Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Complex Roots-Finding Method of Nonlinear Equations

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Received: 16 May 2023; **Revised:** 24 August 2023; **Accepted:** 30 August 2023

Abstract: Various researchers developed numerical methods to solve nonlinear equations. But still, there is a need to improve these methods to get accurate results. Some techniques exist where convergence is not assured. These methods increase the problem's cost using a second/higher-order derivative. So, cost efficiency is also considered an issue in many fast-converging ways. This paper proposes a new numerical method to find complex and real roots of nonlinear equations. By taking an initial guess $w_0 \in C$, the new approach contributes to a series of iterations converging to a real or complex solution. Moreover, the method evaluates complex roots even if a real number is taken as an initial guess. This innovation presents that the proposed method achieves second-order convergence. The number of iterations is also determined to check the performance of the new iterative scheme. Using Python 3.10.9 package, the authors have tested the method's efficacy on several numerical problems, presented the results in the tables, and illustrated them with the help of graphs.

***Keywords*:** transcendental equations, order of convergence, nonlinear equations, complex solution, numerical method, innovation

MSC: 65H04

1. Introduction

Different fields of science and engineering work on several severe problems [1–20] that demand obtaining solutions for nonlinear equations of a scalar function. Scientists generally prefer iterative methods like Newton Raphson's method, Halley's way, Cauchy's form, and more. As a result, numerical algorithms created to solve nonlinear equations using the previously mentioned methods have become increasingly common in modern research. Still, some of the algorithms have high computational costs. Various researchers proposed new strategies to improve the computational cost.

Ostrowski [21] proposed two methods to attain the roots of nonlinear equations, which require calculating two functions and one derivative per iteration. The methods have third and fourth order of convergence. In addition to the above, Traub [22] developed a technique that requires evaluation of the function and two first derivatives at every step, and the process has convergence order three. Afterward, Sharma and Guha [1], with the help of one parameter (i.e., 'a'), obtained a three-step method having six orders of convergence for solving $g(x) = 0$ where $g(x)$ is a nonlinear equation.

DOI: https://doi.org/10.37256/cm.5320243047

$$p_{n+1} = p_n - \frac{h(p_n)}{h'(p_n)} \tag{1}$$

$$t_{n+1} = w_n - \frac{g(w_n)}{g'(t_n)} \frac{g(t_n)}{g(t_n) - 2g(w_n)} \tag{2}$$

$$\widetilde{t_{n+1}} = t_{n+1} - \frac{g(t_{n+1})}{g'(t_n)} \frac{g(t_n) + ag(w_n)}{g(t_n) + (a-2)g(w_n)} \tag{3}$$

Further, Melman [2], with the help of the two-step Newton method, computed the largest or smallest zero of a polynomial with all absolute zeroes. Maheshwari [3] also employed the fourth-order method to evaluate the solution of linear and nonlinear equations. The technique requires less number of iterations and few functional calculations, and the value of absolute error is also less.

$$w_n = t_n - \frac{h(t_n)}{h'(t_n)} \tag{4}$$

$$t_{n+1} = t_n + \frac{1}{h'(t_n)} \left[\frac{\{h(t_n)\}^2}{h(w_n) - h(t_n)} - \frac{\{h(w_n)\}^2}{h(t_n)} \right] \tag{5}$$

Popovski [4] analyzed a three-step method requiring the calculation of one derivative and three functions per iteration. The convergence order of the way is seven. In addition to the above techniques, Singh and Gupta [5] introduced a fourth-order scheme to find the simple root of nonlinear equations. Furthermore, Sharma and Bahl [6] proposed a six-order iterative method that finds the real roots of nonlinear equations. They generated a sequence of iterations by assuming an initial approximation. The total number of iterations and functions evaluated measured the method's performance.

Further, Carlos and Cadenas [7] came down with a new family of Newton-Chebyshev-type schemes to attain the solution for nonlinear equations. They considered quadratic polynomials and evaluated the fixed and critical points. Their study also includes stable and unstable behavior and parameter space. Moreover, the Hansen-Patrick family [8] discussed the method of three convergent orders, defined as

$$t_{n+1} = t_n - \left[\frac{\alpha + 1}{\alpha \pm \left(1 - (\alpha + 1)H^{\frac{1}{2}}\right.} \right] \frac{h(u_n)}{h'(u_n)} \tag{6}$$

