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Research Article

New Qualitative Outcomes for Ordinary Differential Systems of Second Order

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Abstract: This paper deals with a nonlinear ordinary differential system of second order. In the paper, qualitative properties of solutions of the system called asymptotic stability (AS), uniform stability (US), boundedness, ultimately boundedness (UB) and integrability of solutions, are investigated by using the second method of Lyapunov. We give four new qualitative results and an example as a numerical application of the results. The results of this article extend and improve some earlier ones in the literature.

***Keywords*:** differential system, second order, the second method of Lyapunov, integrability, stability, boundedness

MSC: 34K20, 34K06

1. Introduction

From the database of the relevant literature, it can be seen that ordinary differential equations (ODEs) of second order have numerous and effective applications in science and engineering, and there is also an extensive literature on the various qualitative properties of solutions of numerous kind of ODEs, see the books of [1-8] and the papers of [9-43]. For some interesting works on fractional control systems, etc., see also [44-49].

As the reference paper for this work, recently, Adeyanju [12] considered the following nonlinear system of ODEs of second order:

$$\ddot{X} + F(X, \dot{X})\dot{X} + H(X) = P(t, X, \dot{X}).$$

In this paper, motivated from the above system of ODEs of second order (Adeyanju [12]) we consider the following system of ODEs of second order.

$$\ddot{X} + a(t)F(X, \dot{X})\dot{X} + b(t)Q(\dot{X})\dot{X} + c(t)H(X) = P(t, X, \dot{X}). \tag{1}$$

The system of ODEs (1) can be converted to the following system

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$$\dot{X} = Y,$$
$$\dot{Y} = -a(t)F(.)Y - b(t)Q(Y)Y - c(t)H(X) + P(.), \tag{2}$$

where $F(.) = F(X, Y)$, $P(.) = P(t, X, Y)$, $X, Y \in R^n$, $R^+ = [0,\infty)$, $a, b \in C[R^+,(0,\infty)]$, $c \in C^1[R^+,(0,\infty)]$, F, Q are $n \times n$ continuous symmetric and positive definite matrix functions, $H \in C^1(R^n, R^n)$, $H(0) = 0$ and $P \in C[R^+ \times R^n \times R^n, R^n]$. In addition, we assume that the uniqueness of the solutions of the system of ODEs (1) holds. The Jacobian matrix $J_H(X)$ of $H(X)$ is also given by

$$J_H(X) = \left(\frac{\partial h_i}{\partial x_j}\right), i, j = 1, 2, ..., n,$$

where $X = (x_1, x_2, ..., x_n)$, and $J_H(X)$ exists and is continuous. The symbol $\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i$ is used to denote the usual scalar product of any two vectors $X, Y \in R^n$.

In this study, we prove four new theorems in relation to the asymptotic stability, uniform stability, integrability of solutions of the system of ODEs (1) of second order when $P(.) = 0$ and the boundedness, ultimately boundedness of solutions of the same system when $P(.) \neq 0$, respectively. It should be noted that the details of terminologies and definitions of asymptotic stability, uniform stability, integrability, boundedness, ultimately boundedness of solutions of ODEs can be found in the books of Hsu [3], Jordan and Smith [4], Reissig et al. [5], Yoshizawa [8].

Next, the first aim of this paper is to generalize the results of Adeyanju [12, Theorem 4.1, Theorem 4.2, Theorem 4.4] and to add a new result on the integrability of solutions of the system of ODEs (1) of second order. From the information given above, it is seen that the system of ODEs (1) of second order includes and extends the above system of ODEs of second order, which has been investigated by Adeyanju [12]. Hence, we generalize the results of Adeyanju [12]. Next, in particular cases, the qualitative behaviors of solutions such as various stability and boundedness of scalar ODEs of second order have been investigated in the papers [9-11, 20-24, 27, 29-43] and the books [3-5, 8]. For some particular cases, the system of ODEs (1) of second order generalizes various stability and boundedness results in those papers and books. For the sake of brevity, we would not give the details of comparison. Hence, this paper provides new contributions with regard to the stability and boundedness of solutions of ODEs of second order that can be found in the relevant literature. These are the new outcomes and contributions of this paper to the relevant literature.

