The Moments of Real Analytic Functions

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Abstract: We show that if f is real analytic in [0, 1] and it is not the restriction of a periodic real analytic function of period 1, then for all $\alpha \in \mathbb{C}$

$$\limsup_{n\to\infty} \left| \int_0^1 (f(t)-\alpha)^n \, dt \right|^{1/n} > 0$$

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1. Introduction

In a recent article, Müger and Tuset [9] considered the behavior of the moments of a non constant complex polynomial f,

$$M_n(f) = \int_0^1 (f(t))^n dt \tag{1}$$

as $n \to \infty$ showing that

$$\limsup_{n\to\infty} |M_n(f)|^{1/n} > 0 \tag{2}$$

or, equivalently, that the power series

$$\sum_{n=0}^{\infty} M_n(f) z^n \tag{3}$$

do not define an entire function. Their results complement those of Duistermaat and van der Kallen [3] who considered integrals of rational functions over the unit disc and employing this were able to solve a conjecture of Mathieu [7].

In this article we show that if f is real analytic in [0, 1] and it is not the restriction of a periodic real analytic function of period 1, then (2) holds. Our method is based on a simple but useful property of certain points with respect to analytic functionals.

2. Analytic functionals

In this section we recall some basic ideas about analytic functionals. Details can be found in the textbooks [1, 8].

Let U be an open set in $\mathbb{C}$. We denote by $\mathfrak{D}(U)$ the space of analytic functions defined on U. The topology of $\mathfrak{D}(U)$ is that of uniform convergence on compact subsets of U, i.e., the topology generated by the family of seminorms $\|\varphi\|_K = \max\{|\varphi(z)| : z \in K\}$, for K a compact subset of U and $\varphi \in \mathfrak{D}(U)$. Since we can find a sequence of compact subsets of U, $\{K_n\}_{n=1}^{\infty}$, with $K_n \subset \operatorname{int}(K_{n+1})$, $\bigcup_{n=1}^{\infty} K_n = U$, it follows that $\mathfrak{D}(U)$ is a Fréchet space, actually a strict projective limit of Banach spaces.

Let us now consider the dual space $\mathfrak{D}'(U)$, the space of analytic functionals in U. Since $\mathfrak{D}(U)$ is a Fréchet space, if $T \in \mathfrak{D}'(U)$ then there are compact subsets K of U such that T is continuous with respect to the norm $\| \; \|_K$ and consequently admits an extension to $C(K)$. In other words, there are Radon measures on K, μ, not unique in general, such that

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$$\langle T(\omega), \phi(\omega)\rangle = \int_K \phi(\omega) \mathrm{d}\mu(\omega) \ , \ \phi \in \mathfrak{O}(U) \tag{4}$$

In such a case we say that K is a carrier of T. Carriers may seem similar to the supports used in distribution theory [5], but they are actually quite different, since the analytic functionals may have disjoint carriers, as the Cauchy formula already shows for the delta functional $T(\omega) = \delta(\omega - \alpha)$

$$\phi(\alpha) = \langle \delta(\omega - \alpha), \phi(\omega)\rangle = \frac{1}{2\pi i}\int_C \frac{\phi(\omega)\mathrm{d}\mu(\omega)}{\omega - \alpha}, \ \phi \in \mathfrak{O}(U) \tag{5}$$

for any closed curve C in U that goes around once. The set C is a carrier, but so is $\{\alpha\}$. Analytic functional do not have minimal carriers, in general, but minimal carriers with certain extra properties, as minimal convex carriers [6], are sometimes well defined.

In this article we are interested in the dual space $\mathfrak{O}'(\mathbb{C})$, the space of analytic functionals with compact carriers.

A subset S of a topological space X is called locally closed if each $x \in S$ has a neighborhood in X, V_x, such that $S \cap V_x$ is closed in V_x. It can be shown that S is locally closed in X if and only if there exist an open set U and a closed set F such that $S = U \cap F$. If S is locally closed in X, we say that U is an open neighborhood of S if U is open in X and S is closed in U. We denote the set of open neighborhoods of S as $\mathrm{N}(S)$.

