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Research Article

A New Inequality on Jain-Saraswat's Divergence for *S*-Convex Functions

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Abstract: In this article, a new inequality on Jain-Saraswat divergence measure is investigated for s-convex functions, which includes convex functions as a special case. Further, by using this inequality, some special results have also been derived in terms of the different divergences, at distinct values of s. Numerical verification of these results has also been discussed.

***Keywords*:** s-convex function, discrete probability distributions, new inequality, mlogarithmic power mean, identric mean, mathematical verification

MSC: 94A17, 26D15

1. Introduction

Divergence measures are basically measures of distance between two or more than two probability distributions or it is a measure of discrimination between probability distributions. Any arbitrary divergence measure $Ar(\Theta, \Phi)$ represents a natural distance measure from a true probability distribution Θ to an arbitrary probability distribution Φ. Typically, Θ represents an observation or a precisely calculated probability distribution, whereas Φ represents a model, a description, or an approximation of Θ.

Divergence measures are used effectively to resolve different problems in probability theory. The primary purpose of assessing how much information is contained in data is to quantify the amount of meaningful and useful content present in a given set of data. This assessment helps us understand the significance, relevance, and potential insights that can be derived from the data. In other words, it allows us to gauge the richness and value of the data in terms of the knowledge it can provide.

Divergence measures have been illustrated exceptionally valuable in a assortment of disciplines such as: guess of likelihood conveyances [1], choice making [2-3], design acknowledgment [4], examination of possibility tables [5], turbulence stream [6], Medical sciences [7-8], fuzzy sciences [9-10], etc.

Let $h : (0, \infty) \to (-\infty, \infty)$ be a real, differentiable and convex function. Also, $\gamma = \{\Theta = (\theta_1, \theta_2, \theta_3, ..., \theta_p) : \theta_r > 0, \sum_{r=1}^{p} \theta_r = 1\}$, $p > 1$ be the set of all complete finite discrete probability distributions with the assumption in limiting case $0h(0) = 0h\left(\frac{0}{0}\right) = 0$.

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In 1961 [11], Renyi introduced a divergence measure $RL_h(\Theta, \Phi) = \log\left[\sum_{r=1}^{p} h^{-1}\left\{\theta_r h\left(\frac{\theta_r}{\phi_r}\right)\right\}\right]$, after that in 1967 Csiszar [12] and Bregman [13] introduced $C_h(\Theta, \Phi) = \sum_{r=1}^{p} \phi_r h\left(\frac{\theta_r}{\phi_r}\right)$ and $B_h(\Theta, \Phi) = \sum_{r=1}^{p} h(\theta_r) - h(\phi_r) - (\theta_r - \phi_r)$ $h'(\phi_r)$, respectively. Further, Burbea and Rao [14] came with $BR_h(\Theta, \Phi) = \sum_{r=1}^{p} \frac{h(\theta_r) + h(\phi_r)}{2} - h\left(\frac{\theta_r + \phi_r}{2}\right)$, then in 1994 [15] Miquel Salicru defined the functional divergence $MS_h(\Theta, \Phi) = \sum_{r=1}^{p} \left[\sqrt{h(\phi_n)} - \sqrt{h(\theta_r)}\right]^2$. These all are functional or generalized divergence measures for comparing two discrete probability distributions Θ and Φ, at a time, $(\Theta, \Phi) \in \gamma \times \gamma$.

We can obtain several well known divergences by defining a suitable convex function in one of these generalized divergences, like Csiszar's divergence is very useful for generationg different divergences due to its compact formula. Such as: For the convex function $\frac{(u-1)^2}{(u+1)}$, we have Triangular discrimination [16], for $(u-1)\log\left(\frac{u+1}{2}\right)$, we have Relative J-divergence [17], for $(u - 1)^2$, we have Chi square divergence or Pearson divergence [18], similarly for the function $\frac{u+1}{2}\log\left(\frac{u+1}{2\sqrt{u}}\right)$, we have Arithmetic-Geometric mean divergence [19], also for $\frac{u}{2}\log u + \left(\frac{u+1}{2}\right)\log\left(\frac{2}{u+1}\right)$, we have Jensen-Shannon divergence or Capacitory discrimination [14, 20], for the function $u\log u$, we have the famous Kullback-Leibler divergence or Relative entropy or Directed divergence [21], and many more.

We may say that Csiszar divergence behaves like a generator of divergences by using the appropriate convex function as a generating function.

Similarly, in 2013 [22] Jain and Saraswat introduced the following generalized divergence measure:

$$S_h(\Theta, \Phi) = \sum_{r=1}^{p} \phi_r h\left(\frac{\theta_r + \phi_r}{2\phi_r}\right), \tag{1}$$

where θ_r and ϕ_r are probability mass functions corresponding to the discrete distributions Θ and Φ, respectively.

The following fundamental properties (theorem 1.1 and 1.2) on $S_h(\Theta, \Phi)$ can be accessed from the article [22]:

Theorem 1.1 If the given function h is convex and normalized in the interval $(0, \infty)$, i.e., $h''(u) \geq 0$ and $h(1) = 0$, then $S_h(\Theta, \Phi)$ and $S_h(\Phi, \Theta)$ are both non-negative and convex for the probability distributions $\Theta, \Phi \in \gamma$.

Theorem 1.2 Let $D = a_1h_1(u) \pm a_2h_2(u) \pm ... \pm a_ph_p(u)$, where $h_1, h_2, ..., h_p$ are the finite number of convex functions characterized within the interval $(0, \infty)$. Then $S_D(\Theta, \Phi) = a_1S_{h_1}(\Theta, \Phi) \pm a_2S_{h_2}(\Theta, \Phi) \pm ... \pm a_pS_{h_p}(\Theta, \Phi)$, where a_r is constant for each $r = 1, 2, ..., p$ and $(\Theta, \Phi) \in \gamma \times \gamma$.

Other properties, bounds, inequalities in terms of the different divergences and applications of $S_h(\Theta, \Phi)$ can be found in the articles [23-29].

