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Research Article

Solution of Linear PDEs Obtained Using the Laplace Transform

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Abstract: In this research, the solution of linear partial differential equations (PDEs) is obtained using the Laplace transform. The solution of linear PDEs using the Laplace transform has the advantage that an arbitrary constant is specifically displayed as an initial value. Even if the Laplace transform used as a tool is replaced with many other Laplace-type transforms, the attempted method is valid. Incidentally, the solution to Example 2 is obtained using ChatGPT and compared with the exact solution.

***Keywords*:** Laplace transform, integral transform, linear PDEs, ChatGPT

MSC: 44A05, 44A10

1. Introduction

The method of integral transform is generally used to find the solution of ordinary differential equations (ODEs). It is also a reality that this method is not suitable for solving nonlinear equations. Then, we would like to find out to what extent this method can be applied. With this intention, we will try to apply this method to solve linear partial differential equations (PDEs). Consequently, the proposed method has the advantage that it can provide a means to express an arbitrary constant, more specifically as an initial value.

Consider a linear PDE

$$u_{xx} - u = 0,$$

where $u = u(x, y)$. The solution of this equation can be obtained by substituting $u(x, y) = F(x)G(y)$ and separating variables. Another method is a method by trial solution. To show this, consider a plausible trial solution

$$y = e^{\lambda x}.$$

By substitution, we get $\lambda = \pm 1$ and so the bases are e^x and e^{-x}.

This gives the solution

$$u(x, y) = C(y)e^x + D(y)e^{-x}, \tag{1}$$

where C and D are arbitrary functions.

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In this research, for examples of linear PDEs, the solution is obtained by the Laplace transform. We also cover the Laplace transform of u_{xt}, which is easy but not well covered. And the Laplace transform of a two-dimensional heat equation for a steady-state problem is also covered. Incidentally, Example 2 is compared with the solution obtained with ChatGPT [1]. The structure and properties of the generalized Laplace transform were studied in [2]. Laplace transforms of generalized hypergeometric functions [3] were studied, and as an application of the generalized Laplace transform, certain solutions of Abel's integral equations on distribution spaces via the distributional G_α-transform [4] were also studied. Of course, the method used here can be used for Laplace-typed transforms [2-4]. As a related application, brand-new exact solutions to the Sawada-Kotera equation were investigated [5]. Related applied research has been conducted on the Laplace transform and its generalized form. As an example of related applied research, immersed coupled fractional models in spheres and oscillatory pendulums were investigated [6], Laplace-based methods in linearized dynamical models in the presence of a Hilfer fractional operator were studied [7], and general solutions of ODEs related to Chebyshev polynomials of the second kind were also studied [8].

However, since there is no previous study directly related to this proposed method, this study is to proceed with creativity.

2. Solution of linear PDEs obtained using Laplace transform

Let us find the solution of the above equation by the Laplace transform.

Example 1. Find the solution of $u_{xx} - u = 0$.

Taking the Laplace transform on both sides, we get

$$s^2U - su(0, y) - u_x(0, y) - U = 0,$$

where $U = £(u)$. Organizing the equality, we have

$$(s^2 - 1)U = su(0, y) + u_x(0, y).$$

This gives

$$U = \frac{su(0, y) + u_x(0, y)}{s^2 - 1}.$$

Thus, we have the solution

$$\begin{aligned} u &= u(0, y)£^{-1}\left(\frac{s}{s^2 - 1}\right) + u_x(0, y)£^{-1}\left(\frac{1}{s^2 - 1}\right) \\ &= u(0, y)\cosh x + u_x(0, y)\sinh x, \end{aligned} \tag{2}$$

where h is the hyperbolic function.

Since $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$, it is easy to see that (1) and (2) are equivalent. Next, let us check if (2) is a solution of the given equation $u_{xx} - u = 0$.

Since

$$u_x = u(0, y)\sinh x + u_x(0, y)\cosh x$$

and $u_{xx} = u(0, y)\cosh x + u_x(0, y)\sinh x = u$, we have

$$u_{xx} - u = 0.$$

Finally, let us see if the expression of (2) has an advantage over the expression of (1). The solution (2) has the advantage that the arbitrary function of (1) is specifically expressed by the initial value.

Let us look at another example $u_{xy} + u_x = 0$. This can be written as $\frac{u_{xy}}{u_x} = -1$. By integration, we get

$$1_n \, |u_x| = -y + c_1 x,$$

and so $u_x = e^{-y+c_1(x)} = c(x)e^{-y}$, where $c(x) = e^{c_1(x)}$. By integration with respect to x, we have

$$u = p(x)e^{-y} + q(y)$$

where $p(x) = \int c(x)dx$ and p and q are arbitrary. Let us use the Laplace transform again to find the solution.

Theorem 1. Calculate the Laplace transform of u_{xt}.

Proof. Let us put $U = £(u)$. Then,

$$£(u_{xt}) = £((u_x)_t) = s£(u_x) - u_x(x,0) = s\frac{\partial U}{\partial x} - u_x(x,0)$$

because of

$$£(u_x) = \int_0^\infty e^{-st}\frac{\partial u}{\partial x}dt = \frac{\partial}{\partial x}\int_0^\infty e^{-st}u(x,t)dt = \frac{\partial U}{\partial x}.$$

Here, let us consider whether we can change the order of integration and differentiation. Since e^{-st} is continuous, the order can be changed by the result of [9]. Details can be found in [9].