2. Preliminaries

We now state some basic results in the following lines concerning the qualitative concepts to be studied in this paper.

Consider a system of differential equations $x'(t) = F(t, x(t))$, where x is an n-vector and $t \in I$, $I \subset R$. Suppose that $F(t, x(t))$ is continuous in (t, x) on $I \times D$, where D is a connected open set in R^n.

Theorem 2.1. (Yoshizawa [8]). Suppose that there exist a Lyapunov function $V(t, x)$ defined on $0 \leq t < \infty$, $\|x\| < H$ and satisfies the following conditions;

(i) $V(t,0) \equiv 0$,

(ii) $a(\|x\|) \leq V(t, x)$, where $a(r)$ is a continuous, increasing, positive function and $a(r) \to \infty$ as $r \to \infty$,

(iii) $V'(t,\phi) \leq 0$.

Then, the solution $x(t) \equiv 0$ of $x'(t) = F(t, x(t))$ is stable.

Theorem 2.2. (Yoshizawa [8]). If condition (ii) in Theorem 2.1 is replaced by (ii)'$a(\|x\|) \leq V(t, x) \leq b(\|x\|)$, where $a(r)$ is taken from Theorem 2.1 and $b(r)$ is a continuous, increasing, positive function, then the solution $x(t) \equiv 0$ of $x'(t) = F(t, x(t))$ is uniform-stable.

Theorem 2.3. Suppose that there exist a Lyapunov function $V(t, x)$ defined on $0 \leq t < \infty$, $\|x\| \geq R$, where R may be large, which satisfies the following conditions;

(iv) $a(\|x\|) \leq V(t, x) \leq b(\|x\|)$, (where $a(r)$ and $b(r)$ are taken from Theorem 2.2),

(v) $V'(t,x) \leq 0$, then the solutions of $x'(t) = F(t, x(t))$ are uniformly bounded.

We should state that the system of ODEs (1) of second order is included by the system of differential equations $x'(t) = F(t, x(t))$. Hence, Theorems 2.1 to 2.3 hold for the system of ODEs (1) of second order.

3. Qualitative results

Let the following conditions hold.

(T1) There exist positive constants a_0, a_1, b_0, b_1 and c_0 such that

$$a_0 \le a(t) \le a_1,\ b_0 \le b(t) \le b_1,\ 1 \le c(t) \le c_0.$$

(T2) The matrices $J_H(X)$ and $F(.)$ are symmetric, positive definite and their eigenvalues $\lambda_i(J_H(X))$ and $\lambda_i(F(.))$ satisfy the following inequalities, respectively:

$$\delta_h \le \lambda_i(J_H(X)) \le \Delta_h, \forall X \in R^n, \tag{3}$$

$$\alpha - \varepsilon \le \lambda_i(F(.)) \le \alpha, \forall X \in R^n, \tag{4}$$

$$0 < \delta_q \le \lambda_i(Q(Y)) \le \Delta_q, (i = 1, 2, ..., n), \tag{5}$$

where δ_h, α, ε, Δ_h and Δ_q are positive constants and $H(0) = 0$, $H(X) \ne 0$, (whenever $X \ne 0$), such that

$$\delta \ge \frac{\alpha + \varepsilon}{\alpha - \varepsilon} > 1.$$

(T3) There exists a positive finite constant K_2 and a continuous function $\theta(t)$ such that the function $P(.)$ satisfies

$$\|P(.)\| \le \theta(t)\{1 + (\|X\| + \|Y\|)\}, \tag{6}$$

where $\int_0^t \theta(s)ds \le K_2 < \infty, \forall t \ge 0.$

(T4) The function $P(.)$ satisfies

$$\|P(.)\| \le \theta(t),$$

where $\theta(t) \in L^1[0, \infty), \forall t \in R^+$, $L^1(0, \infty)$ is the space of Lebesgue integrable functions.

Let $P(.) \equiv 0$.

The following theorem is the first main result of this study.

Theorem 3.1. Supposed that conditions (T1) and (T2) hold. Then, the zero solution of the system (2) is uniformly stable and asymptotically stable.