If S is locally closed in $\mathbb{C}$ then $\mathfrak{O}(S)$ is the space of germs of analytic functions defined on S. That is, a function φ defined on S belongs to $\mathfrak{O}(S)$ if and only if there exists $U \in \mathrm{N}(S)$ and an analytic function $\tilde{\varphi} \in \mathfrak{O}(U)$ such that, where π_S^U is the restriction operator from U to S. The system of topological vector spaces $\{\mathfrak{O}(U)\}_{U \in \mathrm{N}(S)}$ with operators $\pi_V^U : \mathfrak{O}(U) \to \mathfrak{O}(V)$ for $U \supseteq V$ is actually a directed system and thus we can give $\mathfrak{O}(S)$ the inductive limit topology. When K is compact, then $\mathfrak{O}(K)$ is a strict limit of Banach spaces. If $S \subseteq \mathbb{R}$ is open then $\mathfrak{O}(S)$ is the space of real analytic functions on S; while if $S \subseteq \mathbb{R}$ is locally closed then $\mathfrak{O}(S)$ is the space of germs of the real analytic functions on S.

If $S \subseteq \mathbb{C}$ is locally closed, then the dual space $\mathfrak{O}'(S)$ is called the space of the analytic functionals on S. When $K \subseteq \mathbb{R}$ is compact then $\mathfrak{O}'(K)$ is actually isomorphic to the space $\mathfrak{B}(K)$ of hyperfunctions defined on K, although hyperfunctions are usually constructed by using a dierent approach [8,11,12]. Observe that if $K \subseteq \mathbb{R}$ then the space of distributions $T \in D'(\mathbb{R})$ whose support is contained in K, the space $\mathcal{E}'[K]$, is a subspace of $\mathfrak{B}(K)$.

If K is a compact subset of $\mathbb{C}$, and $T \in \mathfrak{O}'(K)$ then its Cauchy or analytic representation, denoted as $f(z) = \mathcal{C}\{T(\omega); z\}$, is the analytic function $f \in \mathfrak{O}(\overline{\mathbb{C}} \setminus K)$ given by

$$f(z) = \mathcal{C}\{T(\omega); z\} = \frac{1}{2\pi i}\left\langle T(\omega), \frac{1}{\omega - z}\right\rangle \tag{6}$$

Notice that the analytic representation satisfies

$$\lim_{z \to \infty} f(z) = 0 \tag{7}$$

According to a theorem of Silva [8], the operator C is an isomorphism of the space $\mathfrak{O}'(K)$ onto the subspace $\mathfrak{O}_0(\overline{\mathbb{C}} \setminus K)$ of $\mathfrak{O}(\overline{\mathbb{C}} \setminus K)$ formed by those analytic functions that satisfy (7). When $K \subseteq \mathbb{R}$ then the operator $\mathcal{C}$ becomes an isomorphism of the space of hyperfunctions $\mathfrak{B}(K)$ onto $\mathfrak{O}_0(\overline{\mathbb{C}} \setminus K)$.

The inverse operator $\mathcal{C}^{-1}$ is given as follows. Let $\varphi \in \mathfrak{O}(K)$, and let $\tilde{\varphi} \in \mathfrak{O}(U)$ be an analytic extension to some region $U \in \mathrm{N}(K)$; let C be a closed curve in U such that the index of any point of K with respect to C is one. Then if $f \in \mathfrak{O}_0(\overline{\mathbb{C}} \setminus K)$ we define $T = \mathcal{C}^{-1}\{f\} \in \mathfrak{O}'(K)$ by specifying its action on φ as

$$\langle T(\omega), \varphi(\omega)\rangle = -\oint_C f(\xi)\tilde{\varphi}(\xi)\, d\xi \tag{8}$$

Clearly $T = \mathcal{C}^{-1}\{f\}$ is defined if $f \in \mathfrak{O}(\overline{\mathbb{C}} \setminus K)$, but in this space $\mathcal{C}^{-1}$ has a non trivial kernel, namely, the constant functions [8, Section (4)].