Example 2. Find the solution of $u_{xy} + u_x = 0$.

Taking the Laplace transform, we get $£(u_{xy}) + £(u_x) = 0$ because of $£(0) = 0$. From Theorem 1, we get

$$s\frac{\partial}{\partial x}U - u_x(x,0) + \frac{\partial}{\partial x}U = 0,$$

where $U = £(u)$. Organizing the equality, we get

$$(s+1)\frac{\partial U}{\partial x} = u_x(x,0),$$

thus

$$\frac{\partial U}{\partial x} = \frac{u_x(x,0)}{s+1}.$$

Integrating the equation with respect to x, we have

$$U = \frac{1}{s+1}\int u_x(x,0)dx + g(y),$$

where g is arbitrary. Therefore, the solution is

$$u = \int u_x(x,0)dx£^{-1}(\frac{1}{s+1}) + £^{-1}(g(y)) = f(x)e^{-y} + r(y)$$

where $f(x) = \int u_x(x,0)dx$ and $r(y) = £^{-1}(g(y))$.

The fact that this is a solution to the given equation is easily seen by substitution.

Lemma 2. Lagrange's [10] method states that a particular solution y_p of $y'' + p(x)y' + q(x)y = r(x)$ on open interval I is

$$y_p(x) = -y_1\int \frac{y_2 r}{W}dx + y_2\int \frac{y_1 r}{W}dx$$

where y_1 and y_2 form a basis of solutions of the corresponding homogeneous equation $y'' + p(x)y' + q(x)y = 0$ on I, and W is the Wronskian of y_1 and y_2.

Theorem 3. (Laplace transform of a two-dimensional heat equation for a steady problem.) Since $u_t = 0$ in the steady problem, the two-dimensional heat equation becomes $\nabla^2 u = u_{xx} + u_{yy} = 0$. Then, the solution can be expressed as

$$\begin{aligned} u &= u(0,y)\cos sx + \frac{1}{s}\sin sx \\ &+ £^{-1}\left[\frac{\sin sx}{s}\int \cos sx\ [su(x,0) + u_y(x,0)]dx\right] \\ &- £^{-1}\left[\frac{\cos sx}{s}\int \sin sx\ [su(x,0) + u_y(x,0)]dx\right]. \end{aligned} \tag{3}$$

Proof. Taking the Laplace transform with respect to y on $\nabla^2 u = 0$, we get

$$\frac{\partial^2 U}{\partial x^2} + s^2 U = su(x,0) + u_y(x,0),$$

where $U = £(u)$. If the solution of the left-hand side is obtained similarly to the above examples, then the homogeneous solution is

$$u_h = u(0,y)\cos sx + \frac{1}{s}\sin sx.$$

Next, using Lagrange's method of Lemma 2, find a particular solution on the right-hand side.

$$\begin{aligned} U_p &= -\cos sx\int \frac{\sin sx}{s}\ [su(x,0) + u_y(x,0)]dx \\ &+ \sin sx\int \frac{\cos sx}{s}\ [su(x,0) + u_y(x,0)]dx \\ &= \frac{\sin sx}{s}\int \cos sx\ [su(x,0) + u_y(x,0)]dx \\ &- \frac{\cos sx}{s}\int \sin sx\ [su(x,0) + u_y(x,0)]dx, \end{aligned}$$

where $U = £(u)$. Thus, the solution can be expressed as (3).

Note that since $\cos sx$ and $\sin sx$ are variables here, it cannot come out of integral. Here, the particular solution looks rather complicated. However, if the initial conditions are given, it can be expressed simply. Here, $u(0, y)$ can be regarded as a constant because it expresses an arbitrary constant as an initial value.

As discussed above, the solution of linear PDEs by the Laplace transform is meaningful in that an arbitrary constant is expressed more specifically as an initial value. Of course, these results obtained by us can be easily extended to the generalized Laplace transform [9] or several Laplace-type transforms.

Remark. The answer by ChatGPT [1] was sought through Example 2. It is acknowledged that ChatGPT achieves considerable results in the general case. This test was attempted out of curiosity about how accurately the results of ChatGPT would be obtained in higher mathematics. As of April 2023, ChatGPT got the answer of Example 2 as

$$u(x, y) = f(x - y) + C,$$

where f is an arbitrary function and C is a constant. This is clearly the wrong answer. Because $u_x = f'(x - y), u_{xy} = -f''(x - y)$, and hence $u_{xy} + u_x \neq 0$. The intermediate process of detailed calculation is omitted due to data availability. Although ChatGPT gave the wrong result for Example 2, it is believed that it will give an appropriate solution over time. The reason is that ChatGPT learns to get more accurate answers by constantly updating the weights. In the near future, it is expected that the numerical analysis method using artificial intelligence will become a research field in mathematics.

3. Conclusion

The research objective is to determine the advantages of the method of solving linear PDEs using the Laplace transform. There are many cases in which the general solution cannot be obtained from the given conditions in various engineering problems. In this case, this research can provide a means to express an arbitrary constant more specifically as an initial value.

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Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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