Where $H = \dfrac{h''(x_n)\, h(x_n)}{[h'(x_n)]^2}$, $\alpha \in R$ is a widely known method. The family consists of various other methods like the Euler method, Laguerre's method, Halley's method, and Newton-Raphson method. All the methods have three convergence orders except Newton's method. Even with cubic convergence, the above approach is less practical due to its computational cost. The prerequisite is the estimation of the derivative of the second order. Also, based on the Hansen-Patrick method, Sharma et al. [9] developed an improvised double-parameter method. All the plans have convergence order three, except one way, which has four convergence orders. Further, Abbasbandy [10] applied a modified Adomian decomposition method to construct an algorithm to find the solution for a system of two nonlinear variable equations. His system is a modification of Newton's method. Later, Parhi and Gupta [11] developed a process with no requirement for second-order derivatives and was used to find real roots of nonlinear equations. Calculation of two functions and two first-order derivatives per iteration is required. The method has six orders of convergence. 1.565 is the efficiency index. The procedure requires the evaluation few numbers of iterations than Newton's method. Noor and Waseem [12] used quadrature formulas to analyze two new two-step iterative methods. They found roots for a series of nonlinear equations. The techniques have cubic convergence. The efficiency index for the process is $3^{1/\left(n+4n^2\right)}$ for $n \geq 2$. Additionally, Sharma and Sharma [13] developed a technique for analyzing the roots of nonlinear equations with different assortments. The method needs evaluation of three functions. The convergence order is four.

$$w_n = t_n - \frac{2m}{m+2}\frac{f\left(t_n\right)}{f'\left(t_n\right)} \tag{7}$$

$$t_{n+1} = w_n - \frac{\frac{1}{2}m(m-2)\left(\frac{m}{m+2}\right)^{-m} f'\left(w_n\right) - \frac{m^2}{2} f'\left(w_n\right)}{f'\left(t_n\right) - \left(\frac{m}{m+2}\right)^{-m} f'\left(w_n\right)} \frac{f\left(t_n\right)}{f'\left(t_n\right)} \tag{8}$$

Where m is the diversity of roots, Mitlif [14] proposed a three-step method to find the roots of a nonlinear equation. The method's convergence order is 5.

$$w_n = t_n - \frac{h\left(t_n\right)}{h'\left(t_n\right)} \tag{9}$$

$$u_n = w_n - \frac{2h\left(w_n\right)h'\left(w_n\right)}{2h'^2\left(w_n\right) - h\left(w_n\right)h''\left(w_n\right)} \tag{10}$$

$$t_{n+1} = w_n - \frac{2\left[h\left(w_n\right)+h\left(u_n\right)\right]h'\left(w_n\right)}{2h'^2\left(w_n\right) - \left[h\left(w_n\right)+h\left(u_n\right)\right]h''\left(w_n\right)} \tag{11}$$

Fang et al. [15] developed an algorithm to attain the solution of nonlinear equations by a modified quasi-Newton method. Sharma and Kumar [16] obtained a technique with a convergence order of 8 with the help of four evaluations per iteration. Malhotra et al. [17] analyzed fatigue failure. Gong et al. [18] conducted a survey to attain as many roots of a nonlinear equation by Intelligent Optimization Algorithms. Al-Obaidi and Darvishi [19] proposed a new multi-step frozen Jacobian repetitive method technique. The convergence order of the method is 3. The process is highly efficient. Bayrak et al. [20] used the Taylor series' fractional derivative and fractional expansion to construct a modified Newton-Raphson method. They attained the first and second-order fractional Newton-Raphson method.

Furthermore, Ahmad and Singh [23] proposed a four-step iterative method based on Newton, Halley, Householder, and Steffensen methods. The technique has the 36th order of convergence. Jin et al. [24] introduced a fractional differential operator into the uncertain barrier swaption model. Cao et al. [25] discussed the algorithm for the dendritic neuron model to predict water quality. Malhotra et al. [26] addressed the reliability model using the Markov process. Also, Kumar et al. [27] used the machine learning method for learner-centric teaching. The techniques have the fifth and seventh order of convergence. Further, Singh and Sharma [28] developed two multi-step iterative methods using only two Jacobian matrices and one matrix inversion per iteration. Kaur et al. [29, 30] discussed the iterative methods for sixth and seventh-order differential equations.

Various researchers mentioned above developed numerical methods to solve nonlinear equations. But still, there is a need to improve these methods. These methods [14, 31] increase the problem's cost using a second-order derivative.

