Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Fifth-Kind Chebyshev Spectral Collocation Treatment for the Volterra Integro-Differential Equation of the Third Kind

Mahmoud M. Mokhtar

Department of Basic Science, Faculty of Engineering, Modern University for Technology and Information (MTI), Cairo, Egypt
Email: drmmmokhtar@gmail.com

Received: 12 May 2023; **Revised:** 21 July 2023; **Accepted:** 24 June 2023

Abstract: In this research work, we offer an efficient spectral numerical scheme for handling the Volterra integro-differential of the third kind via Chebyshev polynomials of the fifth kind. The celebrated fundamentals and relations were stated with some details. With the aid of the spectral collocation method and the properties of Chebyshev polynomials, the singular Volterra integro-differential equation with its initial condition was transformed into a system of algebraic equations. The convergence and error analyses were discussed in depth. The validity and applicability of the method were tested and verified through three numerical examples, with the absolute errors reported in tables and graphs.

***Keywords*:** Chebyshev polynomials of the fifth-kind, spectral methods, Volterra integro-differential equation of the third-kind

MSC: 65R20, 45J05, 65L60, 41A10, 41A55

1. Introduction

Orthogonal-spectral methods are a class of schemes employed in physics and mathematics to approximately solve differential and integral equations, involving the use of the integral transform. The idea is to write the solution of the differential/integral equation as a sum of certain "basis functions" (for example, Legender, the four kinds of Chebyshev, ultraspherical, and Jacobi polynomials), and then to choose the coefficients in the sum in order to satisfy the differential/integral equation as much as possible. For more details about spectral methods, see [1-10].

A fundamental class of symmetric orthogonal polynomials is introduced in Masjed-Jamei's excellent PhD thesis, which addressed the extended Sturm-Liouville problem to symmetric polynomials. There are four separate parameters in this class. Some properties of these polynomials are also introduced, such as a second-order differential equation, a three-term difference equation, and several other significant relations. The benefit of introducing this class of polynomials is that it generalizes numerous well-known classes of orthogonal polynomials and contains special polynomials that are new varieties of orthogonal polynomials. Specifically, the four well-known Chebyshev polynomial types. Chebyshev polynomials of the fifth and sixth kinds, two further innovative classes of orthogonal polynomials, can also be obtained. One of the main categories of issues in applied mathematics is integral equations. Numerous fields, including biology, business, engineering, and physics, use integral equations. The Volterra-Fredholm integral equations

DOI: https://doi.org/10.37256/cm.5320243030

are derived from the modeling of the spatiotemporal evolution of an epidemic, the parabolic boundary value issues, and the physical and biological models. Integral equations have been more focused on recently for this use. There are two categories of common approaches to solving integral equations: analytical and approximation. The response of integral equations is typically not determinable using analytical methods. As a result, approximate approaches are utilized to gauge how integral equations will respond. The study of Fredholm kernels and operators is based on the solution of the Fredholm integral equation, a mathematical integral equation. Ivar Fredholm looked at the equation for an integral. The Adomian decomposition approach is a helpful strategy for resolving these problems [11-16].

In this work, we are interested in the study of finding an approximate semi-analytic solution to the linear first-order Volterra integro-differential equation of the third kind [17].

$$t^{\beta} y'(t) = a(t)y(t) + g(t) + \int_0^t k(t,x)y(x)dx, \qquad t \in [0,1]$$

subject to the initial condition

$$y(0) = y_0.$$

The organization of the paper is as follows: in Section 2, we report some preliminary and useful relations of Chebyshev polynomials of the fifth kind; Section 3 is devoted to the procedure of the solution; Section 4 is devoted to the study of convergence and error analyses; in Section 5, we report some numerical results; and finally, some concluding remarks are reported in Section 6.

2. Mathematical preliminaries

The basic set of symmetric polynomials that were created in [18] is described in this section. The use of an extended Sturm-Liouville differential problem is the main idea for developing this polynomial class. More specifically, the author in [18] made the supposition that $y = \phi_i(z)$ is a set of symmetric functions that answers the second-order differential equation shown below:

$$A_1(z)\phi_i''(z) + A_2(z)\phi_i'(z) + \left(\mu_i A_3(z) + A_4(z) + \frac{1-\cos(i\pi)}{2} A_5(z)\right)\phi_i(z) = 0, \tag{1}$$

where $A_i(z)$, $1 \le i \le 5$ are independent, and $\{\mu_i\}$ are constants. In [18], it has been shown that $A_i(z)$, $i = 1, 3, 4, 5$ are even, and $A_2(z)$ is odd. You may obtain the desired symmetric category of orthogonal polynomials if $A_i(z)$, $1 \le i \le 5$, $\{\mu_i\}$ are chosen as follows:

$$A_1(z) = z^2\left(rz^2 + s\right),\ A_2(z) = z\left(mz^2 + n\right),\ A_3(z) = z^2,$$

$$A_4(z) = 0,\ A_5(z) = -n,\ \mu_i = -i((i-1)r + m),$$

where the m, n, r, and s parameters are real-free.

