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Research Article

Solving Interval Type-2 Fuzzy Pentagonal Polynomial Equation Using Horner-Muller's Method

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Abstract: The recent boom in interval type-2 fuzzy sets has attracted many researchers to the fields of fuzzy numbers, algebraic expressions, and iterative methods. Interval type-2 fuzzy pentagonal polynomial (IT2 FPP) is one of the algebraic expressions that combine the knowledge of interval type-2 pentagonal fuzzy numbers and polynomial equations. Solving IT2 FPP in the absence of an iterative method is becoming a common encounter as the solution is often difficult, expensive, and inexact. This paper defines IT2 FPP and the iterative method Horner-Muller's used to find the approximate solution of the IT2 FPP. The IT2 FPP is introduced by combining the generalization of interval type-2 pentagonal fuzzy numbers and fuzzy polynomials, whereas the Horner-Muller's method is a combination of Horner's method and Muller's method. A numerical example is presented to illustrate the computational procedures for finding the approximate solution. The approximate solution is obtained after two iterations with zero error. A Comparative analysis is also presented to validate the consistency and efficiency of the proposed method compared to benchmark iterative methods.

***Keywords*:** interval type-2 fuzzy numbers, pentagonal fuzzy number, polynomials, fuzzy polynomials, Horner-Muller's method

MSC: 03-E72, 94-D05

1. Introduction

Numerous iterative methods have been developed to model and solve real-world problems, including Horner's and Muller's methods. A polynomial evaluation method known as Horner's method is named after William George Horner [1]. Horner's method was an algorithm for solving polynomials and a method for approximating the roots of polynomials. For example, Burrus [2] applied Horner's method to evaluate and deflate polynomials, which is the conversion problem into a linear difference equation. In a comparative study, Aziz et al. [3] compared the efficiency of Horner's method in solving nonlinear functions using Bisection, Newton, and Horner's method. Meanwhile, David E. Muller [4] introduced

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Muller's method in 1956 as an extension of the secant method. Muller's method is a root-finding algorithm to solve the equations of the form $f(x) = 0$ where $f(x)$ is a nonlinear function of x. In fact, Muller's method calculates the roots of a continuous function by approximating the root of the quadratic interpolation. Wu [5] developed a new root-finding method that improves Muller and Bisection methods with global convergence for enclosing simple zeros of the nonlinear equation. Muller's method was applied by Chandram [6] to address the profit-based unit commitment. In order to improve the efficiency of obtaining the approximate solution, Horner's and Muller's methods were combined by Nurhakimah et al. [7] in solving interval type-2 fuzzy polynomials. This combination, which considers three initial values has produced faster convergence to the exact solution and minimized the number of iterations.

Turning now to the concept of fuzzy sets, which become parts of fuzzy polynomials. The concept of fuzzy sets was introduced by Zadeh [8]. The fuzzy sets have been extended to interval-valued fuzzy sets [9], intuitionistic fuzzy sets [10], and interval type-2 fuzzy sets (IT2 FS) [11] for dealing with problems of uncertainty, imprecision and vagueness. IT2 FS are generalizations of type-1 fuzzy sets, type-2 fuzzy sets, and interval-valued fuzzy sets. The idea of type-2 fuzzy sets, which is an expansion of the concept of type-1 fuzzy sets, was first described by Zadeh [12] in 1971. Grades of membership in any subset in the range [0, 1] are present in type-2 fuzzy sets. Sambuc [9] proposed interval-valued fuzzy sets that can be used to determine the exact membership values of the given elements, where the intervals serve as the membership values and the exact numerical membership degree is a value inside the considered interval. The notion of IT2 FS, which was formally presented by Zadeh [13], states that each element's membership degree is determined by another fuzzy set that is defined over the referential [0, 1]. Liang and Mendel [14] introduced the definition of IT2 FS as a specific instance of the idea of type-2 fuzzy sets being expressed mathematically where the secondary membership degree is equal to one. Based on the generalization of interval-valued fuzzy sets, Sola [15] described a wider view of the relationship between interval-valued fuzzy sets and IT2 FS. It has been widely believed that interval type-2 fuzzy numbers are applicable to more general cases and simplify mathematical computations [16].