So, cost efficiency is also considered an issue in many fast-converging ways. Hence, in the present study, the authors proposed to develop a new iterative method to undertake a comprehensive approach. The study aims to find the complex roots of nonlinear equations. The analysis aids in identifying the intricate and authentic sources of nonlinear equations of various orders to obtain precision. The authors expressed the proposed method and its convergence.

The study aims to derive one new algorithm for solving nonlinear equations using quadratic equations. After the derivation, the authors also determine the order of convergence.

The advantages of the proposed method in contrast to existing practices are:

• The proposed algorithms offer a way to solve problems fast and efficiently.

• Less number of iterations are required to solve the problem.

• The proposed method is cost-effective as there is no need to evaluate higher-order derivatives.

• This method starts with a complex/real value as the initial guess, the user can find all real or complex roots, but some methods fail to see the same, e.g., Newton Raphson Method fails to find complex roots if the initial guess is a real one.

In addition to the above, the proposed study will lead to a better understanding of basic theory and will strengthen the subject of mathematical sciences. The authors have tested the method's efficacy on several numerical problems, presented the results in the tables, and illustrated them with the help of graphs using the Python 3.10.9 package.

2. Development of method

Let $g(y) = 0$ be a nonlinear equation.

$g(y)$ be a differential function in some interval $D \subset R \subset C$.

Consider a shifted origin parabolic equation.

$$g(y) = c_0 + c_1 y^2 \tag{12}$$

Let us take the n^{th} approximation from equation (12) we obtain.

$$g(y_n) = c_0 + c_1 y_n^2 \tag{13}$$

Differentiating equation (12) w.r.t 'y', we get

$$g'(y) = 2c_1 y \tag{14}$$

Putting $y = y_n$ in equation (14), we get

$$g'(y_n) = 2c_1 y_n \tag{15}$$

Let $y = y_{n+1}$ be an exact root of the equation (12), so

$$g\left(y_{n+1}\right)=c_0+c_1 y_{n+1}^2 \tag{16}$$

$$y_{n+1}^2=-\frac{c_0}{c_1} \tag{17}$$

Equations (13) and (15) are solved to get the values of c_0 and c_1.

$$c_0=\frac{2g\left(y_n\right)-y_n g'\left(y_n\right)}{2} \tag{18}$$

$$c_1=\frac{g'\left(y_n\right)}{2y_n} \tag{19}$$

Substituting these values, equation (17) becomes

$$y_{n+1}=\sqrt{y_n\left(y_n-\frac{2g\left(y_n\right)}{g'\left(y_n\right)}\right)} \tag{20}$$

Equation (20) is the proposed method.

3. Convergence analysis

Theorem Let $d \in D$ be a simple root of sufficiently differentiable function $f: D \rightarrow R$ in an open interval D. The method defined by equation (20) is of second order.

Proof. Let d be a simple root of $g(y)=0$ and substituting $y_n=d+b_n$ in equation (20), we get

$$b_{n+1}+d=\sqrt{\left(b_n+d\right)\left\{\left(b_n+d\right)-\frac{2g\left(b_n+d\right)}{g'\left(b_n+d\right)}\right\}} \tag{21}$$

With the help of Taylor's series expand $g(d+b_n)$ and $g'(d+b_n)$ about the point 'd', we get

$$g\left(d+b_n\right)=g(d)+g'(d)b_n+\frac{b_n^2}{2}g''(d)+O\left(b_n^3\right) \tag{22}$$

Note that $g(d)=0$, substitute in equation (22); we get,

$$g\left(d+b_n\right)=g'(d)b_n+\frac{b_n^2}{2}g''(d)+O\left(b_n^3\right) \tag{23}$$

$$g'(d+b_n) = g'(d) + g''(d) b_n + \frac{b_n^2}{2} g'''(d) + O\left(b_n^3\right) \tag{24}$$

Substituting the values of $g(d+b_n)$ and $g'(d+b_n)$ in equation (21), we get

$$b_{n+1} = \frac{1}{2d}\left(d\frac{g''(d)}{g'(d)} - 1\right) b_n^2 + O\left(b_n^3\right) \tag{25}$$

On omitting b_n^3 and the higher power of b_n, we get

$$b_{n+1} = C\left(b_n^2\right), \text{ where } C = \frac{1}{2a}\left(a\frac{g''(\alpha)}{g'(\alpha)} - 1\right) \tag{26}$$

Hence, the proposed method has second order of convergence.