If $T \in \mathfrak{O}'(K)$, then the power series expansion of its Cauchy representation at infinity takes the form

$$\mathcal{C}\{T(\omega); z\} = -\frac{1}{2\pi i}\sum_{n=0}^{\infty} \frac{\mu_n(T)}{z^{n+1}} \ , \ |z| > \rho \tag{9}$$

where

$$\mu_n(T) = \langle T(\omega), \omega^n \rangle \tag{10}$$

are the moments of T and where $\rho = \max\{|z| : z \in K\}$. Observe that $\mathcal{C}\{T(\omega); z\}$ is defined if $z \in \overline{\mathbb{C}} \setminus K$, but the series in (9) could be divergent if $|z| \leq \rho$.

3. Distributional analytic forms

Let C be a piecewise C^1curve in $\mathbb{C}$, not necessarily simple, given by $z = z(t)$, $a \leq t \leq b$. We say that C is closed if $z(a) = z(b)$, not closed otherwise. If C is not closed, simple means that $t \rightsquigarrow z(t)$ is injective in $[a, b]$, but if C is closed it means that it is injective in $[a, b)$.

We are interested in a certain kind of representation of the analytic functionals along a curve C, given by $z = z(t)$, $a \leq t \leq b$. Let g_0 be a non zero function defined on the interior of C that is analytic along the curve, that is, if $a < c < b$, and $\varepsilon > 0$ is small enough, then g_0 admits an analytic extension to a neighborhood of z ($[c - \varepsilon, c + \varepsilon]$). Notice that g_0 is not defined at the points where the curve intersects itself, but along each branch of the curve g_0 can be defined at such points in a way that the function is analytic on the branch. Notice also that even if $z(a) = z(b) = \xi$ then in general g_0 does not have an analytic extension to a neighborhood of ξ.

Definition 3.1 Let $T \in \mathfrak{D}'(\mathbb{C})$ be an analytic functional. If for all $\varphi \in \mathfrak{D}(\mathbb{C})$

$$\langle T, \varphi \rangle = \langle g, \varphi|_C \rangle_{D'(C) \times D(C)} \tag{11}$$

where g is a distribution of $D'(C)$ that equals g_0, an analytic function along $C \setminus \{z(a), z(b)\}$, in the interior of C, then we say that g is a distributional analytic form of T along C. Our notation for distributions is the standard one [5].

Our first result is based on the following observation: if C is not a closed curve, and T has a distributional analytic form along C then both $z(a)$ and $z(b)$ belong to any set Λ that carries T. In other words, if $f_0(z) = \mathcal{C}\{T(\omega); z\}$ on the outside of a disc, then f_0 cannot be analytically continued to a region that contains either of these points. Let us start with the case of a simple curve.

Lemma 3.2 Suppose C is a simple not closed piecewise C^1 curve from $z(a)$ to $z(b)$. Let T be an analytic functional that has a distributional analytic form along C. Let

$$f(z) = \mathcal{C}\{T(\omega); z\} = \frac{1}{2\pi i}\left\langle T(\omega), \frac{1}{z - \omega}\right\rangle \tag{12}$$

$z \in \mathbb{C} \setminus C$. Then f cannot be analytically continued to a region of the form $\mathbb{C} \setminus F$, F finite.

Proof. Let $r > 0$ be small enough such that C intersects the disc $\mathbb{D}_{z(a),r}$ on a simple curve from the boundary of the disc to its center, $z(a)$. Then f has a jump equal to g_0 along the open part of this intersection, and since g_0 is not zero, f does not admit continuous extensions to $\mathbb{D}_{z(a),r} \setminus \{z(a)\}$.