Proof. We begin by defining a continuously differentiable Lyapunov function (LF), which is given by

$$2V(t) = \|\alpha X + Y\|^2 + \delta\|Y\|^2 + 2(\delta + 1)c(t)\int_0^1 \langle H(\sigma_1 X), X\rangle d\sigma_1. \tag{7}$$

Since

$$\int_0^1 \langle H(\sigma_1 X), X\rangle d\sigma_1 \ge \delta_h \|X\|^2,$$

then

$$2V(t) \geq (\delta+1)\delta_h c(t)\|X\|^2 + \delta\|Y\|^2.$$

Using the condition $1 \leq c(t) \leq c_0$, we have

$$2V(t) \geq \delta_1(\|X\|^2 + \|Y\|^2),$$

where $\delta_1 = \min\{(\delta+1)\delta_h, \delta\}$.

Hence, from the LF (7), we derive

$$\begin{aligned} 2V(t) &\leq \langle \alpha X + Y, \alpha X + Y \rangle + 2(\delta+1)c(t)\int_0^1 \langle H(\sigma_1 X), X \rangle d\sigma_1 + \delta\|Y\|^2 \\ &\leq 2\alpha^2\|X\|^2 + 2(\delta+1)c(t)\int_0^1 \langle H(\sigma_1 X), X \rangle d\sigma_1 + (2+\delta)\|Y\|^2. \end{aligned}$$

Using the condition $1 \leq c(t) \leq c_0$, (T2) and proceeding some mathematical calculations, we get

$$c(t)\int_0^1 \langle H(\sigma_1 X), X \rangle d\sigma_1 \leq c_0\Delta_h\|Y\|^2,$$

$$\begin{aligned} 2V(t) &\leq (2\alpha^2 + 2(\delta+1)c_0\Delta_h)\|X\|^2 + (2+\delta)\|Y\|^2 \\ &\leq \delta_2(\|X\|^2 + \|Y\|^2), \end{aligned}$$

where $\delta_2 = \max\{(2\alpha^2 + 2(\delta+1)c_0\Delta_h), (2+\delta)\}$.

According to the results in the above line, it follows that

$$\delta_1(\|X\|^2 + \|Y\|^2) \leq 2V(t) \leq \delta_2(\|X\|^2 + \|Y\|^2). \tag{8}$$

For the next step, the time derivative of the LF $V(t)$ along the solutions of the system (2) gives that

$$\begin{aligned} 2\dot{V}(t) = &\langle \alpha\dot{X} + \dot{Y}, \alpha X + Y \rangle + 2(\delta+1)c'(t)\int_0^1 \langle H(\sigma_1 X), X \rangle d\sigma_1 + \langle \alpha X + Y, \alpha\dot{X} + \dot{Y} \rangle \\ &+ 2(\delta+1)c(t)\langle H(X), Y \rangle + \delta\langle \dot{Y}, Y \rangle + \delta\langle Y, \dot{Y} \rangle. \end{aligned}$$

Using (T1), (T2), $1 \leq c(t) \leq c_0$, $c'(t) \leq 0$ and the system (2), we derive that

$$\begin{aligned} \dot{V}(t) &= \alpha^2\langle Y, X \rangle + \alpha\langle Y, Y \rangle - \alpha\langle a(t)F(.)Y, X \rangle - (\delta+1)\langle a(t)F(.)Y, Y \rangle - \alpha\langle b(t)Q(Y)Y, X \rangle \\ &\quad - (\delta+1)\langle b(t)Q(Y)Y, Y \rangle - \alpha\langle H(X), X \rangle + \alpha\langle P(.), X \rangle + (\delta+1)\langle P(.), Y \rangle \\ &= -\alpha\langle X, H(X) \rangle - (\delta+1)\langle a(t)F(.)Y, Y \rangle - \langle [(\delta+1)b(t)Q(Y) - \alpha]Y, Y \rangle - \langle [\alpha a(t)F(.) - \alpha^2]Y, X \rangle \\ &\quad - \alpha\langle b(t)Q(Y)Y, X \rangle \\ &= -U_1 - U_2, \end{aligned}$$