Different researchers observed nonlinear equations in various fields of applied sciences, such as astrophysics mechanics, traffic flow, and quantum physics. They worked on other techniques to solve these equations. But only a few studies have been done to evaluate complex roots. Therefore, the proposed research work will serve as a tool to find the complex roots of nonlinear equations. Fast convergence and cost-effectiveness are the second factors in developing a new algorithm. The method is general, so other researchers can use it to solve equations of any order. The authors used different algebraic and transcendental equations to prove the findings. They performed coding and plotted the graphs using Python. Some of the examples are given in the next section.

4. Numerical examples

Here, the authors have shown the performance of the new proposed numerical method to solve nonlinear equations by taking some examples.

Table 1 shows the attribute of the Proposed Method: if the initial approximation is a real number, the proposed method gives the complex solution in less iteration.

Table 1. Take $w_0 = 2$ as an initial guess; complex root of the polynomial is

$f(w)$	α	Proposed method
$w^3 - 9w + 28$	$2 + 1.7321i$	6
$w^3 + 3w^2 + 24w + 364$	$2 + 6.9282i$	5
$w^4 - 6w^3 - 3w^2 - 24w - 28$	$2i$	4
$w^4 + 4w^3 + 9w^2 + 4w + 8$	i	7
$w^6 + 2w^2 - 3w + 8$	$0.1428 - 1.21.6i$	9

Table 2 shows the efficiency of the Proposed Method that if the initial guess is a complex number, the proposed method again gives the complex solution in a lesser number of iterations.

Table 2. Take complex value $w_0 = 1+i$ as an initial guess. The complex root of the polynomial is

$f(w)$	α	Proposed method
$w^3-9w+28$	$2+1.7321i$	4
$w^3+3w^2+24w+364$	$2+6.9282i$	4
$w^4-6w^3-3w^2-24w-28$	$2i$	4
$w^4-18w^3+78w-18w+77$	$-i$	4
w^5+w-5	$0.4913+1.2701i$	5

5. Graphical illustration

The proposed technique is a powerful method to find nonlinear equations' real or complex roots. Figure 1 illustrates that the root of the function $g(w) = \sin(w) - 0.5w$ is 1.895494 when the initial guess is $w_0 = 1.5$. The authors found two values, 'w' (1.5 and 2.5), where respective values of $g(w)$ are opposite in sign. Using the intermediate value theorem, if a function value changes from +ve to $-$ve or from $-$ve to +ve, a matter of 'w' must exist where the functional value $g(w)$ vanishes. Thus, the authors took the initial value of 1.5 because of the less absolute value of $g(1.5)$ than $g(2.5)$. Initially, the function $g(w)$ has a positive value at 1.5, then slightly decreases and approaches zero. The value where $g(w)$ vanishes is called the root of equations. The authors found that the number of iterations is less in the proposed method than in the existing process.

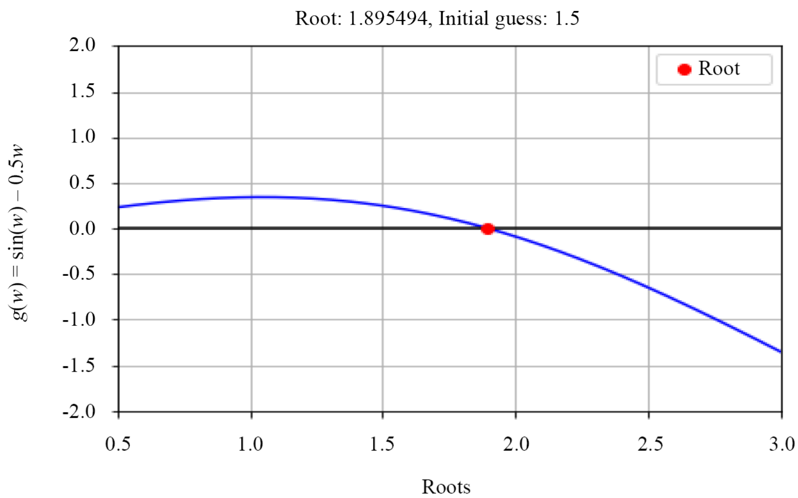

Figure 1. Roots 'w' versus '$g(w) = \sin(w) - 0.5w$'

Figure 2 demonstrates that the root of the function $g(w) = w^3 + 4w^2 - 10$ is 1.365230 when the initial guess is $w_0 = 2$. The authors found two values, 'w' (-1 and 2), where respective values of $g(w)$ are opposite in sign. Using the intermediate value theorem, if a function value changes from +ve to $-$ve or $-$ve to +ve, a value of 'w' must exist where the functional value $g(w)$ vanishes. Thus, the authors took the initial value of 2. Initially, the function $g(w)$ has a positive value at 2, then slightly decreases and approaches zero. The value where $g(w)$ vanishes is called the root of equations. Here also, the authors found that the number of iterations is less in the proposed method than in the existing ones.