Previous authors defined IT2 FS with commonly used fuzzy numbers, especially triangular fuzzy numbers and trapezoidal fuzzy numbers. However, some real-world problems are concerned with more than four parameters [17]. To address these issues, pentagonal fuzzy sets were introduced by Pathinathan and Ponnivalavan [18] along with certain operational properties. They also generalized the concepts of pentagonal fuzzy numbers along with set theoretic operations [19]. The concept of fuzzy centers in pentagonal fuzzy number was presented by Raj Kumar and Pathinathan [20]. Pentagonal fuzzy numbers are defined by Kaur [21] in continuation with the other fuzzy numbers. Vidhya and Ganesan [22] proposed pentagonal fuzzy numbers to solve the fuzzy transportation problem without requiring an initial feasible solution. The study of pentagonal fuzzy numbers has been expanded to the interval-valued fuzzy numbers. Chakraborty et al. [23] studied the different measures of interval-valued pentagonal fuzzy numbers connected to various membership functions by considering the significant exposure of multifarious interval-valued fuzzy numbers in neoteric studies. The interval-valued pentagonal fuzzy number has been introduced by Ajay [24] and applied in the literature. From the generalization of pentagonal fuzzy numbers and interval-valued pentagonal fuzzy numbers, Ajay and Pathinathan [25] proposed interval type-2 pentagonal fuzzy numbers and discussed three types of regular interval type-2 pentagonal fuzzy numbers. However, until recently, there has been no reliable evidence that shows the amalgamation of interval type-2 pentagonal fuzzy numbers and polynomials.

Polynomials are mathematical expressions made up of a sum of terms, with each term including a variable or variables raised to a power and multiplied by a coefficient. On the other hand, fuzzy polynomials are characterized by fuzzy numbers as coefficients while preserving all other characteristics of a polynomial. Three categories of fuzzy polynomials are fuzzy polynomial equations, dual fuzzy polynomial equations, and fuzzy polynomial systems. Fuzzy polynomials have been developed by Rouhparvar [26] and Noor'ani et al. [27]. Nurhakimah and Abdullah [28] have specifically introduced one of the categories of fuzzy polynomials, which is the interval type-2 fuzzy polynomial equation, and solved it using the ranking method. Then, the concepts of value, ambiguity, fuzziness and vagueness are used in the solution of interval type-2 fuzzy polynomials by the ranking method [29]. Although extensive research has been carried out on fuzzy polynomials and their generalizations, to the best of the author's knowledge, there has been no single work investigating the possibility of solving interval type-2 pentagonal polynomials using the iterative Horner-Muller's method. To fill this knowledge gap, the present study aims to propose interval type-2 fuzzy pentagonal polynomials. In addition, Horner-Muller's iterative method is used to find the approximate solution of the proposed polynomials.

This paper is organized as follows: In Section 2, we recall the definitions of interval type-2 fuzzy, interval type-2 pentagonal fuzzy number and fuzzy polynomials. We define the IT2 FPP in Section 3. Section 4. presents the computational procedures for solving IT2 FPP equations using Horner-Muller's method. A numerical example is presented in Section 5. to illustrate the application of Horner-Muller's method. A comparative analysis is given in Section 6. Finally, we present the conclusions in Section 7.

2. Governing model

This section introduces the definitions of related research on interval type-2 fuzzy sets, interval type-2 pentagonal fuzzy number, pentagonal fuzzy number, interval type-2 pentagonal fuzzy number and fuzzy polynomials.

Definition 2.1 Interval type-2 fuzzy sets (IT2 FS) [11]

A type-2 fuzzy sets, A_{T2FS} is characterized by a type-2 membership function, $\mu_{A_{T2FS}}(x, u)$ where $x \in X$ and $u \in J_x \subseteq [0, 1]$. It is defined as:

$$A_{T2FS} = \left\{\left((x, u), \mu_{A_{T2FS}}(x, u)\right) \mid \forall x \in X, \forall u \in J_x \subseteq [0, 1]\right\} \tag{1}$$

where J_x is the primary membership function in [0, 1] and u is the primary membership values in $0 \le \mu_{A_{T2FS}}(x, u) \le 1$. It can also be expressed as follows:

$$A_{T2FS} = \int_{x \in X} \int_{u \in J_x} \mu_{A_{T2FS}}(x, u) / (x, u);\ J_x \subseteq [0, 1] \tag{2}$$

where $\int\int$ denotes union over all admissible x and u. An interval type-2 fuzzy set, A_{IT2FS} is a type-2 fuzzy set when all secondary membership function is unity defined as:

$$A_{T2FS} = \int_{x \in X} \int_{u \in J_x} 1 / (x, u);\ J_x \subseteq [0, 1] \tag{3}$$

Definition 2.2 Interval type-2 fuzzy number [11]

An interval type-2 fuzzy number is an IT2 FS on a real line $\Re$ defined as:

$$A_{IT2FN} = \left[A^U, A^L\right] \tag{4}$$

where A^U and A^L are upper and lower membership functions respectively such that $A^U \subseteq A^L$.