where

$$\begin{aligned} U_1 &= \frac{\alpha}{2}\langle X, H(X) \rangle + \langle [(\delta+1)b(t)Q(Y) - \alpha]Y, Y \rangle + \frac{(\delta+1)}{2}a(t)\langle Y, F(.)Y \rangle, \\ U_2 &= \frac{\alpha}{2}\langle X, H(X) \rangle + \frac{(\delta+1)}{2}a(t)\langle Y, F(.)Y \rangle + \alpha\langle X, (a(t)F(.) - \alpha I)Y \rangle + \alpha\langle b(t)Q(Y)Y, X \rangle. \end{aligned}$$

Using the conditions $a_0 \leq a(t) \leq a_1$, $b_0 \leq b(t) \leq b_1$ and $0 < \delta_q \leq \lambda_i(Q(Y)) \leq \Delta_q$, (see, also [1]), $i = 1, 2, ..., n$, we have

$$U_1 \geq \frac{\alpha}{2}\left\langle X, H(X)\right\rangle + \left\langle [(\delta+1)b_0\delta_q - \alpha]Y, Y\right\rangle + \frac{(\delta+1)}{2}a_0\left\langle Y, F(.)Y\right\rangle,$$

$$U_2 \geq \frac{\alpha}{2}\left\langle X, H(X)\right\rangle + \frac{(\delta+1)}{2}a_0\left\langle Y, F(.)Y\right\rangle + \alpha\left\langle X, (a_0F(.) - \alpha I)Y\right\rangle + \alpha b_0\delta_q\left\langle Y, X\right\rangle.$$

Finally, by virtue of $\delta_h\|X\|^2 \leq \left\langle X, H(X)\right\rangle \leq \Delta_h\|X\|^2$ and $\alpha - \varepsilon \leq \lambda_i(F(.)) \leq \alpha$ of (T2), we derive

$$U_1 \geq \frac{\alpha}{2}\delta_h\|X\|^2 + [(\delta+1)b_0\delta_q - \alpha + \frac{(\delta+1)}{2}a_0(\alpha-\varepsilon)]\|Y\|^2$$
$$\geq \delta_1(\|X\|^2 + \|Y\|^2),$$

where $\delta_1 = \min\left\{\frac{\alpha}{2}\delta_h, \left[(\delta+1)b_0\delta_q - \alpha + \frac{(\delta+1)}{2}a_0(\alpha-\varepsilon)\right]\right\}$.

From (T1) and (T2), it is also clear that

$$\left\langle X, (a_0F(.) - \alpha I)Y\right\rangle = \frac{1}{2}\left\|K_1a_0(F(.) - \alpha I)Y + K_1^{-1}X\right\|^2 - \frac{1}{2K_1^2}\|X\|^2 - \frac{K_1^2}{2}a_0^2(F(.) - \alpha I)^2\|Y\|^2$$
$$\geq -\frac{1}{2K_1^2}\|X\|^2 - \frac{K_1^2}{2}a_0^2 \in^2 \|Y\|^2$$

and

$$\left\langle X, \alpha b_0\delta_qY\right\rangle = \frac{1}{2}\left\|K_2\alpha b_0\delta_qY + K_2^{-1}X\right\|^2 - \frac{1}{2K_2^2}\|X\|^2 - \frac{1}{2}K_2^2\alpha^2b_0^2\delta_q^2\|Y\|^2$$
$$\geq -\frac{1}{2K_2^2}\|X\|^2 - \frac{1}{2}(K_2\alpha b_0\delta_q)^2\|Y\|^2.$$