In addition, the article [27] reveals the following relationship:

Theorem 1.3 Let $h: (0, \infty) \to \mathbb{R}$ be a real, convex function on $(\kappa, \zeta) \subset (0, \infty)$ with $0 < \kappa \leq 1 \leq \zeta < \infty$, $\kappa \neq \zeta$. For $\Theta, \Phi \in \gamma$, we have

$$S_h(\Theta, \Phi) \leq B_h(\kappa, \zeta), \tag{2}$$

where

$$B_h(\kappa, \zeta) = \frac{(\zeta - 1)h(\kappa) + (1-\kappa)h(\zeta)}{\zeta - \kappa}. \tag{3}$$

In this work, we will use the following generalized means (m-Logarithmic power mean (4) and Identric mean (5)), to summarize the long calculations.

$$L_m(\kappa, \zeta) = \begin{cases} \dfrac{\zeta^{m+1} - \kappa^{m+1}}{(m+1)(\zeta - \kappa)} & \text{if } m \neq -1,\ 0 \\ \dfrac{\log\zeta - \log\kappa}{\zeta - \kappa} & \text{if } m = -1,\ \kappa,\ \zeta > 0,\ \kappa \neq \zeta. \\ 1 & \text{if } m = 0 \end{cases} \tag{4}$$

$$I(\kappa, \zeta) = \begin{cases} \dfrac{1}{e}\left(\dfrac{\zeta^{\zeta}}{\kappa^{\kappa}}\right)^{\frac{1}{\zeta-\kappa}} & \text{if } \kappa \neq \zeta \\ \zeta & \text{if } \kappa = \zeta \end{cases} \quad \kappa,\ \zeta > 0. \tag{5}$$

The Definition 1.1, Remark 1.2 and Theorem 1.4 below can be found in the article [30].

Definition 1.1 Let Z be a linear space and s be a fixed positive real number, i.e., $s \in (0, \infty)$. Let $B \subset Z$ be a convex subset. Then, the mapping $h : B \to \mathbb{R}$ (Set of real numbers), is desginated as s-convex on B if

$$h(\delta x + \eta y) \leq \delta^s h(x) + \eta^s h(y), \tag{6}$$

for $\delta, \eta \geq 0$ with $\delta + \eta = 1$ and $x, y \in B$.

Further, for $y_r \in B$ and $\delta_r \geq 0\ \forall\ r = 1, 2, 3, ..., p$ with $\sum_{r=1}^{p} \delta_r = 1$, we have

$$h\left(\sum_{r=1}^{p} \delta_r y_r\right) \leq \sum_{r=1}^{p} \delta_r^s h(y_r). \tag{7}$$

Remark 1.1 At $s = 1$, the convex functions are precisely 1-convex functions or convex functions.

Remark 1.2 Generally, s-convex functions are different from functions that are convex.

(a). There are s-convex mappings in linear spaces which are not convex for some $s \in (0, \infty)$ with $s \neq 1$ (see Example 1.1).

(b). If $0 < s \leq 1$, every non-negative convex function defined on a convex set in a linear space is also an s-convex function. If $s \geq 1$, every non-positive convex function defined on a convex set in a linear space is also an s-convex function.

Example 1.1 Let Z be a normed linear space, also let $B = Z$ and $0 < s < 1$, define $h(x) = \|x\|^s$ for all $x \in B$. For each $x, y \in B$ and $\delta, \eta \geq 0$ with $\delta + \eta = 1$, when $\|\delta x\| = 0$, either $\delta = 0$ or $x = 0$, therefore $\delta x = 0$, and $h(\delta x + \eta y) = h(\eta y) = \|\eta y\|^s = \eta^s \|y\|^s = \eta^s h(y)$; when $\|\delta x\| \neq 0$, from the increasing behaviour of the function $g(u) = 1 + u^s - (1 + u)^s$ in the interval $[0, \infty)$, we have $g(u) \geq g(0) = 0$, $\forall t > 0$, then

$$h(\delta x + \eta y) = \|\delta x + \eta y\|^s \leq (\delta\|x\| + \eta\|y\|)^s = \delta^s \|x\|^s \left[1 + \frac{\eta\|y\|}{\delta\|x\|}\right]^s$$

$$\leq \delta^s \|x\|^s \left[1 + \left(\frac{\eta\|y\|}{\delta\|x\|}\right)^s\right] = \delta^s \|x\|^s + \eta^s \|y\|^s = \delta^s h(x) + \eta^s h(y).$$

Hence function h is s-convex on Z but not a convex function on Z with $0 < s < 1$.

Theorem 1.4 Let $h : B \to \mathbb{R}$ be a s-convex function, and $A_p = \sum_{r=1}^{p} \delta_r^{\frac{1}{s}}$ with $\delta_r \geq 0$, for any $1 \leq r \leq p$. Then we have

$$\frac{1}{A_p^s} \sum_{r=1}^{p} \delta_r h(\lambda_r) \geq h\left(\frac{1}{A_p} \sum_{r=1}^{p} \delta_r^{\frac{1}{s}} \lambda_r\right), \tag{8}$$

for all $\lambda_r \in B \subset Z$, where Z is linear space.

2. New inequality on jain-saraswat's divergence

Now, we will derive the following new inequality on $S_h(\Theta, \Phi)$ for s-convex functions. This inequality will further illustrate the relations in terms of different information divergences.

Theorem 2.1 Let $h : (0, \infty) \to (-\infty, \infty)$ be a s-convex function. For $(\Theta, \Phi) \in \gamma \times \gamma$, we have

$$S_h(\Theta, \Phi) \geq A_p^s h\left[\frac{1}{2A_p} K_s(\Theta, \Phi)\right], \tag{9}$$

where $K_s(\Theta, \Phi) = \sum_{r=1}^{p} \phi_r^{\frac{1-s}{s}} (\theta_r + \phi_r)$.