The options above give the following differential equation:

$$z^2(s + rz^2)\phi_i''(z) + z(n + mz^2)\phi_i'(z) - \left(i\left((i-1)r + m\right)z^2 + (1 + (-1)^{i+1})\frac{n}{2}\right)\phi_i(z) = 0. \tag{2}$$

The solution of (2) is the generalized polynomials $G_i^{m,n,r,s}(z)$, which has the explicit form:

$$G_i^{m,n,r,s}(z) = \sum_{k=0}^{\left\lfloor \frac{i}{2} \right\rfloor} \left(\binom{\left\lfloor \frac{i}{2} \right\rfloor}{k} \left(\prod_{j=0}^{\left\lfloor \frac{i}{2} \right\rfloor - k - 1} \frac{1}{\Gamma_{i,j,m,n,r,s}} \right) z^{2(\frac{i}{2} - k)} \right),$$

where

$$[\Gamma_{i,j,m,n,r,s} = \frac{(2j + (-1)^{i+1} + 2)s + n}{((-1)^{i+1} + 2j + 2\left\lfloor \frac{i}{2} \right\rfloor)r + m}.$$

In addition, the author in [18] introduced the symmetric orthogonal polynomials $\overline{G}_i^{m,n,r,s}(z)$ defined as

$$\overline{G}_i^{m,n,r,s}(z) = \left(\prod_{j=0}^{\left\lfloor \frac{i}{2} \right\rfloor - 1} \Gamma_{i,j,m,n,r,s} \right) G_i^{m,n,r,s}(z).$$

The polynomials $\overline{G}_i^{m,n,r,s}(z)$ satisfy the following recurrence relation:

$$\overline{G}_{i+1}^{m,n,r,s}(t) = z\overline{G}_i^{m,n,r,s}(z) + A_{i,m,n,r,s}\, \overline{G}_{i-1}^{m,n,r,s}(z), \quad i \geq 0, \tag{3}$$

with the initials:

$$\overline{G}_0^{m,n,r,s}(z) = 1, \quad \overline{G}_1^{m,n,r,s}(z) = z,$$

and

$$A_{i,m,n,r,s} = \frac{rsi^2 + \left((m-2r)s - \cos(i\pi)rn\right)i + \frac{1}{2}(m-2r)n\left(1 - \cos(i\pi)\right)}{(2ri + m - r)(2ri + m - 3r)}. \tag{4}$$

Many properties of $\overline{G}_i^{m,n,r,s}(t)$ may be found in [18]. There are many specific categories of important orthogonal polynomials of $\overline{G}_{i,j,m,n,r,s}(z)$. The four different types of Chebyshev polynomials could be formed through to the expressions:

$$\overline{T}_i(z) = \overline{G}_i^{-1,\, 0,\, -1,\, 1}(z), \;\; \overline{U}_i(z) = \overline{G}_i^{-3,\, 0,\, -1,\, 1}(z)$$

$$\overline{V}_i(z) = 2^i\, \overline{G}_{2i}^{-3,\, 2,\, -1,\, 1}\left(\sqrt{\frac{1+z}{2}} \right), \;\; \overline{W}_i(z) = 2^i\, \overline{G}_{2i}^{-3,\, 2,\, -1,\, 1}\left(\sqrt{\frac{1-z}{2}} \right),$$

and $\overline{T}_i(z)$, $\overline{U}_i(z)$, $\overline{V}_i(z)$, $\overline{W}_i(z)$ are located, respectively, the first, second, third, and fourth categories of Chebyshev polynomials. All these other polynomial categories could be obtained as specific categories of $\overline{G}_i^{m,n,r,s}(z)$. The two types of orthogonal polynomials in [18], especially the Chebyshev fourth and sixth types of polynomials, may also be defined, respectively, as:

$$\overline{X}_i(z) = \overline{G}_i^{-3,\, 2,\, -1,\, 1}(z)$$

$$\overline{Y}_i(z) = \overline{G}_i^{-5,\, 2,\, -1,\, 1}(z)$$

We inhibit our study to the Chebyshev fifth type and their shifted polynomials, which have changed. The property of orthogonality of $\overline{X}_i(z)$ is

$$\int_{-1}^{1} \frac{z^2\, \overline{X}_i(z)\, \overline{X}_j(z)}{\sqrt{1-z^2}} dz = \begin{cases} \left((-1)^i \prod_{L=1}^{i} A_{L,-3,2,-1,1} \right) \frac{\pi}{2}, & \text{when } i = j, \\ 0, & \text{when } i \neq j, \end{cases} \tag{5}$$

and $A_{i,m,n,r,s}$ is given in (4).