Combining the results in the above lines, we get

$$U_2 \geq \frac{\alpha}{2}\delta_h\|X\|^2 + \frac{(\delta+1)}{2}a_0(\alpha-\varepsilon)\|Y\|^2 - \frac{\alpha}{2K_1^2}\|X\|^2 - \frac{K_1^2}{2}a_0^2\varepsilon^2\alpha\|Y\|^2 - \frac{1}{2K_2^2}\|X\|^2 - \frac{1}{2}(K_2\alpha b_0\delta_q)^2\|Y\|^2$$
$$\geq (\frac{\alpha}{2}\delta_h - \frac{\alpha}{2K_1^2} - \frac{1}{2K_2^2})\|X\|^2 + (\frac{(\delta+1)}{2}a_0(\alpha-\varepsilon) - \frac{K_1^2}{2}\alpha_0^2\varepsilon^2\alpha - \frac{1}{2}(K_2\alpha b_0\delta_q)^2)\|Y\|^2$$
$$\geq \delta_2(\|X\|^2 + \|Y\|^2),$$

where $\delta_2 = \min\left\{\left(\frac{\alpha}{2}\delta_h - \frac{\alpha}{2K_1^2} - \frac{1}{2K_2^2}\right); \left(\frac{\delta+1}{2}a_0(\alpha-\varepsilon)\right) - \frac{K_1^2a_0^2\varepsilon^2\alpha}{2} - \frac{1}{2}(K_2b_0\alpha\delta_q)^2\right\}$, and

$$U_1 + U_2 \geq \delta_3(\|X\|^2 + \|Y\|^2),$$

where $\delta_3 = \min\{(\delta_1, \delta_2)\}$.

Hence, we conclude that

$$\dot{V}(t) \leq -\delta_3(\|X\|^2 + \|Y\|^2) \leq 0.$$

The inequality in the last line shows that the time derivative the LF $V(t)$ is negative semidefinite. Thus, we can conclude the zero solution of the system ODE (1) is stable and also uniformly stable.

Our next theorem is related to the boundedness of solutions of the system ODE (1) of second order.

Let $P(.) \neq 0$.

Theorem 3.2. If conditions (T1) to (T3) hold, then there exists a positive constant D such that all the solutions of the system (2) are bounded, i.e.,

$$\|X\| \leq D_0, \quad \|Y\| \leq D_0$$

as $t \to \infty$, where D_0 positive constant.

Proof. Since $P(.) \neq 0$, the time derivative of the LF $V(t)$ along the system (2) can be rearranged as the following:

$$\begin{aligned}\dot{V}(t) = \alpha^2 \langle Y, X\rangle + \alpha\langle Y, Y\rangle - \alpha\langle a(t)F(.)Y, X\rangle - (\delta+1)\langle a(t)F(.)Y, Y\rangle - \alpha\langle b(t)Q(Y)Y, X\rangle \\ -(\delta+1)\langle b(t)Q(Y)Y, Y\rangle - \alpha\langle H(X), X\rangle + \alpha\langle P(.), X\rangle + (\delta+1)\langle P(.), Y\rangle.\end{aligned}$$

According to (T1) to (T3), we get

$$\begin{aligned}\dot{V}(t) &\leq -\delta_3(\|X\|^2 + \|Y\|^2) + \alpha\langle P(.), X\rangle + (\delta+1)\langle P(.), Y\rangle \\ &\leq \alpha\langle P(.), X\rangle + (\delta+1)\langle P(.), Y\rangle \\ &\leq \|\alpha X + (\delta+1)Y\|\|P(.)\| \\ &\leq (\alpha\|X\| + (\delta+1)\|Y\|)(\theta(t) + \theta(t)(\|X\| + \|Y\|)) \\ &\leq [\delta_4(\|X\| + \|Y\|)][\theta(t) + \theta(t)(\|X\| + \|Y\|)],\end{aligned}$$

where $\delta_4 = \max\{\alpha, (\delta+1)\}$.

Next, using some well know mathematical inequalities, it follows that

$$\begin{aligned}\dot{V}(t) &\leq \delta_4\theta(t)(2 + \|X\|^2 + \|Y\|^2) + 2\delta_4\theta(t)(\|X\|^2 + \|Y\|^2) \\ &= 2\delta_4\theta(t) + 3\delta_4\theta(t)(\|X\|^2 + \|Y\|^2) \\ &\leq 2\delta_4\theta(t) + 6\delta_1^{-1}\delta_4\theta(t)V(t) \\ &\leq \delta_5\theta(t) + \delta_6\theta(t)V(t),\end{aligned} \tag{9}$$

where $\delta_5 = 2\delta_4, \delta_6 = 6\delta_1^{-1}\delta_4$.