Proof: By taking δ_r as ϕ_r and λ_r as $\frac{\theta_r + \phi_r}{2\phi_r}$ for $r = 1, 2, 3, ..., p$ in inequality (8), we get the desired relation (9).

Remark 2.1 If function h is normalized, i.e., $h(1) = 0$, then at $s = 1$ the inequality (9) will convert into $S_h(\Theta, \Phi) \geq 0$.

Remark 2.2 We denote $K_s(\Theta, \Phi) = \sum_{r=1}^{p} \phi_r^{\frac{1-s}{s}} (\theta_r + \phi_r)$ to summerize the calculations.

3. Some special results

By using the inequalities (2) and (9) together, we may have the significant results in terms of the different divergences, like: Proposition 3.1 gives the result in terms of the Triangular Discrimination.

Case I For $u \in (0, \infty)$ and $s \in (0, 1]$:

Proposition 3.1 For $\Theta, \Phi \in \gamma$ and $0 < \kappa \leq 1 \leq \zeta < \infty$, $\kappa = \zeta$, we have

$$\frac{2(\zeta-1)(1-\kappa)}{\kappa\zeta} \geq \sum_{r=1}^{p} \frac{(\theta_r - \phi_r)^2}{\theta_r + \phi_r} \geq 4A_p^{s+1} \frac{\left(\frac{1}{2A_p} K_s(\Theta, \Phi) - 1\right)^2}{K_s(\Theta, \Phi)}. \tag{10}$$

Proof: Let $h(u) = \frac{2(u-1)^2}{u}$, $u \in (0, \infty)$, $h'(u) = \frac{2(u^2-1)}{u^2}$ and $h''(u) = \frac{4}{u^3}$. The function is strictly convex by definition because $h''(u) > 0 \ \forall \ u > 0$. Also, for the function $h(u) = \frac{2(u-1)^2}{u}$, we have $S_h(\Theta, \Phi) = \sum_{r=1}^{p} \frac{(\theta_r - \phi_r)^2}{\theta_r + \phi_r}$, which is the famous triangular divergence measure [16] and

$$B_h(\kappa, \zeta) = \frac{\frac{2(\zeta-1)(\kappa-1)^2}{\kappa} + \frac{2(1-\kappa)(\zeta-1)^2}{\zeta}}{\zeta - \kappa} = \frac{2(\zeta-1)(\kappa-1)}{\zeta-\kappa}\left(\frac{\kappa-1}{\kappa} - \frac{\zeta-1}{\zeta}\right) = \frac{2(\zeta-1)(1-\kappa)}{\kappa\zeta}.$$

Moreover, by Remark 1.2 (b), we can conclude that the function $h(u)$ is s-convex for $s \in (0, 1]$ since $h(u) \geq 0 \ \forall \ u > 0$. So, we get the desired result (10) by using the inequalities (2) and (9), after a small simplification.

Proposition 3.2 For $\Theta, \Phi \in \gamma$ and $0 < \kappa \leq 1 \leq \zeta < \infty$, $\kappa \neq \zeta$, we have

$$2(\zeta-1)(1-\kappa)L_{-1}(\kappa, \zeta) \geq \sum_{r=1}^{p} (\theta_r - \phi_r) \log\left(\frac{\theta_r + \phi_r}{2\phi_r}\right) \geq A_p^{s-1}\left[K_s(\Theta, \Phi) - 2A_p\right] \log\left[\frac{K_s(\Theta, \Phi)}{2A_p}\right]. \tag{11}$$

Proof: Let $h(u) = 2(u-1)\log u$, $u > 0$, $h'(u) = 2\left(1 - \frac{1}{u} + \log u\right)$ and $h''(u) = 2\left(\frac{1}{u} + \frac{1}{u^2}\right)$. The function is strictly convex by definition because $h''(u) > 0 \ \forall \ u > 0$. Also, for the function $h(u) = 2(u-1)\log u$, we have

$$S_h(\Theta, \Phi) = \sum_{r=1}^{p} (\theta_r - \phi_r) \log\left(\frac{\theta_r + \phi_r}{2\phi_r}\right),$$

which is the Relative J-divergence measure [17] and

$$B_h(\kappa, \zeta) = \frac{2(\zeta - 1)(\kappa - 1)\log\kappa + 2(1-\kappa)(\zeta - 1)\log\zeta}{\zeta - \kappa}$$

$$= \frac{2(\zeta - 1)(\kappa - 1)}{\zeta - \kappa}(\log\kappa - \log\zeta) = 2(\zeta - 1)(1-\kappa)L_{-1}(\kappa, \zeta).$$

Moreover, by Remark 1.2 (b), we can conclude that the function $h(u)$ is s-convex for $s \in (0, 1]$ since $h(u) > 0\ \forall\ u > 0$. So, we get the desired result (11) by using the inequalities (2) and (9), after a small simplification.

By using the same procedure, we have the following results For $u \in (0, \infty)$ and $s \in (0, 1]$, by omitting the proofs:

Proposition 3.3 For the function $h(u) = 2|u - 1|$, $u > 0$, we have

$$\frac{4(\zeta - 1)(1-\kappa)}{\zeta - \kappa} \geq \sum_{r=1}^{p} |\theta_r - \phi_r| \geq A_p^{s-1}\left|K_s(\Theta, \Phi) - 2A_p\right|, \tag{12}$$

where $\sum_{r=1}^{p}|\theta_r - \phi_r|$ is the Variational divergence measure [31].

Proposition 3.4 For the function $h(u) = 4(u - 1)^2$, $u > 0$, we have

$$4(\zeta - 1)(1-\kappa) \geq \sum_{r=1}^{p} \frac{(\theta_r - \phi_r)^2}{\phi_r} \geq A_p^{s-2}\left[K_s(\Theta, \Phi) - 2A_p\right]^2, \tag{13}$$

where $\sum_{r=1}^{p}\frac{(\theta_r - \phi_r)^2}{\phi_r}$ is the Chi-square divergence measure [18].