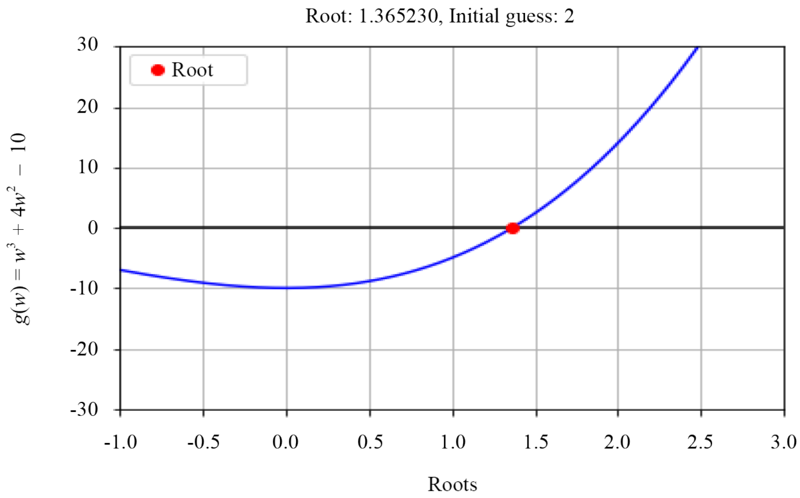

Figure 2. Roots 'w' versus '$g(w) = w^3 + 4w^2 - 10$'

Finally, Figure 3 shows that the root of the function $f(w) = \log(w) - 1$ is $w = 2.718282$ when the initial guess is $w_0 = 0.5$. The authors determined the root in the same manner and got the value. They compared the result with existing methods and found that the number of iterations in the proposed process is less than the earlier methods.

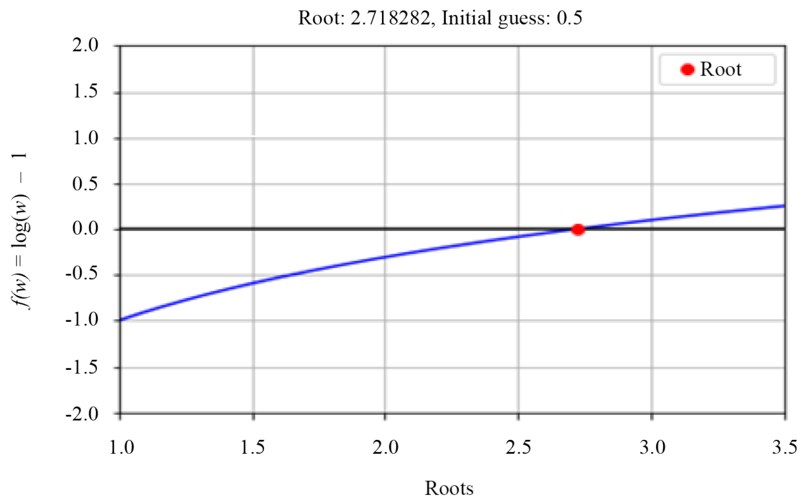

Figure 3. Roots 'w' versus '$f(w) = \log(w) - 1$'

Thus, the authors have taken different types of functions (algebraic/transcendental/polynomials) and found the proposed study best suited to finding real and complex roots.

6. Conclusion

In this paper, the authors propose a new numerical method to find complex and real roots of nonlinear equations. Firstly, they develop a new algorithm considering a quadratic equation to solve nonlinear equations. They also assess the convergence order. This technique has second order of convergence. Moreover, the process evaluates complex roots even if a real number is an initial guess. The number of iterations is also determined to check the performance of the new iterative scheme.

In contrast to existing methods, the proposed algorithm offers a way to solve problems fast and efficiently. It proves less number of iterations are required to solve the problem. The authors performed coding and plotted graphs to determine the roots using Python 3.10.9 package. The proposed method is cost-effective as there is no need to evaluate higher-order derivatives. It's a general approach that can solve any nonlinear equation. Graphical analysis will help to visualize the results more carefully.

In addition to the above, the proposed study will lead to a better understanding of basic theory and will strengthen the subject of mathematical sciences.

Conflict of interest

The authors declare no competing financial interest.

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