Alternatively, the orthogonality relation above can be written as

$$\int_{-1}^{1} \frac{z^2\, \overline{X}_i(z)\, \overline{X}_j(z)}{\sqrt{1-z^2}} dz = \begin{cases} h_i, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \tag{6}$$

and

$$h_i = \begin{cases} \frac{\pi}{2^{2i+1}}, & i \text{ even}, \\ \frac{\pi(i+2)}{i 2^{2i+1}}, & i \text{ odd}. \end{cases} \tag{7}$$

It is more reasonable to normalize fifth type Chebyshev polynomials. For this specific purpose, we describe

$$X_i(z) = \frac{1}{\sqrt{h_i}} \overline{X}_i(z).$$

Accordingly, $X_i(z)$ are orthonormal on $I = [-1, 1]$:

$$\int_{-1}^{1} \frac{z^2}{\sqrt{1-z^2}} X_i(z)\, X_j(z) dz = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{8}$$

2.1 *Fifth type of shifted orthonormal Chebyshev polynomials (5SOCP)*

The 5SOCP $C_i(z)$ can be defined on $I^* = [0, 1]$ by:

$$C_i(z) = \frac{1}{\sqrt{h_i}} \overline{X}_i(2z-1), \tag{9}$$

and h_i is known in (7).

From (8), it is easy to note that $C_i(z),\ i \geq 0$ are orthonormal on I^*. Directly, we have

$$\int_0^1 w^*(z)C_i(z)C_j(z)dx = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{10}$$

and $w^*(z) = \frac{(2z-1)^2}{\sqrt{z-z^2}}$. The following formulae are needed in the sequel.

Theorem 1 The polynomials $C_i(z)$ in (9) are connected with $T_i^*(z)$ by the formula

$$C_i(z) = \sum_{j=0}^{i} g_{i,j}\, T_j^*(z), \tag{11}$$

where

$$g_{i,j} = 2\sqrt{\frac{2}{\pi}}(-1)^{\frac{i-j}{2}} \begin{cases} \delta_j, & i, j \text{ even}, \\ \frac{j}{i}, & i, j \text{ odd}, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\delta_j = \begin{cases} \frac{1}{2}, & \text{if } j = 0, \\ 1, & \text{if } j > 0. \end{cases} \tag{12}$$

Proof. See [19].

Theorem 2 [19] The polynomials $C_i(z)$ in (9) are connected with $T_i^*(z)$ by two formulae below:

$$C_{2i}(z) = 2\sqrt{\frac{2}{\pi}} \sum_{r=0}^{i} (-1)^{i+r} \delta_r\, T_{2r}^*(z), \tag{13}$$

where δ_r is defined in (12), and

$$C_{2i+1}(z) = \frac{2\sqrt{2}}{\sqrt{\pi(2i+3)(2i+1)}} \sum_{r=0}^{i} (-1)^{i+r} (2r+1)\, T_{1+2r}^*(z). \tag{14}$$

The next corollary is important in the sequel: $\overline{X}_{2i}(z)$ and $\overline{X}_{2i+1}(z)$. The following corollary shows these important illustrations.

Corollary 1 The trigonometric representations hold:

$$X_{2i}(\cos\phi) = \sqrt{\frac{2}{\pi}} \frac{\cos((1+2i)\phi)}{\cos\phi}, \tag{15}$$

and

$$X_{2i+1}(\cos\phi) = \sqrt{\frac{2}{\pi(2i+1)(2i+3)}} \frac{(3+2i)\cos((2+2i)\phi)\cos\phi - \cos((3+2i)\phi)}{\cos^2\phi}. \tag{16}$$

Proof. See [19].

The following connection theorem is needed in sequel.

Theorem 3 The polynomials $T_r^*(z)$ is linked to a polynomial $C_i(z)$ by two formulae below:

$$T_{2r}^*(z) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\frac{C_{2r}(z)}{\delta_r} + C_{2r-2}(z)\right), \quad r \geq 0, \tag{17}$$

and

$$T_{2r+1}^*(z) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\sqrt{\frac{3+2r}{1+2r}}C_{2r+1}(z) + \sqrt{\frac{2r-1}{1+2r}}C_{2r-1}(z)\right), \quad r \geq 1. \tag{18}$$

Proof. See [19].

The following two theorems are important.