Definition 2.3 Pentagonal fuzzy number [30]

A pentagonal fuzzy number is defined as:

$$\tilde{A}_{PFN} = \left(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5\right) \tag{5}$$

where membership functions, $\tilde{\mu}_{\tilde{A}}(x)$ should be satisfied by the following conditions:

i. $\tilde{\mu}_{\tilde{A}}(x)$ is a continuous function in the interval [0, 1].

ii. $\tilde{\mu}_{\tilde{A}}(x)$ is strictly increasing and continuous function on $[\tilde{a}_1, \tilde{a}_2]$ and $[\tilde{a}_2, \tilde{a}_3]$.

iii. $\tilde{\mu}_{\tilde{A}}(x)$ is strictly decreasing and continuous function on $[\tilde{a}_3, \tilde{a}_4]$ and $[\tilde{a}_4, \tilde{a}_5]$.

Definition 2.4 Interval type-2 pentagonal fuzzy number [25]

An interval type-2 pentagonal fuzzy number is an IT2 FS of $\Re$ defined as:

$$\tilde{\tilde{A}}_{IT2PFN} = \left[\tilde{\tilde{A}}^U, \tilde{\tilde{A}}^L \right] = \left[\left(\overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_4, \overline{a}_5 \right), \left(\underline{a}_1, \underline{a}_2\ \underline{a}_3, \underline{a}_4, \underline{a}_5 \right) \right] \tag{6}$$

where $\tilde{\tilde{A}}_U = \left(\overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_4, \overline{a}_5 \right)$ and $\tilde{\tilde{A}}_L = \left(\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4, \underline{a}_5 \right)$ are upper and lower pentagonal fuzzy numbers.

The membership function of the upper interval type-2 pentagonal fuzzy number, $\overline{\mu}_{\tilde{\tilde{A}}^U}$ is defined as follows:

$$\overline{\mu}_{\tilde{\tilde{A}}^U}(x) = \begin{cases} \dfrac{\overline{h}\left(x - \overline{a}_1\right)}{\overline{a}_2 - \overline{a}_1} & ; \overline{a}_1 \le x \le \overline{a}_2 \\ \dfrac{\overline{h}\left(x - \overline{a}_2\right)}{2\left(\overline{a}_3 - \overline{a}_2\right)} & ; \overline{a}_2 \le x \le \overline{a}_3 \\ \overline{h} & ; x = \overline{a}_3 \\ \dfrac{\overline{h}\left(\overline{a}_4 - x\right)}{2\left(\overline{a}_4 - \overline{a}_3\right)} & ; \overline{a}_3 \le x \le \overline{a}_4 \\ \dfrac{\overline{h}\left(\overline{a}_5 - x\right)}{\left(\overline{a}_5 - \overline{a}_4\right)} & ; \overline{a}_4 \le x \le \overline{a}_5 \end{cases} \tag{7}$$

and the membership function of the lower interval type-2 pentagonal fuzzy number $\underline{\mu}_{\tilde{\tilde{A}}^L}$ is defined as below:

$$\underline{\mu}_{\tilde{\tilde{A}}^L}(x) = \begin{cases} \dfrac{\underline{h}\left(x - \underline{a}_1\right)}{\underline{a}_2 - \underline{a}_1} & ; \underline{a}_1 \le x \le a_2 \\ \dfrac{\underline{h}\left(x - \underline{a}_2\right)}{2\left(\underline{a}_3 - \underline{a}_2\right)} & ; a_2 \le x \le \underline{a}_3 \\ \underline{h} & ; x = \underline{a}_3 \\ \dfrac{\underline{h}\left(\underline{a}_4 - x\right)}{2\left(\underline{a}_4 - \underline{a}_3\right)} & ; \underline{a}_3 \le x \le \underline{a}_4 \\ \dfrac{h\left(\underline{a}_5 - x\right)}{\left(\underline{a}_5 - \underline{a}_4\right)} & ; a_4 \le x \le \underline{a}_5 \end{cases} \tag{8}$$

where $\overline{h}$ and $\underline{h}$ are the height of $\tilde{\tilde{A}}^U$ and $\tilde{\tilde{A}}^L$, respectively and $\overline{h}, \underline{h} \in [0, 1]$.

