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Research Article

Asymptotic Behavior of Trajectories in Some Models of Ecnomic Dynamics

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Abstract: Asymptotic behavior of trajectories in Neumann type models of economic dynamics with average growth rate is studied. Index set is introduced and the sequence of cones generated by the cone Z of the Neumann-Gale model is considered. The concept of quasi rate of the model z_i is introduced. The relationship between the concepts of average growth rate and quasi rate is found. The relationship between the turnpikes M_α for different values of x and the set A_z of conic hulls of the sets of all angular distance limit points is examined. Upper and lower estimates for a non-empty turnpike are obtained. Under some additional conditions, more accurate lower estimate is obtained which allows to conclude that in most cases the set M_α is a subset of the set A_z. Algorithm is proposed for constructing a trajectory X_t, which has the point x among its angular distance limit points. Theorem on the existence of a turnpike which has the point x among its angular distance limit points is proved.

***Keywords*:** asymptotics, trajectory, growth rate, turnpike

MSC: 37M05, 91B55, 91B02.

1. Introduction

Asymptotic behavior of trajectories of various classes is of great interest in Neumann type models of economic dynamics. In this work, the asymptotic behavior of trajectories with average growth rate α is studied. These trajectories represent both an independent and an applied interest as in many cases they help to describe asymptotic behavior of optimal trajectories. A lot of research have been dedicated to related problems [1-5]. The papers study the asymptotic properties of solutions of matrix linear models in discrete time. In [6], the limiting behavior of trajectories for some classes of Neumann-Gale models is considered. In [7], a theorem on the asymptotic behavior of trajectories in a Leontief-type nonlinear model under certain conditions was proved. Dynamic systems with trajectories given by the sequences of sets are studied [8]. In [9], the behavior of optimal trajectories in economic growth models with Cobb-Douglas production functions is studied. In [10], the asymptotic behavior of solutions of linear dynamical models with discrete time was studiet. In [11-13], several theorems on the turnpike properties of trajectories for some models of economic dynamics of the Leontief type were proved.

R. Radner and L. Mckenzie [13] proved several theorems on the main properties of trajectories for some models of

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economic dynamics of the Leontief type. In [], the turnpike properties of optimal trajectories of Leontief-type models were studied. A theorem is proved that, under certain conditions, the turnpike of the production mapping is a ray.

Similar results were also obtained in [13-15].

First, we give a definition of some concepts and notation used in this paper. Models of the Neumann type of economic dynamics are given by multi-valued mappings $a(x)$, which are defined as follows:

$$a : R_+^n \to \pi\left(R_+^n\right)$$

so that

$$a(x) = \{y|(x, y) \in Z\}, Z \subset R_+^n \times R_+^n,$$

where R_+^n the positive orthant of the space R_n and Z is a polyhedral convex closed cone. The vector is called the resource vector, the vector y is called the output vector.

A sequence of vectors $\{x_t\}_{t=1}^{\infty}$ is called a trajectory of the Z model if $x_{t+1} \in a\left(x_t\right)$.

The purpose of this work is to obtain upper and lower bounds for a non-empty turnpike and propose an algorithm for constructing a trajectory that has a point among all limit points of the angular distance.

Definition [6]. The trajectory of the $\{x_t\}_{t=0}^{\infty}$ model Z is said to have an average growth rate α, (α > 0) if for some $p \in riQ_\alpha$ where $Q_\alpha = \left\{p > 0 | \ p \in a'(p)\right\}$; in other words, if it $\{x_t\}$ is consistent with the trajectory of the dual model Z' of the form, for $\varphi_p = (p, \alpha^{-1}p, ..., \alpha^{-1}p, ...)$ at $p \in riQ_\alpha$.

Definition [6]. The number α is called the quasi-growth rate of the Neumann-Gale model if there is a process $(x, y) \in Z$ such that $\alpha x \le y$ the inequality $y \le \left(y' - \alpha x'\right)$ has no solution $\left(x', y'\right) \in Z$.

The set of indices is introduced as follows: a sequence of cones $Z_1, ..., Z_n$ is constructed in Z_i, $i = \overline{1, N}$ such a way that each is a projection of the previous one onto some face of the cone $Z \subset R_+^n \times R_+^n$.

Our work is based on the construction method presented in [6, 11]. We describe it below.

Let Z be a convex cone in $R_+^n \times R_+^n$ such that $P_r Z \cap \operatorname{int} R_+^n \neq \varnothing$. By Neumann growth rate of the cone Z we mean a number

$$\alpha = \sup_{\substack{(x, y) \in Z \\ (x, y) \neq 0}} \min_{i \in I} \frac{y^i}{x^i}$$

where $I = \{1, 2, ..., n\}$.