Lemma 1 [19] The $T_n^*(z)$ power form and its inversion formula are given as:

$$T_\ell^*(z) = \ell\sum_{r=0}^{\ell}\frac{(-1)^{\ell+r}\,2^{2r}(\ell+r-1)!}{(2r)!(\ell-r)!}z^r \tag{19}$$

and

$$z^\ell = 2^{1-2\ell}(2\ell)!\sum_{j=0}^{\ell}\frac{\delta_j}{(\ell-j)!(j+\ell)!}T_j^*(z). \tag{20}$$

Theorem 4 [19] The analytical form $C_i(z)$ is specifically given as

$$C_i(z) = \sum_{r=0}^{i}\varrho_{r,i}\,z^r, \tag{21}$$

where

$$\varrho_{r,i} = \frac{2^{2r+\frac{3}{2}}}{\sqrt{\phi}(2r)!}\begin{cases} 2\displaystyle\sum_{j=\left\lfloor\frac{r+1}{2}\right\rfloor}^{\frac{i}{2}}\frac{(-1)^{\frac{i}{2}+j-r}\,j\delta_j(2j+r-1)!}{(2j-r)!}, & i \text{ even},\\[2ex] \dfrac{1}{\sqrt{i(i+2)}}\displaystyle\sum_{j=\left\lfloor\frac{r}{2}\right\rfloor}^{\frac{i-1}{2}}\frac{(-1)^{\frac{i+1}{2}+j-r}(2j+1)^2(2j+r)!}{(2j-r+1)!}, & i \text{ odd}.\end{cases} \tag{22}$$

Theorem 5 The inversion formula (21) of the analytical formula may be written as

$$z^m = \sum_{\ell=0}^{m}q_{m,\ell}\,C_\ell(z), \tag{23}$$

where

$$q_{m,\ell} = \sqrt{\pi}\,2^{-2m-\frac{1}{2}}(2m)!\begin{cases} \dfrac{2\left(m^2+m+(1+\ell)^2\right)}{(m-\ell)!(\ell+m+2)!}, & \ell \text{ even}, \\ \dfrac{\sqrt{\dfrac{\ell}{2+\ell}}}{(m-(2+\ell))!(m+2+\ell)!}+\dfrac{\sqrt{\dfrac{2+\ell}{\ell}}}{(m+\ell)!(m-\ell)!}, & \ell \text{ odd}. \end{cases}$$

Proof. See [19].

The following derivative formula is needed.

Corollary 2 These two identities hold for all non-negative integer q:

$$D^q C_i(z)\Big|_{t=1} = \alpha_{i,q}, \tag{24}$$

$$D^q C_i(z)\Big|_{t=0} = \cos((i+q)\pi)\alpha_{i,q}, \tag{25}$$

where

$$\alpha_{i,q} = \sum_{j=0}^{i} \bar{\alpha}_{i,j,q}, \tag{26}$$

and

$$\bar{\alpha}_{i,j,q} = \frac{2\sqrt{2}(-1)^{\frac{i-j}{2}}\Gamma(j+q)(1+j-q)_q}{\Gamma(j)\Gamma\left(q+\frac{1}{2}\right)}\begin{cases} \delta_j, & \text{if } i,\ j \text{ even}, \\ \dfrac{j}{i}\sqrt{\dfrac{i}{2+i}}, & i,\ j \text{ odd}, \\ 0, & \text{otherwise} \end{cases} \tag{27}$$

where $(j)_q$ denotes the known Pochhammer symbol.

Proof. See [19].

3. Procedures of solution

In this section, we build an efficient collocation algorithm for numerically handling the following Volterra-Fredholm integral equation of the third kind:

$$t^{\beta} y'(t) = a(t)y(t) + g(t) + \int_0^t k(t,x)y(x)dx, \qquad t \in [0,1] \tag{28}$$

subject to the initial condition

$$y(0)=y_0 \tag{29}$$

using the change of variables $x = t\mu$, we have

$$t^{\beta} y'(t)=a(t)y(t)+g(t)+\int_0^t k(t,t\mu)y(t\mu)d\mu. \tag{30}$$

Based on the interesting Rombergs' integration formula, we have

$$\int_o^1 z_t(\mu)d\mu \approx r_{n,m}(z_t) \tag{31}$$

and $r_{n,m}(Z_t)$ are given by

$$r_{0,0}(z_t)=\frac{z_t(0)+z_t(1)}{2}$$

$$r_{n,m}(z_t)=\frac{1}{2}r_{n-1,0}+2^{-n}\sum_{k=1}^{2^{n-1}} z_t\left(2^{-n}(2k-1)\right)$$

$$r_{n,m}(z_t)=r_{n,m-1}+\frac{1}{4^m-1}\left(r_{n,m-1}-r_{n-1,m}\right)$$

with $n \geq m \geq 1$. Now, equation (28) will be

$$t^{\beta} y'(t)=a(t)y(t)+g(t)+r_{n,m}(z_t); \qquad y(0)=y_0 \tag{32}$$

Now, consider the following approximate solution:

$$y(t)\approx y_N(t)=\sum_{i=0}^{N} v_i C_i(t). \tag{33}$$

Equation (32) can be written in the following discretized form:

$$\sum_{i=1}^{N} v_i t^{\beta} C_i'(t)\approx a(t)\sum_{i=0}^{N} v_i C_i(t)+g(t)+r_{n,m}(z_t) \tag{34}$$