Proposition 3.5 For the function $h(u) = \frac{1}{3}\frac{(4u^2 - 2u + 1)}{u}$, $u > 0$, we have

$$\frac{2\kappa\zeta + \kappa + \zeta - 1}{3\kappa\zeta} \geq \sum_{r=1}^{p} \frac{\theta_r^2 + \theta_r\phi_r + \phi_r^2}{\theta_r + \phi_r} \geq A_p^{s-1}\left[\frac{K^2(\Theta, \Phi) - A_pK_s(\Theta, \Phi) + A_p^2}{K_s(\Theta, \Phi)}\right], \tag{14}$$

where $\frac{2}{3}\sum_{r=1}^{p}\frac{\theta_r^2 + \theta_r\phi_r + \phi_r^2}{\theta_r + \phi_r}$ is the Centroidal mean divergence measure [32].

Proposition 3.6 For the function $h(u) = \sqrt{2u^2 - 2u + 1}$, $u > 0$, we have

$$C(\kappa, \zeta) \geq \sum_{r=1}^{p} \sqrt{\theta_r^2 + \phi_r^2} \geq A_p^{s-1}\sqrt{K^2(\Theta, \Phi) - 2A_pK_s(\Theta, \Phi) + 2A_p^2}, \tag{15}$$

where $\sum_{r=1}^{p}\sqrt{\frac{\theta_r^2 + \phi_r^2}{2}}$ is the Root mean square divergence measure [32] and

$$C(\kappa, \zeta) = \frac{(\zeta - 1)\sqrt{2\kappa^2 - 2\kappa + 1} + (1-\kappa)\sqrt{2\zeta^2 - 2\zeta + 1}}{\zeta - \kappa}.$$

Proposition 3.7 For the function $h(u) = \frac{2u^2 - 2u + 1}{u}$, $u > 0$, we have

$$\frac{\kappa + \zeta - 1}{\kappa\zeta} \geq \sum_{r=1}^{p} \frac{\theta_r^2 + \phi_r^2}{\theta_r + \phi_r} \geq A_p^{s-1}\left[\frac{K^2(\Theta, \Phi) - 2A_pK_s(\Theta, \Phi) + 2A_p^2}{K_s(\Theta, \Phi)}\right], \tag{16}$$

where $\sum_{r=1}^{p} \frac{\theta_r^2 + \phi_r^2}{\theta_r + \phi_r}$ is the Contra harmonic mean divergence measure [32].

Case II For $u \in (\frac{1}{2}, \infty)$ and $s \in (0, 1]$:

Proposition 3.8 For $\Theta, \Phi \in \gamma$ and $\frac{1}{2} < \kappa \le 1 \le \zeta < \infty,\ \kappa \ne \zeta$, we have

$$\left(\frac{1}{2} + \kappa\zeta\right) L_{-1}(2\kappa - 1,\ 2\zeta - 1) - \kappa\zeta L_{-1}(\kappa,\ \zeta) + \log \frac{eI(\kappa,\ \zeta)}{\sqrt{I(2\kappa - 1,\ 2\zeta - 1)}}$$

$$\ge \sum_{r=1}^{p} \left(\frac{\theta_r + \phi_r}{2}\right) \log \frac{\theta_r + \phi_r}{2\sqrt{\theta_r \phi_r}} \ge \frac{A_p^{s-1}}{2} K_s(\Theta,\ \Phi) \log \frac{K_s(\Theta,\ \Phi)}{2\sqrt{A_p}\sqrt{K_s(\Theta,\ \Phi) - A_p}}. \tag{17}$$

Proof: Let $h(u) = u \log \frac{u}{\sqrt{2u-1}},\ u > \frac{1}{2},\ h'(u) = \log \frac{u}{\sqrt{2u-1}} + \frac{u-1}{2u-1}$ and $h''(u) = \frac{2u^2 - 2u + 1}{u(1-2u)^2}$. The function is strictly convex by definition because $h''(u) > 0\ \forall\ u > 0$. Also, for the function $h(u) = u \log \frac{u}{\sqrt{2u-1}}$, we have $S_h(\Theta, \Phi) = \sum_{r=1}^{p} \left(\frac{\theta_r + \phi_r}{2}\right) \log \frac{\theta_r + \phi_r}{2\sqrt{\theta_r \phi_r}}$, which is the AGM divergence measure [19] and

$$B_h(\kappa,\ \zeta) = \frac{(\zeta - 1)\kappa \log \frac{\kappa}{\sqrt{2\kappa - 1}} + (1 - \kappa)\zeta \log \frac{\zeta}{\sqrt{2\zeta - 1}}}{\zeta - \kappa}$$

$$= \frac{\kappa\zeta\left[\log\kappa - \frac{1}{2}\log(2\kappa - 1) - \log\zeta + \frac{1}{2}\log(2\zeta - 1)\right]}{\zeta - \kappa} + \frac{\zeta\left[\log\zeta - \frac{1}{2}\log(2\zeta - 1)\right]}{\zeta - \kappa} - \frac{\kappa\left[\log\kappa - \frac{1}{2}\log(2\kappa - 1)\right]}{\zeta - \kappa}$$

$$= \frac{\kappa\zeta(\log\kappa - \log\zeta)}{\zeta - \kappa} + \frac{(\zeta\log\zeta - \kappa\log\kappa)}{\zeta - \kappa} + \frac{\kappa\zeta\left[\log(2\zeta - 1) - \log(2\kappa - 1)\right]}{2(\zeta - \kappa)} + \frac{\left[\kappa\log(2\kappa - 1) - \zeta\log(2\zeta - 1)\right]}{2(\zeta - \kappa)}$$

$$= \left(\frac{1}{2} + \kappa\zeta\right) L_{-1}(2\kappa - 1,\ 2\zeta - 1) - \kappa\zeta L_{-1}(\kappa,\ \zeta) + \log eI(\kappa,\ \zeta) - \frac{1}{2}\log I(2\kappa - 1,\ 2\zeta - 1).$$

Moreover, by Remark 2 (b), we can conclude that the function $h(u)$ is s-convex for $s \in (0, 1]$ since $h(u) > 0\ \forall\ u > \frac{1}{2}$. So, we get the desired result (17) by using the inequalities (2) and (9), after a small simplification.