Then, interval type-2 pentagonal fuzzy number have been generalized in Figure 1 by Ajay and Pathinathan [25]. Suppose that $A^U_{-IT2P} = \left(a_1, a_2, a_3, a_4, a_5\right)$ and $A^L_{-IT2P} = \left(b_1, b_2, b_3, b_4, b_5\right)$ are two generalized interval type-2 pentagonal fuzzy number with λ^U and λ^L are height of A^U_{-IT2P} and A^L_{-IT2P} as below:

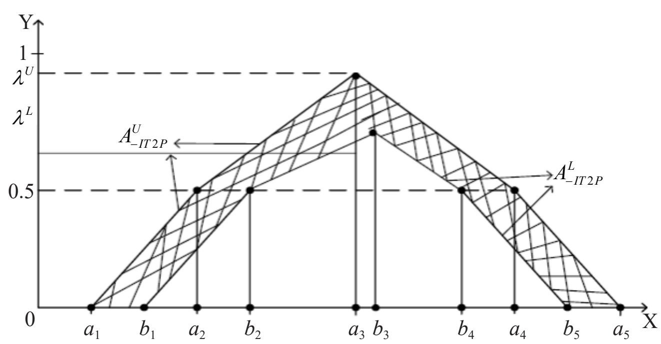

Figure 1. Generalized interval type-2 pentagonal fuzzy number [25]

According to this definition, it can be concluded that the parametric form for pentagonal fuzzy numbers can be obtained as below:

$$\overline{u}(x) = \overline{a}_3 - \overline{a}_4 + \overline{a}_5$$

$$\underline{u}(x) = \underline{a}_1 - \underline{a}_2 + \underline{a}_3 \tag{9}$$

Definition 2.5 Fuzzy polynomial [26]

Polynomial can be written as $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. Then, a fuzzy polynomial equation can be defined as follows:

$$A_1 x + A_2 x^2 + \ldots + A_n x^n = A_0 \tag{10}$$

where $x \in \mathfrak{R}$, the coefficient $A_1, A_2, \ldots, A_n$ and A_0 are fuzzy numbers.

These preliminaries are being used in defining the new IT2 FPP and also in constructing the integration of IT2 FPP and Horner-Muller's method.

3. Interval type-2 fuzzy pentagonal polynomials

This section defines the interval type-2 fuzzy pentagonal polynomials (IT2 FPP) that are generated from the definition of interval type-2 pentagonal fuzzy numbers (see Definition 2.4) and the definition of fuzzy polynomials (see Definition 2.5).

Let $\hat{A}^U = \left(a_1^U, a_2^U, a_3^U, a_4^U, a_5^U\right)$ and $\hat{A}^L = \left(a_1^L, a_2^L, a_3^L, a_4^L, a_5^L\right)$ are the upper and lower for IT2 FPP. Then, polynomial for interval type-2 pentagonal fuzzy is given as:

$$\hat{A}_1 x + \hat{A}_2 x^2 + \ldots + \hat{A}_n x^n = \hat{A}_0 \tag{11}$$

where $x \in R$ and the coefficient $\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_n$ and $\hat{A}_0$ are interval type-2 pentagonal fuzzy numbers.

Then, the IT2 FPP is defined as:

$$\left(\hat{A}_1^U, \hat{A}_1^L\right)x + \left(\hat{A}_2^U, \hat{A}_2^L\right)x^2 + \ldots + \left(\hat{A}_n^U, \hat{A}_n^L\right)x^n = \left(\hat{A}_0^U, \hat{A}_0^L\right) \tag{12}$$

It also can be written as follows:

$$\left\{\left(\overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4}, \overline{a_5}\right)\left(\underline{a_1}, \underline{a_2}, \underline{a_3}, \underline{a_4}, \underline{a_5}\right)\right\}x + \left\{\left(\overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4}, \overline{a_5}\right)\left(\underline{a_1}, \underline{a_2}, \underline{a_3}, \underline{a_4}, \underline{a_5}\right)\right\}x^2 + \ldots +$$

$$\left\{\left(\overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4}, \overline{a_5}\right)\left(\underline{a_1}, \underline{a_2}, \underline{a_3}, \underline{a_4}, \underline{a_5}\right)\right\}x^n = \left\{\left(\overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4}, \overline{a_5}\right)\left(\underline{a_1}, \underline{a_2}, \underline{a_3}, \underline{a_4}, \underline{a_5}\right)\right\} \tag{13}$$