Our aim is to prove a corresponding result for arbitrary not closed piecewise C^1 curves. In order to do so we need to point out that the analyticity of g_0 allows us to obtain from an analytic form of T along a curve C other analytic forms of T along curves C_1 from $z(a)$ to $z(b)$ that are obtained by deforming C inside simply connected regions of analyticity of g_0. Before considering the general case, it is instructive to consider the case of a curve that intersects itself just once.

Lemma 3.3 Suppose C is a not closed piecewise C^1 curve from $z(a)$ to $z(b)$ that intersects itself once, at a point ξ. Let T be an analytic functional that has an analytic form along C. Let $f(z) = \mathcal{C}\{T(\omega); z\}$, z in the unbounded component of $\mathbb{C} \setminus C$. Then f cannot be analytically continued to a region of the form $\mathbb{C} \setminus F$, F finite.

Proof. Let us assume that f admits an analytic continuation to $\mathbb{C} \setminus F$, F finite, and see that this leads to a contradiction. Let $C = C_{z(a)} \cup C_\xi \cup C_{z(b)}$, where $C_{z(a)}$ is the part from $z(a)$ to ξ, C_ξ is the closed path that starts and ends at ξ, and $C_{z(b)}$ is the part from ξ to $z(b)$. The three parts are simple curves.

We shall consider the situation at $z(a)$. Let Λ_{ext} be the unbounded component of $\mathbb{C} \setminus C_\xi$, Λ_{int} the bounded component. The contradiction follows as in the Lemma 3.2 if $C_{z(a)} \subset \Lambda_{\text{ext}}$; in fact this remains true no matter how many times C intersects itself. Let us then suppose that $C_{z(a)} \subset \Lambda_{\text{int}}$. The function g_0 is analytic in $C_\xi \setminus \{\xi\}$, and may be continuous or have a jump discontinuity at ξ. Let us consider the function

$$h(z)=\frac{1}{2\pi i}\int_{C_\xi}\frac{g_0(\omega)d\omega}{\omega-z}\ , \qquad z\in\mathbb{C}\setminus C_\xi \tag{13}$$

We have

$$f(z) = h(z) + f_a(z) + f_b(z),\ z \in \Lambda_{\text{ext}}\ , \tag{14}$$

where f_a and f_b are the evaluations of the restrictions of $g(\omega)$ to $C_{z(a)}$ and $C_{z(b)}$ at the test function $(2\pi i(\omega-z))^{-1}$. Since the jump of h across $C_\xi \setminus \{\xi\}$ is g_0 and both f_a and f_b are continuous on $C_\xi \setminus \{\xi\}$, it follows that $h_{\text{int}} = h|_{\Lambda_{\text{int}}}$ has an analytic continuation to a neighborhood of $C_\xi \setminus \{\xi\}$ and in that neighborhood,

$$g_0(z) = h_{\text{int}}(z) - f(z) \tag{15}$$

Therefore g_0 admits an analytic continuation to the region $\Lambda_{\text{int}} \setminus F$ for some finite set F, given that f does and h_{int} is analytic in Λ_{int}. Let us now deform C_ξ to another simple closed curve from ξ to itself, say $\tilde{C}_\xi$, within Λ_{int}, in such a way that $z(a)$ is in the exterior of $\tilde{C}_\xi$. Calling $\tilde{h}$ the corresponding integral over $\tilde{C}_\xi$ as in (13) we would have

$$f(z)=\tilde{h}(z)+f_a(z)+f_b(z)+\sum_{a\in\tilde{F}}G_a\left(\frac{1}{z-a}\right), \qquad z \text{ outside } \tilde{C}_\xi \tag{16}$$

where $\tilde{F}\subset F$ and where G_a are entire functions. This contradicts what we already proved since for the curve $\tilde{C}=C_{z(a)}\cup\tilde{C}_\xi\cup C_{z(b)}$, part of $C_{z(a)}$ is in the unbounded component of $C\setminus\tilde{C}$.