In a similar manner, we have the following results For $u \in (\frac{1}{2}, \infty)$ and $s \in (0, 1]$, by omitting the proofs:

Proposition 3.9 For the function $h(u) = \left(u - \frac{1}{2}\right)\log(2u - 1) - u\log u,\ u > \frac{1}{2}$, we have

$$\left(\frac{1}{2} - 2\kappa\zeta\right) L_{-1}(2\kappa - 1,\ 2\zeta - 1) + \log\left[\frac{I(2\kappa - 1,\ 2\zeta - 1)}{I(\kappa,\ \zeta)\sqrt{eI\left(\frac{1}{2\kappa - 1},\ \frac{1}{2\zeta - 1}\right)}}\right]$$

$$\ge \frac{1}{2}\left[\sum_{r=1}^{p} \theta_r \log \frac{2\theta_r}{\theta_r + \phi_r} + \sum_{r=1}^{p} \phi_r \log \frac{2\phi_r}{\theta_r + \phi_r}\right] \ge \frac{1}{2A_p}\left[A_p^s\{K_s(\Theta,\ \Phi) - A_p\}\log \frac{K_s(\Theta,\ \Phi) - A_p}{A_p}\right]$$

$$-\frac{1}{2A_p}\left[K_s(\Theta,\ \Phi)\log \frac{K_s(\Theta,\ \Phi)}{2A_p}\right], \tag{18}$$

where $\frac{1}{2}\left[\sum_{r=1}^{p}\theta_r \log\frac{2\theta_r}{\theta_r+\phi_r}+\sum_{r=1}^{p}\phi_r \log\frac{2\phi_r}{\theta_r+\phi_r}\right]$ is the JS divergence measure [14, 20].

Proposition 3.10 For the function $h(u)=\frac{\left(1-\sqrt{2u-1}\right)^2}{2}$, $u>\frac{1}{2}$, we have

$$1+\frac{\sqrt{2\kappa-1}(1-\zeta)+\sqrt{2\zeta-1}(\kappa-1)}{\zeta-\kappa}\geq\sum_{r=1}^{p}\left(\sqrt{\theta_r}-\sqrt{\phi_r}\right)^2\geq A_p^{s-1}\left(\sqrt{A_p}-\sqrt{K_s(\Theta,\ \Phi)-A_p}\right)^2, \tag{19}$$

where $\frac{1}{2}\sum_{r=1}^{p}\left(\sqrt{\theta_r}-\sqrt{\phi_r}\right)^2$ is the Hellinger divergence measure [33].

Proposition 3.11 For the function $h(u)=\frac{8u(u-1)^2}{2u-1}$, $u>\frac{1}{2}$, we have

$$\frac{8(\zeta-1)(1-\kappa)\left[1+2\kappa\zeta-(\kappa+\zeta)\right]}{(2\kappa-1)(2\zeta-1)}\geq\sum_{r=1}^{p}\frac{(\theta_r-\phi_r)^2(\theta_r+\phi_r)}{\theta_r\phi_r}\geq A_p^{s-2}\frac{\left[K_s(\Theta,\ \Phi)-2A_p\right]^2K_s(\Theta,\ \Phi)}{\left[K_s(\Theta,\ \Phi)-A_p\right]}, \tag{20}$$

where $\sum_{r=1}^{p}\frac{(\theta_r-\phi_r)^2(\theta_r+\phi_r)}{\theta_r\phi_r}$ is the Symmetric Chi-square divergence measure [34].

Proposition 3.12 For the function $h(u)=2(u-1)\log(2u-1)$, $u>\frac{1}{2}$, we have

$$4(\zeta-1)(1-\kappa)L_{-1}(2\kappa-1,\ 2\zeta-1)\geq\sum_{r=1}^{p}(\theta_r-\phi_r)\log\frac{\theta_r}{\phi_r}\geq A_p^{s-1}\left[K_s(\Theta,\ \Phi)-2A_p\right]\log\left[\frac{K_s(\Theta,\ \Phi)}{A_p}-1\right], \tag{21}$$

where $\sum_{r=1}^{p}(\theta_r-\phi_r)\log\frac{\theta_r}{\phi_r}$ is the JK divergence measure [21, 35].

Proposition 3.13 For the function $h(u)=4\frac{(u-1)^2}{\sqrt{2u-1}}$, $u>\frac{1}{2}$, we have

$$\frac{4(\zeta-1)(1-\kappa)}{(\zeta-\kappa)}\left[\frac{1-\kappa}{\sqrt{2\kappa-1}}+\frac{\zeta-1}{\sqrt{2\zeta-1}}\right]\geq\sum_{r=1}^{p}\frac{(\theta_r-\phi_r)^2}{\sqrt{\theta_r\phi_r}}\geq A_p^{s-\frac{3}{2}}\frac{\left(K_s(\Theta,\ \Phi)-2A_p\right)^2}{\sqrt{K_s(\Theta,\ \Phi)-A_p}}, \tag{22}$$

where $\sum_{r=1}^{p}\frac{(\theta_r-\phi_r)^2}{\sqrt{\theta_r\phi_r}}$ is the Jain-Srisvastava divergence measure [36].