The sequence (x_k, y_k) of elements of the cone Z is called a Neumann sequence if

$$\min_{i \in I} \frac{y_k^i}{x_k^i} \to \alpha$$

Consider the set of indices $I_Z \subset I$. The relation $i \in I_Z$ is true if and only if there exists a Neumann sequence (x_k, y_k) such that $y_k^i > 0$ $(k = 1, 2, \ldots)$.

Let Z be a Neumann-Gale model. The cone generates the finite sequence of cones $Z_1, Z_2, ..., Z_n$ in the following way. Let $Z_1 = Z$, denote $R_+^n = K_1$. So, $Z_1 \subset K_1 \times K_1$. If $I_1 = I_{Z1} = I$, then the process is over; if $I^1 \neq I$, then we consider the face K_2 of the cone R_+^n, stretched on the unit vectors with the numbers from $I \setminus I^1$, and define Z_2 as a projection of the cone K_1 onto the face $K_2 \times K_2$ of the cone $R_+^n \times R_+^n$

If $I^2 = I_{Z_2} = I \setminus I^1$, then the process is over; if otherwise, we consider the face K_3 of the cone R_+^n, stretched on the unit vectors with the numbers from $I \setminus \left(I^1 \cup I^2\right)$, and denote by Z_3 the projection of Z_2 onto the face $K_3 \times K_3$ of the cone. $R_+^n \times R_+^n$. If $I^3 \equiv I_{Z_3} = I \setminus \left(I^1 \cup I^2\right)$, the we construct the cone Z_4, etc.

This process will be over after some step N.

As a result, we will have constructed the cones Z_j and the sets of indices I^j (j = 1, 2, ... N), with $I^j \cap I^{j_1} = \varnothing$, $I^j \equiv I_{Z_j}$ $\left(j \neq j_1\right)$ Denote by α_j the Neumann growth rate of the cone Z_j. The numbers α_j will be called the growth quasi rates of the model. It is known [9] that $\alpha_{j-1} > \alpha_j$.

Throughout this paper, we will use the terminology of [6, 16].

2. Main part

The triple $(\alpha, (\bar{x}, \bar{y}), \bar{p})$, where α is a positive number, $(\bar{x}, \bar{y}) \in Z$, and $\bar{p} \in \left(R_+^n\right)^*$ is a positive functional, is called an equilibrium state of the model Z, if the following conditions hold:

$$\alpha\bar{x} \leq \bar{y},$$

$$\bar{p}(y) \leq \alpha\bar{p}(x) \text{ for every } (x, y) \in Z,$$

$$\bar{p}(\bar{y}) > 0.$$

The number α, appearing in the definition of the equilibrium state, is called a growth rate of the model . Every growth rate of the model is a quasi rate, the opposite is not true [6].

The trajectory X is said to have an average growth rate α, if there exists a trajectory φ_α of the dual model of the form

$$\varphi_\alpha = \left(p,\ \alpha^{-1}p,\ \ldots,\ \alpha^{-n}p,\ \ldots\right)$$

coordinated wtih x.

As is known [6], if for some α there exists a trajectory with an average growth rate α, then α is a growth quasi rate of the model.

Obviously, for the growth rate α there always exists a trajectory with an average growth rate α.

By turnpike M_α we mean [22] a conical hull of the set of all angular distance limit points of all trajectories with average growth rate . Angular distance between the points and is defined as $\left\| \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\|$.

Denote by A_z the conical hull of the set of all angular distance limit points of all optimal trajectories.

The conditions which provide $M_\alpha \subset A_z$ and even $M_\alpha \subset A_z$ are stated in [8, 9].

In this work, we continue studying the relationship between the sets M_a and A_z for different values of a. We obtain upper and lower estimates for a non-empty turnpike (Corollary of Lemma 1, Lemma 2). Also, under some conditions we get a more precise lower estimate, which allows us to conclude that in a fairly large range of cases the relation $M_\alpha \subset A_z$ is true not only when is a quasi rate, but also when is a gowth rate of the model. (Corollary of Theorem 2).

Besides, we show Theorem 1 that if M_{aj} and M_{ak} are non-empty, then for $j > k$

$$M_{\alpha j} \subset M_{\alpha\mu}.$$

Let's introduce the following notations:

R^j is a space stretched on the unit vectors with the numbers from the set of indices J.

R_+^J is a positive orthant of the space R^j.

We consider the Neumann-Gale model $Z \subset R_+^n \times R_+^n$, which has quasi rates, and its dual model Z'.