Let $\{t_\eta\}_{\eta=1}^{N}$ be the first N distinct roots of $C_{N+1}(t)$ collocating (34) at t_v, we have

$$\sum_{i=1}^{N} v_i (t_\eta)^{\beta} C_i'(t_\eta))\approx \sum_{i=0}^{N} v_i a(t_\eta))C_i(t_\eta)+g(t_\eta)+r_{n,m}(z_{t_\eta}); \qquad 1\leq \eta \leq N \tag{35}$$

$$\sum_{i=0}^{N} v_i C_i(0)=y_0 \tag{36}$$

The system (35) is a system consist of $N + 1$ algebraic equation in the unknown expansion coefficients $\{v_i\}_{i=0}^{N}$, and consequently,

$$y_N(t) = \sum_{i=0}^{N} v_i C_i(t) \tag{37}$$

4. Discussion of convergence and error analyses

Here, following the extensive work of Abd-Elhameed and Youssri [19]. The following lemma is needed.

Lemma 2 Let $C_\ell(t)$ dominated on i^*. The following inequality holds:

$$|C_\ell(t)| < (2+\ell)\sqrt{\frac{2}{\pi}}, \forall t \in I^*. \tag{38}$$

Theorem 6 If $f(t) \in L^2_{w^*}(I^*)$, $|f^{(3)}(t)| \leq L$, and if its expansion is

$$f(t) = \sum_{\ell=0}^{\infty} a_\ell C_\ell(t), \tag{39}$$

the series in (39) uniformly converges to $f(t)$. Also, we have

$$|a_\ell| \lessapprox \frac{1}{\ell^4}, \text{for all } \ell > 3. \tag{40}$$

Proof. Following the steps of the proof in [19], by integrating by parts one more time, we get the desired result.

Theorem 7 [19] Let $u(t)$ satisfy the conditions of Theorem 6, $u_{N+1}(t)$ and $u_N(t)$ are two approximate solutions of $u(t)$, we define $\overline{e}_N(t) = u_{N+1}(t) - u_N(t)$. Then, we get the following estimation of the error:

$$\|\overline{e}_N(t)\|_{2,w^*} = O(1/N^3),$$

where

$$\|\overline{e}_N(t)\|_{2,w^*} \text{ means the } L^2\text{-norm of } \overline{e}_N(t).$$

Theorem 8 Let $f(t)$ ascertain conditions of Theorem 6. Let $e_N(t) = \sum_{\ell=N+1}^{\infty} a_\ell C_\ell(t)$ be the global error. $e_N(t)$ can be dominated as:

$$|e_N(t)| < 3L/N.$$

Theorem 9 The residual of equation (28) satisfies the following global error estimate:

$$t^\beta y_{N'}(t) - a(t)y_N(t) - \int_0^t k(t,x)y_N(x)dx - g(t)| \lessapprox N^{-1}. \tag{41}$$

Proof. We have

$$t^{\beta} y'(t) - a(t) y(t) - \int_0^t k(t,x) y(x) dx = g(t)$$

and

$$t^{\beta} y_{N'}(t) - a(t) y_N(t) - \int_0^t k(t,x) y_N(x) dx = g(t).$$

Therefore, the residual of equation (28) is given by

$$R(x)| = |t^{\beta}[y'(t) - y_N(x)] - a[y - y_N] - \int_0^t k(t,x)[y(x) - y_n(x)]dx|. \tag{42}$$

By the triangle inequality, we have

$$R(x)| \le t^{\beta} |y'(t) - y'_N(x)| + |a||y - y_N| + \int_0^t |k(t,x)||y(x) - y_n(x)| \, dx.$$

By the triangle inequality, we have

$$|R(x)| \le t^{\beta} |y'(t) - y'_N(x)| + |a||y - y_N| + \int_0^t |k(t,x)||y(x) - y_n(x)| \, dx. \tag{43}$$

Since $a(t)$, $k(t, x)$ are continuous, they are bounded, say

$$|a(t)| \le A, |k(t,x)| \le K,$$

and noting that $t^{\beta} \le 1$, we get

$$|R(x)| \le |y'(t) - y'_N(x)| + A|y - y_N| + \int_0^t K|y(x) - y_n(x)| \, dx \tag{44}$$

Based on Lemma 1, Lemma 2, and Theorem 8, we have

$$|R(x)| \le N^{-1} + AN^{-2} + KN^{-2}, \tag{45}$$

and hence, $|R(x)| \lessapprox N^{-1}$, which complete the proof of the theorem.

