groups. *AKCE International Journal of Graphs and Combinatorics*. 2021; 18(1): 16-24. Available from: https://doi.org/10.1080/09728600.2021.1878868.

[20] Vukičević D, Furtula B. Topological index based on the ratios of geometrical and arithmetical means of end-vertex degrees of edges. *Journal of Mathematical Chemistry*. 2009; 46: 1369-1376. Available from: https://doi.org/10.1007/s10910-009-9520-x.

[21] Das KC, Gutman I, Furtula B. On atom-bond connectivity index. *Chemical Physics Letters*. 2011; 511(4-6): 452-454. Available from: https://doi.org/10.1016/j.cplett.2011.06.049.

[22] Gutman I. Selected properties of the Schultz molecular topological index. *Journal of Chemical Information and Computer Sciences*. 1994; 34(5): 1087-1089. Available from: https://doi.org/10.1021/ci00021a009.

[23] Hayat S, Wang S, Liu JB. Valency-based topological descriptors of chemical networks and their applications. *Applied Mathematical Modelling*. 2018; 60: 164-178. Available from: https://doi.org/10.1016/j.apm.2018.03.016.

[24] Hayat S, Imran M, Liu JB. An efficient computational technique for degree and distance based topological descriptors with applications. *IEEE Access*. 2019; 7: 32276-32296. Available from: https://doi.org/10.1109/

ACCESS.2019.2900500.
[25] Liu, JB, Pan XF, Hu FT, Hu FF. Asymptotic Laplacian-energy-like invariant of lattices. *Applied Mathematics and Computation*. 2015; 253: 205-214. Available from: https://doi.org/10.1016/j.amc.2014.12.035.
[26] Sharma V, Goswami R, Madan AK. Eccentric connectivity index: a novel highly discriminating topological descriptor for structure property and structure activity studies. *Journal of Chemical Information and Computer Sciences*. 1997; 37(2): 273-282. Available from: https://doi.org/10.1021/ci960049h.
[27] Schultz HP. Topological organic chemistry. 1. Graph theory and topological indices of alkanes. *Journal of Chemical Information and Computer Sciences*. 1989; 29(3): 227-228. Available from: https://doi.org/10.1021/ci00063a012.
[28] Plavšić D, Nikolić S, Trinajstić N, Mihalić Z. On the Harary index for the characterization of chemical graphs. *Journal of Mathematical Chemistry*. 1993; 12: 235-250. Avaliable from: https://doi.org/10.1007/BF01164638.
[29] Gutman I, Trinajstić N. Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons. *Chemical Physics Letters*. 1972; 17(4): 535-538. Available from: https://doi.org/10.1016/0009-2614(72)85099-1.
[30] Wiener H. Structural determination of paraffin boiling points. *Journal of the American Chemical Society*. 1947; 69(1): 17-20. Available from: https://doi.org/10.1021/ja01193a005.
[31] Klein DJ, Lukovits I, Gutman I. On the definition of the hyper-Wiener index for cycle-containing structures. *Journal of Chemical Information and Computer Sciences*. 1995; 35(1): 50-52. Available from: https://doi.org/10.1021/ci00023a007.