We can now consider the general case.

Theorem 3.4 Suppose C is a not closed piecewise C^1 curve from $z(a)$ to $z(b)$. Let T be an analytic functional that has a distributional analytic form along C. Let $f(z)=\mathcal{C}\{T(\omega);z\}$ for z in the unbounded component of $\mathbb{C}\setminus C$. Then f cannot be analytically continued to a region of the form $\Omega\setminus\{z(a),z(b)\}$ if either $z(a)$ or $z(b)$ belongs to Ω.

Proof. Let us suppose that f can be analytically continued to a pointed neighborhood of one of the endpoints of C, say $z(a)$. Then there is a simply connected region Ω formed by a disc around $z(a)$, the exterior of another disc, and a neighborhood (which we can ask to have a C^1 boundary) of a path from the circle about $z(a)$ to the circle that bounds the disc at infinity, such that f can be defined as an analytic function in $\Omega \setminus \{z(a)\}$.

The open set $\Omega \setminus C$ has a finite number of components, separated by parts of C. Let Γ be a component adjacent to the unbounded component of $\Omega \setminus C$, separated by a part γ of C. Since $g_0(z) = f_\Gamma(z) - f(z)$ in a neighborhood of γ, where $f_\Gamma(z)=\mathcal{C}\{T(\omega);z\}$ for $z\in\Gamma$, and since both f and f_Γ are analytic in Γ, it follows that so is g_0. Therefore we can deform the path γ to the part of the boundary of Γ that bounds the rest of $\Omega \setminus C$ obtaining a new distributional analytic form for T with a new curve C_1 such that $\Omega \setminus C_1$ has one component less than $\Omega \setminus C$. Repeating this procedure we will eventually obtain a distributional analytic form along a curve C_n such that $\Omega \setminus C_n$ has just one component, and in fact $\Omega \cap C_n$ is a simple C^1 path from the boundary of Ω to $z(a)$. However, as we saw in the proof in the Lemma 3.2 in such a case f has a non zero jump across this path, and hence it does have continuous extensions to Ω : a contradiction.

3.1 Moments

If $T\in\mathfrak{O}'(\mathbb{C})$, then the power series expansion of its Cauchy representation at infinity takes the form

$$\mathcal{C}\{T(\omega);z\}=-\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{\mu_{n,\alpha}(T)}{(z-\alpha)^{n+1}} \tag{17}$$

for $|z-\alpha|$ large enough, where

$$\mu_{n,\alpha}(T)=\left\langle T(\omega),(\omega-\alpha)^n\right\rangle=\sum_{j=0}^{n}\binom{n}{j}(-\alpha)^{n-j}\mu_j(T) \tag{18}$$

are the translated moments of T.

If T has a distributional analytic form along a not closed piecewise C^1 curve from $z(a)$ to $z(b)$ then f cannot be analytically continued with a set of the form $\mathbb{C} \setminus F$, F finite. Therefore the radius of convergence of the series (17) cannot be infinite, this radius of convergence being at most the smallest of $|z(a)-\alpha|^{-1}$ and $|z(b)-\alpha|^{-1}$.

Theorem 3.5. Suppose C is a not closed piecewise C^1 curve from z (a) to z (b). Let T be an analytic functional that has a distributional analytic form along C. Then $\forall\, \alpha \in \mathbb{C}$

$$\limsup_{n\to\infty} |\mu_{n,\alpha}(T)|^{1/n} \geq \max\{|z(a)-\alpha|, |z(b)-\alpha|\} > 0 \tag{19}$$

Let us now consider the analytic functional T_φ given for Φ entire by

$$\langle T_\varphi(\omega), \Phi(\omega)\rangle = \int_0^1 \Phi(\varphi(t))dt \tag{20}$$

where φ is an analytic function in a complex region that contains [0, 1], or, equivalently, it is real analytic in an open interval that contains [0, 1]. The moments of the analytic functional T_φ are the moments of φ in the following sense,

$$\mu_n(T_\varphi) = \langle T_\varphi(\omega), \omega^n\rangle = \int_0^1 (\varphi(t))^n dt \tag{21}$$

Both interpretations of the word 'moment' are easy to find in the literature.