5. Numerical tests and comparison

In this section, we provide three numerical examples to check the applicability of the method and verify the accuracy of the solutions. One of the main advantages of the method is that highly efficient solutions are obtained with only a small number of retained modes.

Example 1 Consider the following Volterra-Fredholm of the third kind:

$$t^{\frac{2}{3}} y''(t) = g(t) + \int_0^t xy(x)dx, \ t \in [0, 1], \tag{46}$$

subject to the initial condition

$$y(0) = 0$$

in which

$$g(t) = t^{\frac{2}{3}} \left(\frac{10}{3} t^{\frac{7}{3}} - \frac{3}{16} t^{\frac{14}{3}} \right).$$

In Table 1, we list the maximum absolute errors (MAE) of Example 1 for various values of N. In Figure 1, we depict the absolute error of Example 1.

Table 1. MAE for Example 1

N	4	6	8	10	12
MAE	2.2×10^{-3}	2.2×10^{-4}	5.3×10^{-5}	1.8×10^{-5}	7.8×10^{-6}
CPU (seconds)	12.3	13.5	15.7	20.5	22.8

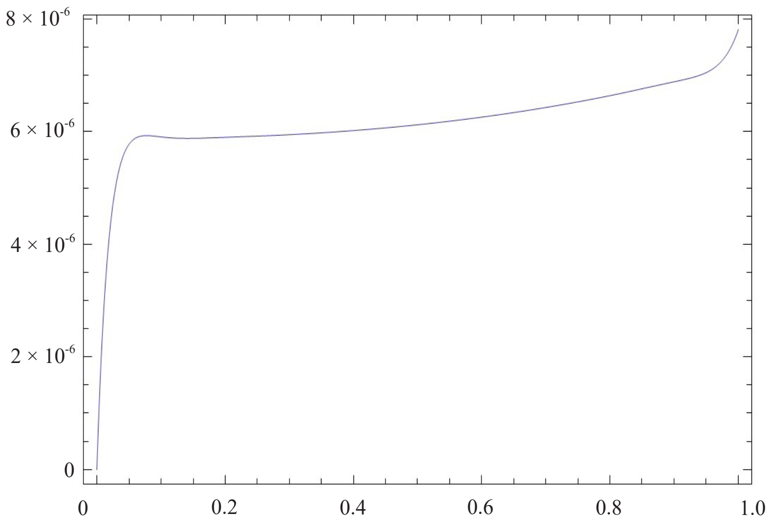

Figure 1. Absolute error of Example 1 for $N = 12$

Example 2 Consider the following Volterra-Fredholm of the third kind:

$$t^{\frac{1}{2}}y'(t)=\frac{1}{20}ty(t)-\frac{9}{2}t^4-\frac{1}{20}t^{\frac{11}{2}}-\frac{1}{30}t^6+\int_0^t x^{\frac{1}{2}}y(x)dx,\ t\in[0,1],\ \text{with }\ y(0)=0.$$

In Table 2, we list the MAE of Example 2 for various values of N. In Figure 2, we depict the absolute error of Example 2.

Table 2. MAE for Example 2

N	4	6	8	10	12
MAE	1.1×10^{-2}	1.5×10^{-4}	1.7×10^{-5}	3.5×10^{-6}	1.1×10^{-6}
CPU (seconds)	14.7	13.5	16.2	22.6	24.2

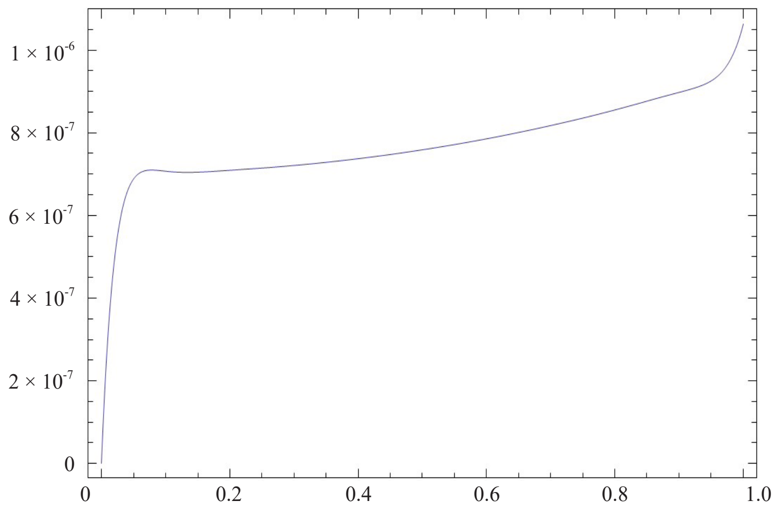

Figure 2. Absolute error of Example 2 for $N = 12$

Example 3 Consider the following Lane-Emden equation [20]:

$$ty''+2y'+ty=0;\ t\in(0,\ 1),\ y(0)=1,\ y'(0)=0. \tag{47}$$

with the exact solution $y(t)=\frac{\sin t}{t}$.