Proposition 3.14 For the function $h(u)=\frac{8u^2(u-1)^2}{(2u-1)^{\frac{3}{2}}}$, $u>\frac{1}{2}$, we have

$$\frac{8(\zeta-1)(1-\kappa)}{(\zeta-\kappa)}\left[\frac{\kappa^2(1-\kappa)}{(2\kappa-1)^{\frac{3}{2}}}+\frac{\zeta^2(\zeta-1)}{(2\zeta-1)^{\frac{3}{2}}}\right]\geq\sum_{r=1}^{p}\frac{\left(\theta_r^2-\phi_r^2\right)^2}{(\theta_r\phi_r)^{\frac{3}{2}}}\geq\frac{A_p^{s-\frac{5}{2}}K^2(\Theta,\ \Phi)\left(K_s(\Theta,\ \Phi)-2A_p\right)^2}{\left(K_s(\Theta,\ \Phi)-A_p\right)^{\frac{3}{2}}}, \tag{23}$$

where $\frac{1}{2}\sum_{r=1}^{p}\frac{\left(\theta_r^2-\phi_r^2\right)^2}{(\theta_r\phi_r)^{\frac{3}{2}}}$ is the Pranesh-Johnson divergence measure [37].

Proposition 3.15 For the function $h(u)=\frac{8u(u-1)^2}{(2u-1)}\log\frac{u}{\sqrt{2u-1}}$, $u>\frac{1}{2}$, we have

$$D(\kappa,\ \zeta)\geq\sum_{r=1}^{p}\frac{(\theta_r+\phi_r)(\theta_r-\phi_r)^2}{\theta_r\phi_r}\log\frac{\theta_r+\phi_r}{2\sqrt{\theta_r\phi_r}}$$

$$\geq \frac{A_p^{s-2} K_s(\Theta, \Phi)\left(K_s(\Theta, \Phi)-2A_p\right)^2}{\left(K_s(\Theta, \Phi)-A_p\right)} \log\left[\frac{K_s(\Theta, \Phi)}{2\sqrt{A_p}\sqrt{K_s(\Theta, \Phi)-A_p}}\right], \tag{24}$$

where $\sum_{r=1}^{p} \frac{(\theta_r+\phi_r)(\theta_r-\phi_r)^2}{\theta_r\phi_r} \log\frac{\theta_r+\phi_r}{2\sqrt{\theta_r\phi_r}}$ is the Pranesh-Chhina divergence measure [38] and

$$D(\kappa, \zeta) = \frac{8(\zeta-1)(1-\kappa)}{(\zeta-\kappa)}\left[\frac{\kappa(1-\kappa)}{(2\kappa-1)}\log\frac{\kappa}{\sqrt{2\kappa-1}} + \frac{\zeta(\zeta-1)}{(2\zeta-1)}\log\frac{\zeta}{\sqrt{2\zeta-1}}\right].$$

Proposition 3.16 For the function $h(u) = \frac{2(u-1)^2}{u}\log\frac{u}{\sqrt{2u-1}}$, $u > \frac{1}{2}$, we have

$$E(\kappa, \zeta) \geq \sum_{r=1}^{p} \frac{(\theta_r-\phi_r)^2}{(\theta_r+\phi_r)}\log\frac{\theta_r+\phi_r}{2\sqrt{\theta_r\phi_r}} \geq \frac{A_p^{s-1}\left(K_s(\Theta, \Phi)-2A_p\right)^2}{K_s(\Theta, \Phi)}\log\left[\frac{K_s(\Theta, \Phi)}{2\sqrt{A_p}\sqrt{K_s(\Theta, \Phi)-A_p}}\right], \tag{25}$$

where $\sum_{r=1}^{p} \frac{(\theta_r-\phi_r)^2}{(\theta_r+\phi_r)}\log\frac{\theta_r+\phi_r}{2\sqrt{\theta_r\phi_r}}$ is the Pranesh-Hunter divergence measure [39] and

$$E(\kappa, \zeta) = 2(\zeta-1)(1-\kappa)\left[\left(\frac{1}{2\kappa\zeta}-1\right)L_{-1}(2\kappa-1,\ 2\zeta-1) + L_{-1}(\kappa, \zeta) - \frac{1}{\kappa\zeta}\log\left\{e^{\frac{3}{2}} I\left(\frac{1}{\kappa}, \frac{1}{\zeta}\right)\sqrt{I\left(\frac{1}{2\kappa-1}, \frac{1}{2\zeta-1}\right)}\right\}\right].$$

Case III For $s \in (0, 1]$:

Proposition 3.17 For $\Theta, \Phi \in \gamma$, we have

$$\log eI\left(\frac{1}{\kappa}, \frac{1}{\zeta}\right) - L_{-1}(\kappa, \zeta) \geq \sum_{r=1}^{p} \phi_r \log\frac{2\phi_r}{\theta_r+\phi_r} \leq A_p^s \log\left[\frac{1}{2A_p}K_s(\Theta, \Phi)\right]. \tag{26}$$

Proof: Let $h(u) = -\log u$, $u > 0$, $h'(u) = -\frac{1}{u}$ and $h''(u) = \frac{1}{u^2}$ The function is strictly convex by definition because $h''(u) > 0\ \forall\ u > 0$. Also, for the function $h(u) = -\log u$, we have $S_h(\Theta, \Phi) = \sum_{r=1}^{p} \phi_r \log\frac{2\phi_r}{\theta_r+\phi_r}$, which is Adjoint of the Relative JS divergence measure [21] and

$$B_h(\kappa, \zeta) = \frac{-(\zeta-1)\log\kappa - (1-\kappa)\log\zeta}{\zeta-\kappa} = -\left[\frac{\zeta\log\kappa - \log\kappa + \log\zeta - \kappa\log\zeta}{\zeta-\kappa}\right] = \log eI\left(\frac{1}{\kappa}, \frac{1}{\zeta}\right) - L_{-1}(\kappa, \zeta).$$

Moreover, by Remark 1.2 (b), we can conclude that the function $h(u)$ is s-convex for $s \in (0, 1]$ if $h(u) \geq 0\ \forall\ u \leq 1$ and it is s-convex for $s \in [1, \infty)$ if $h(u) \leq 0\ \forall\ u \geq 1$. So, we get the desired result (26) by using the inequalities (2) and (9), after a small simplification.