Integrating both sides of (9) from 0 to t, we obtain

$$\begin{aligned}V(t) &\leq V(0) + \delta_5 K_2 + \delta_6\int_0^t V(s)\theta(s)ds \\ &\leq \delta_7 + \delta_6\int_0^t V(s)\theta(s)ds,\end{aligned}$$

where $\delta_7 = V(0) + \delta_5 K_2$.

From (T3), we have $\int_0^\infty \theta(s)ds \leq K_2 < \infty$. Then, applying the Gronwall-Bellman inequality, we get

$$V(t) \leq \delta_7 \exp(\delta_6\int_0^t \theta(s)ds) = D_1.$$

Considering the discussion above, we arrive the following results, respectively:

$$\|X\|^2 + \|Y\|^2 \leq 2\delta_1^{-1}V(t) \leq 2\delta_1^{-1}D_1 = D_0^2$$

and

$$\|X\| \le D_0 \text{ and } \|Y\| \le D_0$$

The last inequalities complete the proof of Theorem 3.2, i.e., the boundedness of solutions of the system (2) is verified and hence that of the ODE (1) of second order.

Our next result deals with the ultimately boundedness, which is given in Theorem 3.3.

Theorem 3.3. If conditions (T1), (T2) and (T4) hold, then there exists a positive constant D_1 such that all solutions of the system (2) ultimately satisfies

$$\|X\| \le D_1, \|Y\| \le D_1$$

as $t \to \infty$.

Proof. Following the way of Theorem 3.2, in the light of (T1), (T2) and (T4), we can obtain

$$\begin{aligned}\dot{V}(t) &\le -\delta_3(\|X\|^2 + \|Y\|^2) + \langle \alpha X + (\delta+1)Y, P(.)\rangle \\ &\le \langle \alpha X + (\delta+1)Y, P(.)\rangle \\ &\le \theta(t)(\alpha\|X\| + (\delta+1)\|Y\|) \\ &\le \delta_4\theta(t)(\|X\| + \|Y\|),\end{aligned}$$

where $\delta_4 = \max\{(\alpha, (\delta+1)\}$. Applying the inequality

$$\|X\| + \|Y\| \le \sqrt{2}\sqrt{\|X^2\| + \|Y\|^2},$$

we find that

$$\begin{aligned}\frac{d}{dt}V(t) &\le \delta_4\theta(t)2^{\frac{3}{2}}\delta_1^{-\frac{1}{2}}V^{\frac{1}{2}}(t) \\ &\le \delta_8\theta(t)V(t),\end{aligned}$$

where $\delta_8 = 2^{\frac{3}{2}}\delta_4\delta_1^{-\frac{1}{2}}$.

An integration of the inequality above leads that

$$V(t) \le V(0)\exp(\delta_8\int_0^t \theta(s)ds) \le D_2.$$

Since

$$\delta_1(\|X\|^2 + \|Y\|^2) \le 2V(t),$$

then

$$\|X\|^2 + \|Y\|^2 \le 2\delta_1^{-1}V(t) \le 2\delta_1^{-1}D_2 = D_1.$$

Thus, we arrive that

$$\begin{aligned}\|X(t)\| &\le \sqrt{D_1}, \\ \|Y(t)\| &\le \sqrt{D_1}.\end{aligned}$$

This ends the proof of the Theorem 3.3.

Let $P(.) \equiv 0$.

Our next result deals with the integrability of solutions of the system (2), which is given in Theorem 3.4.

Theorem 3.4. If the conditions Theorem 3.1 hold, then the square of the norm of solutions of the system (2) are integrable on R^+ in the sense of Lebesgue.

Proof. From Theorem 3.1, we have

$$\dot{V}(t) \le -\delta_3(\|X\|^2 + \|Y\|^2) \le 0.$$

Integrating this inequality, we obtain

$$V(t) \le V(0) - \delta_3 \int_0^t (\|X\|^2 + \|Y\|^2) ds.$$

Hence, we have that

$$\begin{aligned} \delta_3 \int_0^t (\|X\|^2 + \|Y\|^2) ds &\le V(t) + \delta_3 \int_0^t (\|X\|^2 + \|Y\|^2) ds \\ &\le V(0) = K, K > 0,\ K \in R. \end{aligned}$$

Clearly, it follows that

$$\int_0^\infty (\|X\|^2 + \|Y\|^2)\, ds < +\infty.$$

The proof of this theorem is completed.