By integrating (47) and using the initial conditions, we get Volterra-Fredholm of the third kind:

$$ty'(t)=1-y(t)-\int_0^t xy(x)dx,\ t[0,\ 1],\ \text{with }\ y(0)=1.$$

In Table 3, we list the MAE of Example 3 for various values of N. In Figure 3, we show the absolute error of

Example 3 for $N = 8$. In Table 4, we compare our method with the method offered in [20].

Table 3. MAE for Example 3

N	4	5	6	7	8
MAE	9.66×10^{-6}	9.29×10^{-7}	2.45×10^{-8}	1.84×10^{-9}	8.24×10^{-13}
CPU (seconds)	17.3	18.2	19.7	25.8	29.5

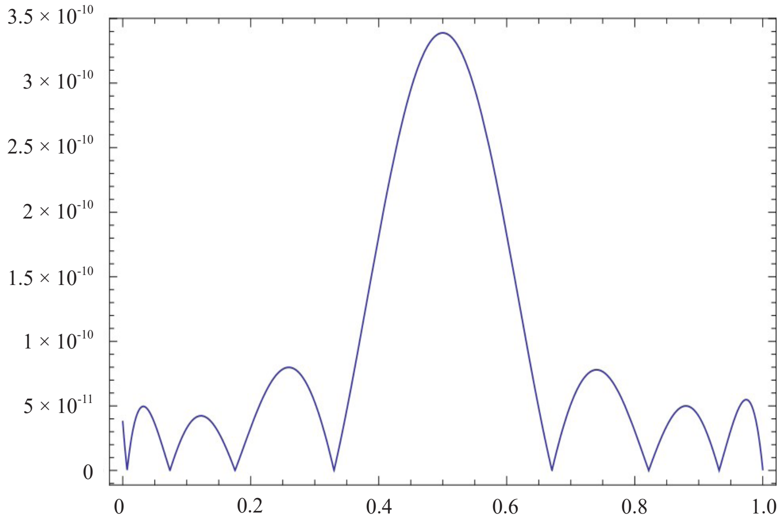

Figure 3. Absolute error of Example 3 for $N = 8$

Table 4. Best errors for Example 3 for $N = 8$

Method	[20]	Our method
Error	2.79×10^{-8}	8.24×10^{-13}

Example 4 Consider the following integral equation:

$$t\,y' = 2\operatorname{erf}(t) - 2t^2 e^{-t^2} + \frac{4}{\sqrt{\pi}} \int_0^t y(x)dx; \quad t \in (0, 1), \quad y(0) = 1 \tag{48}$$

with no known exact solution.

Because of the absence of the exact solution, we report in Table 5, the residual error:

$$R_N = |\,t\,y_{N'} - 2\operatorname{erf}(t) - 2t^2 e^{-t^2} - \frac{4}{\sqrt{\pi}} \int_0^t y_N(x)dx\,|$$

Table 5. Residual error for Example 4

N	4	5	6	7	8
MAE	2.4×10^{-5}	3.5×10^{-6}	4.6×10^{-7}	6.7×10^{-8}	2.5×10^{-12}
CPU (seconds)	5.2	7.5	9.2	11.5	12.6

6. Conclusion

Herein, we have developed an efficient and reliable algorithm for numerically solving the third-kind Volterra-Fredholm integro-differential equations. The convergence and error analyses were discussed in detail, and three numerical examples were presented to ascertain the applicability of the method.

Acknowledgment

The author thanks the editor and three anonymous reviewers for their constructive comments which greatly enhanced the manuscript.

Conflict of interest

The author declares no competing financial interest.

References

[1] Boyd JP. *Chebyshev & Fourier spectral methods*. Berlin, Heidelberg: Springer; 1989.

[2] Canuto C, Hussaini MY, Quarteroni A, Zang TA. *Spectral methods: Fundamentals in single domains*. Berlin, Heidelberg: Springer; 2006. Available from: https://doi.org/10.1007/978-3-540-30726-6.

[3] Atta AG, Abd-Elhameed WM, Moatimid GM, Youssri YH. A fast Galerkin approach for solving the fractional Rayleigh-Stokes problem via sixth-kind Chebyshev polynomials. *Mathematics*. 2022; 10(11): 1843. Available from: https://doi.org/10.3390/math10111843.