Our results immediately give the ensuing.

Theorem 3.6. Let φ be real analytic in [0, 1]. If $\varphi(0) \neq \varphi(1)$ then for all $\alpha \in \mathbb{C}$

$$\limsup_{n\to\infty} \left|\int_0^1 (\varphi(t)-\alpha)^n dt\right|^{1/n} \geq \max\{|\varphi(0)-\alpha|, |\varphi(1)-\alpha|\} > 0 \tag{22}$$

Proof. We may assume, by deforming the interval [0, 1] to another path from 0 to 1 if needed, that $\varphi'(t) = 0$ for $0 < t < 1$. Then

$$\langle T_\varphi(\omega), \Phi(\omega)\rangle = \int_C g_0(z)\Phi(z)\mathrm{d}z \tag{23}$$

where the curve C is given by $z = \varphi(t)$, $0 \leq t \leq 1$, and where

$$g_0(z) = \frac{1}{\varphi'(\eta(z))} \tag{24}$$

η being an appropiate branch of φ^{-1}, namely the one with $\eta(\varphi(t)) = t$ if $\varphi(t)$ is not a point where C intersects itself. This is a distributional analytic form of T_φ along the non closed curve C, and therefore (22) follows from (19).

4. Closed curves

Our next task is to consider the case of closed curves, when $z(a) = z(b) = \xi$. In such a case it may be the case that if $f_0(z) = \mathcal{C}\{T(\omega); z\}$ on the outside of a disc, then f_0 can be analytically continued to a region that contains ξ. Nevertheless, as we shall see, there are conditions that ensure that such analytic extension does not exist, but before we study them it is useful to give an example.

Example 4.1. Let us consider the curve $z(t) = e^{2\pi i t}$ for $0 \leq t \leq 1$. Let g_0 (ω) be a fixed branch of $\sqrt{1-\omega} - 1/\omega$. Then

$$\frac{1}{2\pi i}\int_C \frac{g_0(\omega)\mathrm{d}\omega}{\omega - z} = \frac{1}{z}, \quad |z| > 1 \tag{25}$$

admits an analytic continuation to $\mathbb{C}\setminus\{0\}$ even though g_0 is not real analytic at $\xi = 1$.

We now define a type of singularity at ξ that guarantees that the Cauchy transform cannot be analytically continued across a part of C that contains ξ.

Definition 4.2. We say that ξ is a two sided singularity of g if for no $r > 0$ the function g_0 admits an analytic continuation to one of the sides of $\mathbb{D}_{\xi,r} \setminus C$.

In the Example 4.1 the point $\xi = 1$ is not a two sided singularity. We shall now prove that no analytic continuation to

$\mathbb{C}\setminus F$, F finite, is possible if ξ is a two sided singularity of g.

Lemma 4.3. Suppose C is a simple closed piecewise C^1 curve from $z(a) = \xi$ to $z(b) = \xi$. Suppose ξ is a two sided singularity of g. Let T be an analytic functional that has the analytic form g along C. Let $f(z) = \mathcal{C}\{T(\omega); z\}$, z in the unbounded component of $\mathbb{C}\setminus C$. Then f cannot be analytically continued to a region of the form $\mathbb{C}\setminus F$, F finite.

Proof. Let $f_{\text{int}}(z) = \mathcal{C}\{T(\omega); z\}$, z in the bounded component of $\mathbb{C}\setminus C$. If f had an analytic extension to a region of the form $\mathbb{C}\setminus F$, F finite, then g_0, that equals $f_{\text{int}} - f$ in a neighborhood of $C\setminus\{\xi\}$, would also have an analytic extension to the bounded component minus F and thus ξ would not be singular from the inside side, a contradiction.