Example 3.1. Let $n = 2$. As a particular case of the system (2), we consider the system (2) with the following data:

$$F(.) = \begin{bmatrix} 1 + \dfrac{4}{2 + x_1^2 + y_1^2} & 0 \\ 0 & 1 + \dfrac{4}{2 + x_2^2 + y_2^2} \end{bmatrix},$$

$$H(X) = \begin{bmatrix} 2x_1 - \cos x_1 \\ 3x_2 - \cos x_2 \end{bmatrix},$$

$$Q(Y) = \begin{bmatrix} 3 + \dfrac{8}{1 + e^{y_1^2}} & 0 \\ 0 & 3 + \dfrac{8}{1 + e^{y_2^2}} \end{bmatrix}$$

and

$$P(.) = e^{-5t} \begin{bmatrix} x_1 e^{-x_1^2} + y_1 e^{-y_1^2} \\ x_2 e^{-x_2^2} + y_2 e^{-y_2^2} \end{bmatrix}.$$

After some elementary mathematical calculations, we obtain the eigenvalues of matrix $F(.)$ as

$$\lambda_1 = 1 + \frac{4}{2 + x_1^2 + y_1^2}, \lambda_2 = 1 + \frac{4}{2 + x_2^2 + y_2^2}.$$

Then,

$$1 \le \lambda_i(F(.)) \le 3, \quad (i = 1, 2, 3, ...).$$

Next, the Jacobian matrix of vector $H(X)$ is given by

$$J_H(X) = \begin{bmatrix} 2 + \sin x_1 & 0 \\ 0 & 3 + \sin x_2 \end{bmatrix}.$$

Hence, we obtain the bounds of the eigenvalues of the matrix H as

$$1 \le \lambda_i(J_H(X)) \le 4, \quad (i = 1, 2, 3, ...).$$

Furthermore, we obtain the eigenvalues of matrix $Q(Y)$ as the following:

$$\lambda_1 = 3 + \frac{8}{1 + e^{y_1^2}}, \lambda_2 = 3 + \frac{8}{1 + e^{y_2^2}}.$$

Thus, it is clear that

$$3 \le \lambda_i(Q(Y)) \le 7, \ (i = 1, 2, 3, ...).$$

Thus, all the conditions of Theorem 3.1 and Theorem 3.4 are satisfied. Then, for the particular case of (2), the zero solution of the considered system is asymptotic and uniform stable and all solutions of the same system are integrable when $P(.) = 0$.

Finally, it is obvious that

$$\|P(.)\| \le e^{-5t}(1 + \|X\| + \|Y\|) = \theta(t)(1 + \|X\| + \|Y\|),$$

where

$$\theta(t) = e^{-5t} \text{ with } \int_0^\infty \theta(s)ds = \int_0^\infty e^{-5s}ds = \frac{1}{5}.$$

Hence, all the conditions of Theorem 3.2 and Theorem 3.3 hold. Thus, we arrive that for the particular case of the system (2), all solutions are bounded and ultimately bounded when $P(.) \neq 0$.

4. Conclusion

In this study, we consider a general system of ODEs of second order. The considered system of ODEs of second order is more general and includes several scalar ODEs and systems of ODEs of second order, which can be found in the database of the literature. In this paper, we investigate various qualitative concepts of solutions of the considered system, which are known as the asymptotic stability (AS), the uniform stability (US), the boundedness, the ultimately boundedness (UB) and the integrability of solutions. Here, four new results including sufficient conditions, are proved regarding these concepts using the second method of the Lyapunov. To achieve the aim of this paper, we define a new Lyapunov function. In particular, the applications of the new results are verified by a numerical example. The results of this work are new and have new contributions to the qualitative theory of ODEs of second order.

Conflict of interest

The author declares no competing financial interest.

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