Similarly we have the following cases for $s \in (0, \infty)$ by skipping the detail proof:

Proposition 3.18 For the function $h(u) = u\log u$, $u > 0$ and taking into consideration the Remark 1.2 (b), the function $h(u)$ is s-convex for $s \in (0, 1]$ if $h(u) \geq 0\ \forall\ u \geq 1$ and it is sconvex for $s \in [1, \infty)$ if $h(u) \leq 0\ \forall\ u \leq 1$. So, we have the result

$$\log eI(\kappa, \zeta) - \kappa\zeta L_{-1}(\kappa, \zeta) \geq \sum_{r=1}^{p}\left(\frac{\theta_r+\phi_r}{2}\right)\log\frac{\theta_r+\phi_r}{2\phi_r} \geq \frac{A_p^{s-1}}{2}K_s(\Theta, \Phi)\log\frac{K_s(\Theta, \Phi)}{2A_p}, \tag{27}$$

where $\sum_{r=1}^{p}\left(\frac{\theta_r+\phi_r}{2}\right)\log\frac{\theta_r+\phi_r}{2\phi_r}$ is adjoint of the Relative AG divergence measure [19].

Proposition 3.19 For the function $h(u) = (2u - 1)\log(2u - 1)$, $u > \frac{1}{2}$ and taking into consideration the Remark 1.2 (b), the function $h(u)$ is s-convex for $s \in (0, 1]$ if $h(u) \geq 0\ \forall\ u \geq 1$ and it is s-convex for $s \in [1, \infty)$ if $h(u) \leq 0\ \forall\ \frac{1}{2} < u \leq 1$. So, we have the result

$$F(\kappa,\ \zeta) \geq \sum_{r=1}^{p}\theta_r\log\frac{\theta_r}{\phi_r} \geq A_p^{s-1}\left[K_s(\Theta,\ \Phi) - A_p\right]\log\left[\frac{K_s(\Theta,\ \Phi)}{A_p} - 1\right], \tag{28}$$

where $\sum_{r=1}^{p}\theta_r\log\frac{\theta_r}{\phi_r}$ is the KL divergence measure [21] and

$$F(\kappa,\ \zeta) = (1 - 4\kappa\zeta)L_{-1}(2\kappa - 1,\ 2\zeta - 1) + \log\left[\frac{eI^2(2\kappa - 1,\ 2\zeta - 1)}{I\left(\frac{1}{2\kappa - 1},\ \frac{1}{2\zeta - 1}\right)}\right].$$

Proposition 3.20 For the function $h(u) = \left(\frac{u+1}{2}\right)\log\left(\frac{u+1}{2u}\right)$, $u > 0$ and taking into consideration the Remark 1.2 (b), the function $h(u)$ is s-convex for $s \in (0, 1]$ if $h(u) \geq 0\ \forall\ u \leq 1$ and it is s-convex for $s \in [1, \infty)$ if $h(u) \leq 0\ \forall\ u \geq 1$. So, we have the following result

$$G(\kappa,\ \zeta) \geq \sum_{r=1}^{p}\left(\frac{\theta_r + 3\phi_r}{4}\right)\log\left[\frac{\theta_r + 3\phi_r}{2(\theta_r + \phi_r)}\right] \geq \frac{A_p^{s-1}\left(K_s(\Theta,\ \Phi) + 2A_p\right)}{4}\log\left[\frac{K_s(\Theta,\ \Phi) + 2A_p}{2K_s(\Theta,\ \Phi)}\right], \tag{29}$$

where $\sum_{r=1}^{p}\left(\frac{\theta_r + 3\phi_r}{4}\right)\log\left[\frac{\theta_r + 3\phi_r}{2(\theta_r + \phi_r)}\right]$ is the Jain-Chhabra divergence measure [27] and

$$G(\kappa,\ \zeta) = \frac{(\zeta - 1)\left(\frac{\kappa + 1}{2}\right)\log\frac{\kappa + 1}{2\kappa} + (1 - \kappa)\left(\frac{\zeta + 1}{2}\right)\log\frac{\zeta + 1}{2\zeta}}{\zeta - \kappa}.$$

4. Verification of the results

In order to be sure that the obtained results are authentic, it is necessary to take appropriate data. Also, we cannot take into account all 20 results for this process, so we will take only three results from each case. The remaining results can be verified using the same procedure. Also, it is not possible to validate the outcomes at each value of s for the given domain, so we fix the value of s as $\frac{1}{2}$. Of course, a similar procedure can be used for other values of s.

Let us have two discrete probability distributions Θ (Binomial) and Φ (Poisson), with finite number of trials ($N = 10$), probability of success of one trial ($\theta = 0.7$). So the probability of failure of one trial will be $\phi = 1 - \theta = 1 - 0.7 = 0.3$ and the Poisson parameter will be $N\theta = 10 \times 0.7 = 7$. The Binomial distribution represents real information, while the Poisson distribution shows its approximated form. Now, by using the probability mass function of Binomial distribution $\left[\Theta(T = t_r = r) = \theta_r = {}^N C_r\theta^r\phi^{N-r}\right]$ and Poisson distribution $\left[\Phi(T = t_r = r) = \phi_r = \frac{e^{-N\theta}(N\theta)^r}{r!}\right]$, we have the following evaluation for the random variable T:

Table 1. Evaluation of discrete probability distributions for $N = 10$, $\theta = 0.7$, $\phi = 0.3$

t_r	0	1	2	3	4	5	6	7	8	9	10
$\theta_r \approx$	0.0000059	0.000137	0.00144	0.009	0.036	0.102	0.200	0.266	0.233	0.121	0.0282
$\phi_r \approx$	0.000911	0.00638	0.022	0.052	0.091	0.177	0.199	0.149	0.130	0.101	0.0709
$\frac{\theta_r + \phi_r}{2\phi_r}$	0.503	0.510	0.532	0.586	0.697	0.788	1.002	1.392	1.396	1.099	0.698