For illustrate this definition of IT2 FPP, we consider one example for IT2 FPP. This example is primarily retrieved from Nurhakimah and Abdullah [28] where the interval type-2 fuzzy polynomial is extended to IT2 FPP

$$\{(0, 1, 1, 1, 1; 1, 1, 1)(0, 0.5, 0.5, 0.5; 0.5; 1, 1, 1)\}x + \{(0, 2, 2, 2, 2; 1, 1, 1)(0, 1, 1, 1, 1; 1, 1, 1)\}x^2 +$$

$$\{(1, 1, 1, 1, 1, 1; 1, 1, 1)(0, 0.5, 0.5, 0.5, 0.5; 1, 1, 1)\}x^3 = \{(-1, 4, 4, 4, 4; 1, 1, 1)(1, 2, 3, 4, 5, 1; 1, 1, 1\}$$

It is good to note that interval type-2 pentagonal fuzzy numbers are the coefficients of the IT2 FPP.

4. Proposed computational procedures

The definition of the IT2 FPP coupled with a computational procedure is synergistically used to find the approximate solution of IT2 FPP equation using the iterative Horner-Muller's method. In this section, the proposed computational procedures are carried out in two phases to solve the IT2 FPP equation using Horner-Muller's method. Phase 1 defines the development of IT2 FPP equations. The computational procedure of Horner-Muller's method is applied in Phase 2 where eventually the approximate solution can be obtained. The computational procedures for solving the IT2 FPP equation using Horner-Muller's method are summarized in Figure 2.

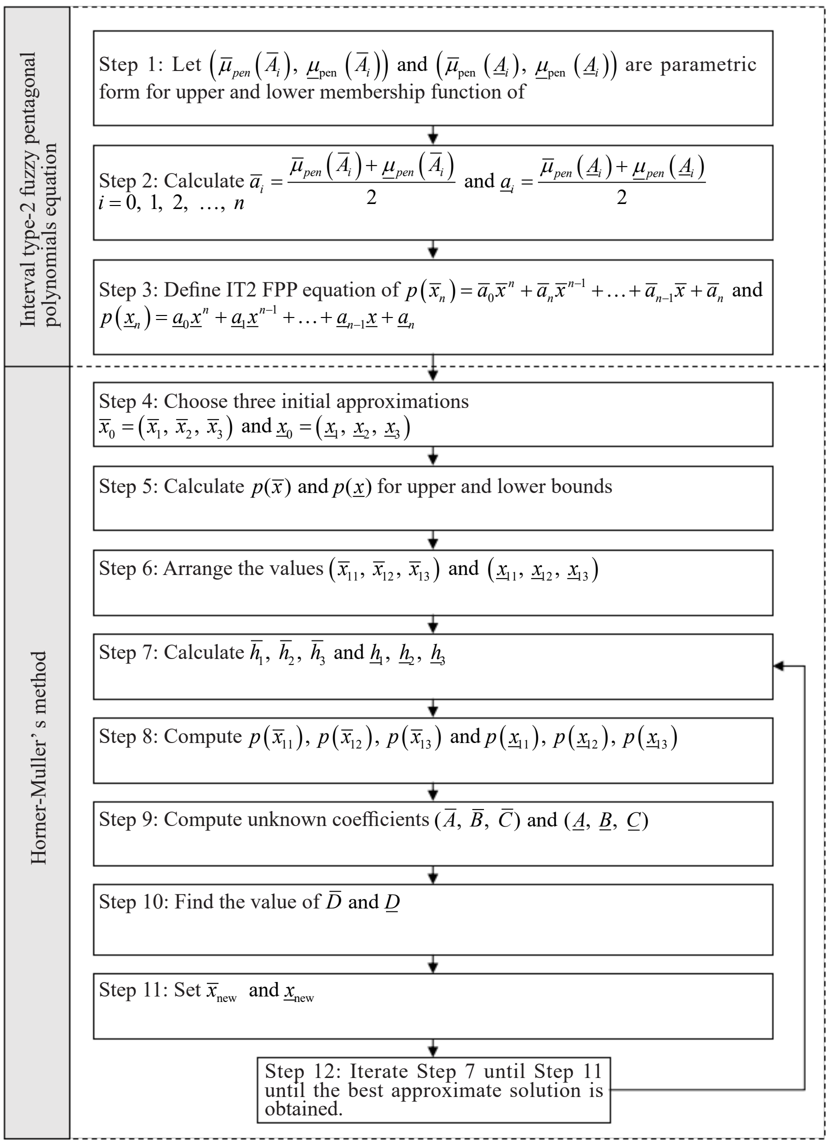

Figure 2. Framework of solution IT2 FPP using Horner-Muller's method

5. Example of finding approximate solution

To illustrate the computational procedures proposed in Section 4, an example of IT2 FPP equation and its computational procedures for finding an approximate solution using Horner-Muller's method are presented.