We may employ the same techniques of the Section 3 of deforming the path C along simply connected regions of analyticity of g_0 to obtain the ensuing general result.

Theorem 4.4. Suppose C is a closed piecewise C^1 curve from $z(a) = \xi$ to $z(b) = \xi$. Suppose ξ is a two sided singularity of g. Let T be an analytic functional that has the analytic form g along C. Let $f(z) = \mathcal{C}\{T(\omega); z\}$, z in the unbounded component of $\mathbb{C}\setminus C$. Then f cannot be analytically continued to a region of the form $\Omega\setminus\{\xi\}$ if ξ belongs to Ω.

4.1 Moments

The Theorem 4.4 allows us to conclude that if T has a two sided singularity at ξ then the moments cannot decrease very fast at infinity.

Theorem 4.5. Suppose C is a closed piecewise C^1 curve from $z(a) = \xi$ to $z(b) = \xi$. Suppose ξ is a two sided singularity of g. Let T be an analytic functional that has the analytic form g along C. Then $\forall\,\alpha\in\mathbb{C}$

$$\limsup_{n\to\infty} |\mu_{n,\alpha}(T)|^{1/n} > 0 \qquad (26)$$

In case $T = T_\varphi$ for a real analytic function from [0, 1] to $\mathbb{C}$ with $\varphi(0) = \varphi(1)$ then if g_0 given by (24) has an analytic extension to any side of a $\mathbb{D}_{\xi,r}\setminus C$ then it would follow that $\varphi^{(n)}(0) = \varphi^{(n)}(1)$ for all n and consequently φ would be the restriction of a periodic function of period 1 to [0, 1].

Lemma 4.6. Let φ be a real analytic function from [0, 1] to $\mathbb{C}$ such that $\varphi(0) = \varphi(1)$ but such that φ is not the restriction of a periodic real analytic function of period 1 to [0, 1]. Let $g_0(z) = 1/\varphi'(\eta(z))$, η being the branch of φ^{-1} with $\eta(\varphi(t)) = t$ if $\varphi(t)$ is not a point where C intersects itself. Then $\xi = \varphi(0) = \varphi(1)$ is a two sided singularity of g_0.

Proof. Functions that are the distributional boundary values of analytic functions cannot have certain types of singularities, particularly, they cannot have jumps, not even in the distributional or more general senses [2, 4, 10]. If g_0 is the boundary value of an analytic function from one side in a neighborhood of ξ, then so would $\varphi'(\eta(z))$, and consequently this function would not have a jump at ξ, that is, $\varphi'(0) = \varphi'(1)$. Using that $d/dz(\varphi'(\eta(z)))$ cannot have a jump at ξ we then obtain that $\varphi''(0) = \varphi''(1)$. An inductive argument, using higher order derivatives yields $\varphi^{(n)}(0) = \varphi^{(n)}(1)$ for all n, and this implies that φ is the restriction to [0, 1] of a periodic function of period 1.

We immediately obtain the following.

Theorem 4.7. Let φ be a real analytic function from [0, 1] to $\mathbb{C}$ such that $\varphi(0) = \varphi(1)$ but such that φ is not the restriction of a periodic real analytic function of period 1 to [0, 1]. Then for all $\alpha\in\mathbb{C}$

$$\limsup_{n\to\infty} \left|\int_0^1 (\varphi(t)-\alpha)^n\,dt\right|^{1/n} > 0 \qquad (27)$$

It would also be interesting to obtain a lower bound for the superior limit $\limsup_{n\to\infty}\left|\int_0^1 (\varphi(t)-\alpha)^n\,dt\right|^{1/n}$ similar to (22).

The Theorem 4.7 says that (27) is satisfied if φ is not the restriction of a periodic function of period 1 to [0, 1]. However, it is also true for some restrictions of the periodic functions. The analysis of when (27) holds for $\alpha = 0$ and φ a rational function of $e^{2\pi i t}$ can be seen in [3].

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