Since $(\kappa, \zeta) \subset (0, \infty)$ with $0 < \kappa \le 1 \le \zeta < \infty$, $\kappa \ne \zeta$, also $\frac{1}{2} < \kappa \le \frac{\theta_r + \phi_r}{2\phi_r} \le \zeta < \infty$, so we can conclude from the Table 1 that the values of κ and ζ will be 0.503 and 1.396, respectively. Also, by using the data from the Table 1 above, we have the following illustrations:

$$\sum_{r=1}^{11} |\theta_r - \phi_r| = |\theta_1 - \phi_1| + |\theta_2 - \phi_2| + \ \dots\ + |\theta_{11} - \phi_{11}|$$

$$= |0.0000059 - .000911| + |0.000137 - 0.00638| + \ \dots\ + |0.0282 - 0.0709| \approx 0.4844. \tag{30}$$

$$\sum_{r=1}^{11} \frac{(\theta_r - \phi_r)^2 (\theta_r + \phi_r)}{\theta_r \phi_r} = \frac{(\theta_1 - \phi_1)^2 (\theta_1 + \phi_1)}{\theta_1 \phi_1} + \frac{(\theta_2 - \phi_2)^2 (\theta_2 + \phi_2)}{\theta_2 \phi_2} + \ \dots\ + \frac{(\theta_{11} - \phi_{11})^2 (\theta_{11} + \phi_{11})}{\theta_{11} \phi_{11}}$$

$$= \frac{(0.0000059 - 0.000911)^2 \times (0.0000059 + 0.000911)}{0.0000059 \times 0.000911} + \ \dots$$

$$+ \frac{(0.0282 - 0.0709)^2 \times (0.0282 + 0.0709)}{0.0282 \times 0.0709} \approx 1.5558. \tag{31}$$

$$\sum_{r=1}^{11} \left(\frac{\theta_r + 3\phi_r}{4} \right) \log \left[\frac{\theta_r + 3\phi_r}{2(\theta_r + \phi_r)} \right] = \left(\frac{\theta_1 + 3\phi_1}{4} \right) \log \left[\frac{\theta_1 + 3\phi_1}{2(\theta_1 + \phi_1)} \right] + \ \dots\ + \left(\frac{\theta_{11} + 3\phi_{11}}{4} \right) \log \left[\frac{\theta_{11} + 3\phi_{11}}{2(\theta_{11} + \phi_{11})} \right]$$

$$= \left(\frac{0.0000059 + 3 \times 0.000911}{4} \right) \log \left[\frac{0.0000059 + 3 \times 0.000911}{2 \times (0.0000059 + 0.000911)} \right] + \ \dots$$

$$+ \left(\frac{0.0282 + 3 \times 0.0709}{4} \right) \log \left[\frac{0.0282 + 3 \times 0.0709}{2 \times (0.0282 + 0.0709)} \right] \approx 0.01154. \tag{32}$$

Now, at $s = \frac{1}{2}$, we also have the following evaluations:

$$K_{s=\frac{1}{2}}(\Theta, \Phi) = \sum_{r=1}^{11} \phi_r^{\frac{1-\frac{1}{2}}{\frac{1}{2}}} (\theta_r + \phi_r) = \sum_{r=1}^{11} \phi_r (\theta_r + \phi_r) = \phi_1 (\theta_1 + \phi_1) + \ \dots\ + \phi_{11} (\theta_{11} + \phi_{11})$$

$$= 0.000911 \times (0.0000059 + 0.000911) + \ \dots\ + 0.0709 \times (0.0282 + 0.0709) = 0.2825. \tag{33}$$

$$(A_p)_{s=\frac{1}{2}} = \sum_{r=1}^{11} \phi_r^{\frac{1}{1/2}} = \sum_{r=1}^{11} \phi_r^2 = \phi_1^2 + \ \dots\ + \phi_{11}^2 = (0.000911)^2 + \ \dots\ + (0.0709)^2 = 0.1367. \tag{34}$$

Now, put the data from the equations (30), (33) and (34), together with the values of κ and ζ, into the inequality (12) at $s = \frac{1}{2}$, we have

$$0.881576\left(=\frac{4(\zeta-1)(1-\kappa)}{\zeta-\kappa}\right) \geq 0.4844\left(=\sum_{r=1}^{11}\left|\theta_r-\phi_r\right|\right) \geq 0.02461.$$

And, put the values from the equations (31), (33) and (34), together with the values of κ and ζ, into the inequality (20) at $s = \frac{1}{2}$, we have

$$74.0059\left(=\frac{8(\zeta-1)(1-\kappa)\left[1+2\kappa\zeta-(\kappa+\zeta)\right]}{(2\kappa-1)(2\zeta-1)}\right) \geq 1.5558\left(=\sum_{r=1}^{11}\frac{(\theta_r-\phi_r)^2(\theta_r+\phi_r)}{\theta_r\phi_r}\right) \geq 0.003174.$$

Also, put the values from the equations (32), (33) and (34), together with the values of κ and ζ, into the inequality (29) at $s = \frac{1}{2}$, we have

$$0.03180(=G(\kappa,\ \zeta)) \geq .01154\left(=\sum_{r=1}^{11}\left(\frac{\theta_r+3\phi_r}{4}\right)\log\left[\frac{\theta_r+3\phi_r}{2(\theta_r+\phi_r)}\right]\right) \geq -0.006103.$$

Thus, validate the results (12), (20), and (29).

Remark 4.1

a. The results can be verified at other values of the number of trials (N) and probability of success of one trial (θ).

b. The results can be verified by taking other discrete probability distributions.

5. Conclusion

Several articles have defined the information inequality using divergence measures on different convex functions, but in this article the inequality is defined on s-convex functions and novel results are found in terms of different well-known divergence measures. The author believes that these results have significant implications for information theory at different levels. These implications include signal processing, statistical data analysis, pattern recognition, analysis of contingency tables, testing of statistical hypotheses, and others.

Conflict of interest

The author declares no competing financial interest.

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