Example: Let us consider the interval type-2 fuzzy pentagonal for cubic polynomial equations as follows:

$$\{(0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5;\ 1,\ 1,\ 1)(0,\ 0.2,\ 0.3,\ 0.4,\ 0.5;\ 1,\ 1,\ 1)\}x$$

$$+\{(0,\ 0.1,\ 0.3,\ 0.2,\ 0.3;\ 1,\ 1,\ 1)(0.1,\ 0.2,\ 0.3,\ 0.2,\ 0.1;\ 1,\ 1,\ 1)\}x^2$$

$$+\{(0.1,\ 0.1,\ 0.2,\ 0.3,\ 0.1;\ 1,\ 1,\ 1)(0.2,\ 0.2,\ 0.4,\ 0.3,\ 0.2;\ 1,\ 1.1)\}x^3$$

$$=\{(0.2,\ 0.4,\ 0.8,\ 0.9,\ 0.9;\ 1,\ 1,\ 1)(0.3,\ 0.6,\ 1.0,\ 0.9,\ 0.8;\ 1.1,\ 1\}$$

We want to find its approximate solution using Horner-Muller's method.

Step 1: The parametric form for the upper membership functions is given as:

$$\left[\overline{A}_1\right]_l^\alpha = 0.1-0.2+(0.3)(0.5)=0.05 \quad \left[\overline{A}_1\right]_U^\alpha = 0.3-0.4+(0.5)(0.5)=0.45$$

$$\left[\overline{A}_2\right]_l^\alpha = 0-0.1+(0.3)(0.5)=0.05 \quad \left[\overline{A}_2\right]_U^\alpha = 0.3-0.2+(0.3)(0.5)=0.35$$

$$\left[\overline{A}_3\right]_l^\alpha = 0.1-0.1+(0.2)(0.5)=0.1 \quad \left[\overline{A}_3\right]_U^\alpha = 0.2-0.3+(0.1)(0.5)=0.45$$

$$\left[\overline{A}_0\right]_l^\alpha = 0.2-0.4+(0.8)(0.5)=0.2 \quad \left[\overline{A}_0\right]_U^\alpha = 0.8-0.9+(0.9)(0.5)=1.25$$

and the parametric form for the lower membership functions is given as:

$$\left[\underline{A}_1\right]_l^\alpha = -0.05 \quad \left[\underline{A}_1\right]_U^\alpha = 0.45$$

$$\left[\underline{A}_2\right]_l^\alpha = 0.05 \quad \left[\underline{A}_2\right]_U^\alpha = 0.45$$

$$\left[\underline{A}_3\right]_l^\alpha = 0.2 \quad \left[\underline{A}_3\right]_U^\alpha = 0.6$$

$$\left[\underline{A}_0\right]_l^\alpha = 0.2 \quad \left[\underline{A}_0\right]_U^\alpha = 1.5$$

Step 2: Divide the parametric forms of the upper membership functions and the lower membership functions by 2.

$$\overline{a}_1=\frac{0.05+0.45}{2}=0.25,\ \overline{a}_2=\frac{0.05+0.35}{2}=0.2,\ \overline{a}_3=\frac{0.1+0.45}{2}=0.275,\ \overline{a}_0=\frac{0.2+1.25}{2}=0.725$$

and $\underline{a}_1=0.2,\ \underline{a}_2=0.25,\ \underline{a}_3=0.4,\ \underline{a}_0=0.85$

Step 3: Therefore, the IT2 FPP is defined as follows:

$$p(\overline{x}) = 0.275x^3 + 0.2x^2 + 0.25x - 0.725 \text{ and } p(x) = 0.4x^3 + 0.25x^2 + 0.2x - 0.85$$

Step 4: We choose three initial approximations as below:

$$\overline{x}_1 = \underline{x}_1 = 0.15,\ \overline{x}_2 = \underline{x}_2 = 0.55,\ \overline{x}_3 = \underline{x}_3 = 0.85$$

Step 5: For the upper membership function, we substitute the initial values $\overline{x}_{11}$, $\overline{x}_{12}$ and $\overline{x}_{13}$ consecutively into the following arrangements:

$\overline{x}_1 = 0.15$	0.275	0.2	0.25	-0.725
		0.04125	0.03619	0.04293
$\overline{x}_1 = 0.15$	0.275	0.24125	0.28619	-0.68207
		0.04125	0.04238	
	0.275	0.28250	0.32856	

$$p\left(\overline{x}_1\right) = -0.68207 \text{ and } p'\left(\overline{x}_1\right) = 0.32856, \text{ hence } \overline{x}_{11} = 0.15 - \left(\frac{-0.68207}{0.32856}\right) = 2.22593$$

$\overline{x}_1 = 0.55$	0.275	0.2	0.25	-0.725
		0.15125	0.19319	0.24375
$\overline{x}_1 = 0.55$	0.275	0.35125	0.44319	-0.48125
		0.15125	0.27638	
	0.275	0.50250	0.71956	

$$p\left(\overline{x}_1\right) = -0.48125 \text{ and } p'\left(\overline{x}_1\right) = 0.71956, \text{ hence } \overline{x}_{11} = 0.55 - \left(\frac{-0.48125}{0.71956}\right) = 1.21880$$

$\overline{x}_1 = 0.85$	0.275	0.2	0.25	-0.725
		0.23375	0.36869	0.52588
$\overline{x}_1 = 0.85$	0.275	0.43375	0.61869	-0.19912
		0.23375	0.56738	
	0.275	0.66750	1.18606	

$$p\left(\overline{x}_1\right) = -0.19912 \text{ and } p'\left(\overline{x}_1\right) = 1.18606, \text{ hence } \overline{x}_{11} = 0.85 - \left(\frac{-0.19912}{1.18606}\right) = 1.01788$$

Iterate the procedure for the lower membership function $\underline{x}_1$.
Step 6: From Step 5, arrange the values as follows:

$$\overline{x}_{11} = 2.22593 \qquad \underline{x}_{11} = 2.84214$$

$$\overline{x}_{12} = 1.21880 \quad \text{and} \quad \underline{x}_{12} = 1.26339$$

$$\overline{x}_{13} = 1.01788 \qquad \underline{x}_{13} = 1.02006$$

Step 7: Calculate $\overline{h}_1$, $\overline{h}_2$, $\overline{h}_3$ and $\underline{h}_1$, $\underline{h}_2$, $\underline{h}_3$ as below:

$$\overline{h}_1 = 2.22593 - 1.01788 = 1.20805 \qquad \underline{h}_1 = 1.82208$$

$$\overline{h}_2 = 1.21880 - 1.01788 = 0.20092 \quad \text{and} \quad \underline{h}_2 = 0.24334$$

$$\overline{h}_3 = (1.06096)(0.18636)(1.06096 - (0.18636)) = 0.24445 \qquad \underline{h}_3 = 0.69997$$

Step 8: Compute the range of the polynomials as follows:

$$p\left(\overline{x}_{11}\right) = p(2.22593) \to (0.275)(2.22593)^3 + (0.2)(2.22593)^2 + (0.25)(2.22593) - 0.725 = 3.85540$$

$$p\left(\overline{x_{12}}\right) = p(1.21880) \to (0.275)(1.21880)^3 + (0.2)(1.21880)^2 + (0.25)(1.21880) - 0.725 = 0.37468$$

$$p\left(\overline{x_{13}}\right) = p(1.01788) \to (0.275)(1.01788)^3 + (0.2)(1.01788)^2 + (0.25)(1.01788) - 0.725 = 0.02670$$

$$\underline{p}\left(\underline{x}_{11}\right) = 10.92112$$

$$\underline{p}\left(\underline{x}_{12}\right) = 0.60834$$

$$\underline{p}\left(\underline{x}_{13}\right) = 0.03870$$

Step 9: Compute the coefficients as below:

$$\overline{A} = \frac{(0.20092)(3.85540 - 0.02670) - (1.20805)(0.37468 - 0.02670)}{0.24445} = 1.42722$$

$$\overline{B} = \frac{(1.20805)(0.37468 - 0.02670) - (0.20092)^2(3.85540 - 0.02670)}{0.24445} = 1.44517$$

$$\overline{C} = 0.02670 \ \underline{A} = 2.30024 \ \underline{B} = 1.78131 \ \underline{C} = 0.03870$$

Step 10: Find the value of $\overline{D}$ and $\underline{D}$ as below:

$$\overline{D} = \sqrt{(1.44517)^2 - 4(1.42722)(0.02670)} = 1.39143$$

$$\underline{D} = 1.67839$$

Step 11: Find the solution.

$$\overline{x}_{\text{new}} = 1.017880 - \frac{2(0.02670)}{1.44517 + 1.39143} = 0.99905$$

$$\underline{x}_{\text{new}} = 0.99769$$

Step 12: Iterate Step 7 to Step 11 until the best approximate solution is obtained. Table 1 shows the approximate solutions in the form of the upper membership function and the lower membership function.