[4] Atta AG, Abd-Elhameed WM, Youssri YH. Shifted fifth-kind Chebyshev polynomials Galerkin-based procedure for treating fractional diffusion-wave equation. *International Journal of Modern Physics C*. 2022; 33(8): 2250102. Available from: https://doi.org/10.1142/S0129183122501029.

[5] Atta AG, Abd-Elhameed WM, Moatimid GM, Youssri YH. Advanced shifted sixth-kind Chebyshev tau approach for solving linear one-dimensional hyperbolic telegraph type problem. *Mathematical Sciences*. 2022; 17: 415-429. Available from: https://doi.org/10.1007/s40096-022-00460-6.

[6] Abd-Elhameed WM, Youssri YH. New formulas of the high-order derivatives of fifth-kind Chebyshev polynomials: Spectral solution of the convection-diffusion equation. *Numerical Methods for Partial Differential Equations*. 2024; 40(2): e22756. Available from: https://doi.org/10.1002/num.22756.

[7] Abd-Elhameed WM, Alkhamisi SO, Amin AK, Youssri YH. Numerical contrivance for Kawahara-type differential equations based on fifth-kind Chebyshev polynomials. *Symmetry*. 2023; 15(1): 138. Available from: https://doi.org/10.3390/sym15010138.

[8] Youssri YH. Two Fibonacci operational matrix pseudo-spectral schemes for nonlinear fractional Klein-Gordon equation. *International Journal of Modern Physics C*. 2022; 33(4): 2250049. Available from: https://doi.org/10.1142/S0129183122500498.

[9] Youssri YH. Orthonormal ultraspherical operational matrix algorithm for fractal-fractional Riccati equation with

generalized Caputo derivative. *Fractal and Fractional*. 2021; 5(3): 100. Available from: https://doi.org/10.3390/fractalfract5030100.

[10] Abd-Elhameed WM, Ali A, Youssri YH. Newfangled linearization formula of certain nonsymmetric Jacobi polynomials: Numerical treatment of nonlinear Fisher's equation. *Journal of Function Spaces*. 2023; 2023: 6833404. Available from: https://doi.org/10.1155/2023/6833404.

[11] Polyanin AD, Manzhirov AV. *Handbook of integral equations*. Boca Raton: CRC Press; 1998.

[12] Maleknejad K, Rashidinia J, Eftekhari T. Existence, uniqueness, and numerical solutions for two-dimensional nonlinear fractional Volterra and Fredholm integral equations in a Banach space. *Computational and Applied Mathematics*. 2020; 39(4): 271. Available from: https://doi.org/10.1007/s40314-020-01322-4.

[13] Maleknejad K, Kalalagh HS. An iterative approach for solving nonlinear Volterra-Fredholm integral equation using tension spline. *Iranian Journal of Science and Technology, Transactions A: Science*. 2020; 44: 1531-1539. Available from: https://doi.org/10.1007/s40995-020-00963-8.

[14] Rashidinia J, Maleknejad K, Jalilian H. Convergence analysis of non-polynomial spline functions for the Fredholm integral equation. *International Journal of Computer Mathematics*. 2020; 97(6): 1197-1211. Available from: https://doi.org/10.1080/00207160.2019.1609669.

[15] Sweilam NH, Nagy AM, Youssef IK, Mokhtar MM. New spectral second kind Chebyshev wavelets scheme for solving systems of integro-differential equations. *International Journal of Applied and Computational Mathematics*. 2017; 3(2): 333-345. Available from: https://doi.org/10.1007/s40819-016-0157-8.

[16] Youssri YH, Hafez RM. Chebyshev collocation treatment of Volterra-Fredholm integral equation with error analysis. *Arabian Journal of Mathematics*. 2020; 9(2): 471-480. Available from: https://doi.org/10.1007/s40065-019-0243-y.

[17] Shayanfard F, Dastjerdi HL, Ghaini FMM. Collocation method for approximate solution of Volterra integro-differential equations of the third-kind. *Applied Numerical Mathematics*. 2020; 150: 139-148. Available from: https://doi.org/10.1016/j.apnum.2019.09.020.

[18] Masjed-Jamei M. *Some new classes of orthogonal polynomials and special functions: A symmetric generalization of Sturm-Liouville problems and its consequences*. PhD thesis. University of Kassel; 2006.

[19] Abd-Elhameed WM, Youssri YH. Fifth-kind orthonormal Chebyshev polynomial solutions for fractional differential equations. *Computational and Applied Mathematics*. 2018; 37: 2897-2921. Available from: https://doi.org/10.1007/s40314-017-0488-z.

[20] Doha EH, Abd-Elhameed WM, Youssri YH. Second kind Chebyshev operational matrix algorithm for solving differential equations of Lane-Emden type. *New Astronomy*. 2013; 23-24: 113-117. Available from: https://doi.org/10.1016/j.newast.2013.03.002.