Table 1. The approximate solutions

Upper membership function			Lower membership function		
t	$\overline{x}$	e	t	$\underline{x}$	e
0	0.9990530426	0.0009469574	0	0.9976876869	0.0023123131
1	1.000000691	-0.000000691	1	1.000002575	-0.000002575
2	1.000000000	0	2	1.000000000	0

According to Table 1, the upper membership function of IT2 FPP produced an approximate solution, $\overline{x} = 1.00000000$ after 2 iterationsand (t) errors, $e = 0$. Then, the approximate solutions for lower membership functions are also obtained after 2 iterations, which is $\underline{x} = 1.000000000$ and also has $e = 0$. The result shows that it has the lowest number of iterations to converge to the approximated solution. Therefore, it can be concluded that Horner-Muller's method successfully produced an approximate solution to solve IT2 FPP.

6. Comparative analysis

Based on the numerical example in Section 5, the proposed method is compared with two existing iterative methods under the IT2 FPP. Horner's method and Muller's method are incorporated into this comparative analysis. In this comparative analysis, three efficiency measures, which are approximate solutions, number of iterations and errors, are compared. In this comparative analysis, the number of iterations reflects the repetition of a process to generate the best approximate solution. Error is also used to maximize the possibility of the closeness of exact solutions and approximate solutions. Table 2 summarizes the results of approximate solutions, errors and number of iterations that are solved by using three iterative methods.

Table 2. The comparison with other iterative methods

Criteria	Horner's method		Muller's method		Horner-muller's method	
	$\overline{x}$	$\underline{x}$	$\overline{x}$	$\underline{x}$	$\overline{x}$	$\underline{x}$
Approximated solution	0.9999999999	1.000000000	0.9999999999	1.000000000	1.0000000000	1.0000000000
Error	0.0000000001	0	0.0000000001	0	0	0
Number of iterations	4	4	3	3	2	2

The following conclusions are drawn from a comparison between different iterative methods:

It can be seen that the proposed method shows a different result compared to the other two iterative methods.

The upper membership function of IT2 FPP, Horner-Muller's method, produced an approximate solution, $\overline{x} = 1.000000000$ where Horner's and Muller's methods have the same approximate solution, $\overline{x} = 1.9999999999$. For the lower membership function of IT2 FPP, these three iterative methods have the same approximate solution, which is $\overline{x} = 1.00000000$.

For the upper and lower membership functions of IT2 FPP, Horner-Muller's method has errors, $e = 0$ where the upper membership function of Horner's and Muller's methods has the same error, $e = 0.0000000001$ and also have the same error $e = 0$ for the lower membership function of IT2 FPP. This shows Horner-Muller's method has the smallest error for the upper and lower membership functions of IT2 FPP.

Horner-Muller's method has 2 iterations, whereas Horner's method has 4 iterations and Muller's method has 3 iterations. It can be seen that Horner-Muller's method has the lowest number of iterations to converge to the approximated solutions.

Therefore, it can be concluded that Horner-Muller's is the most efficient method for solving IT2 FPP equations compared to Horner's method and Muller's method.

7. Conclusions

Previous literature indicates that there is limited research about solving IT2 FPP equations using iterative methods. In this study, we defined an IT2 FPP and applied Horner-Muller's method as an iterative method to solve the IT2 FPP. In addition, this study illustrated the computational procedure for finding an approximate solution of IT2 FPP using one numerical example. Based on this example, it shows that the solutions for IT2 FPP have produced an approximate solution after two iterations. Hence, Horner-Muller's method can be concluded to be one of the most feasible iterative methods to solve the IT2 FPP. This study also presented a comparative analysis of performance between Horner-Muller's method and other iterative methods, which are Horner's method and Muller's method to find the consistency and efficiency of IT2 FPP. Based on the efficiency results from this study, it is suggested that the IT2 FPP could be solved using other iterative methods such as Newton Raphson method, Bisection method, Secant method and Brent's method as one of the projects in future research directions.

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Conflicts of interest

The authors declare no competing financial interest.

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