Lemma 1 For every quasi rate α_j, every number $\lambda > \alpha_j$, every index $i \in \bigcup_j^N I_\mu$ and every trajectory $X = (x_t)$ there exists a limit

$$\lim \lambda^{-t} \chi_t^i = 0,$$

Proof. Assume that there exist a trajectory $X = (x_t)$, the quasi rates a_j, a number $\lambda_0 = \alpha_j$ and an index $i \in \bigcup_j^N I_\mu$ such that the above lemma is not true, i.e. $\overline{\lim}\lambda_0^{-t} x_t^{i_0} = c > 0$.

Choose $\lambda_1 \in \left(\lambda_0^{-1},\ \alpha_j^{-1}\right)$. Using the results of [6, 16], it is not difficult to show the existence of prices p satisfying the following conditions:

$$\left(p,\ \lambda_1 p\right) \in Z',$$

$$p^i = 0 \text{ for every } i \in \bigcup_1^{j-1} I_\mu,$$

$$p^i > 0 \text{ for every } i \in \bigcup_j^N I_\mu.$$

In fact, by Theorem 2 of [6], the numbers $\dfrac{1}{\alpha_j}$ (and only them!) are the growth quasi rates of the dual model Z'. On the other hand, by Lemma 1 of [6], for every $\alpha : \dfrac{1}{\alpha_j} < \alpha < \dfrac{1}{\alpha_{j-1}}$ there exists a functional $p \in \left(R_+^n\right)^*$ such that

$$(p,\ \alpha p) \in Z'$$

$$p^i = 0 \text{ for every } i \in \cup_1^{j-1} I_\mu,$$

$$p^i > 0 \text{ for every } i \in \bigcup_j^N I_\mu.$$

Taking into account that $\dfrac{1}{\alpha_j} < \dfrac{1}{\lambda_0}$ and, we obtain the existence of the prices p for λ_1 with the above properties.

Consider the sequence $\varphi = (p_t)$, where p_0, $p_t = \lambda_1 p_{t-1}$. It is clear that $p_t x_t$ does not increase monotonically with the growth of t.

On the other hand, assuming that the lemma is not true, we obtain

$$p_t x_t \geq p_t^{i_0} x_t^{i_0} = \lambda_1^t p_0^{i_0} x_t^{i_0} = \left(\frac{\lambda_1}{\lambda_0^{-1}}\right)^t p_0^{i_0} \lambda_0^{-t} x_t^{i_0},$$

where $\dfrac{\lambda_1}{\lambda_0^{-1}} > 1$, $\lambda_0^{-t} x_t^{i_0} = c > 0$. Consequently, $\overline{\lim}\lambda_0^{-t} x_t^{i_0} = c > 0$, which is impossible. This contradiction proves the lemma.

Corollary. For every angular distance limit point x of any trajectory X with average growth rate α_{j-1} for $1 < j \leq N$, the following relation is true:

$$x^i = 0 \text{ for every } i \in U_j^N I_\mu.$$

Proof. As the quasi rates are decreasing monotonically with the growth of j, applying the theorem for $\lambda = \alpha_{j-1}$ we get the validity of the corollary.

Lemma 2 If the turnpike $M_{\alpha j}$ is non-empty, then every point x of the form

$$x^i > 0 \text{ for every } i \in \bigcup_1^{j-1} I_\mu,$$

$$x^i = 0 \text{ for every } i \in \bigcup_j^N I_\mu \tag{1}$$

lies on the turnpike $M_{\alpha j}$.

Proof. Let $\overline{x} \in R_+^n$ be an arbitrary point satisfying (1). Let's construct the trajectory $X = (x_k)$ with average growth rate , for which $\overline{x}$ will be an angular distance limit point.

Recall that $\alpha_{j-1} > \alpha_j$. Choose the number $\lambda \in \left(\alpha_j,\ \alpha_{j-1}\right)$ and consider the sequence of positive numbers β_k which converges monotonically to $+\infty$.

As $\lambda > \alpha_{j-1}$, from the definition of a quasi rate it follows that there exists a point $\overline{x} \in R_+^n$ which satisfies (1) and. $(x,\ \lambda x) \in Z$.

The points x and $\overline{x}$ have the same non-zero coordinates. Consequently, for some θ_1, θ_2 the inequality

$$\theta_2 x < \theta_1 \overline{x} < x \tag{2}$$

holds.

Choose $\overline{\lambda} \in \left(\alpha_j,\ \lambda\right)$. For every β_k there exists a positive integer m_k such that

$$\theta_2 \left(\frac{\lambda}{\overline{\lambda}}\right)^{m_k} > \beta_k$$

Let's construct the trajectory $X_1 = (y_k)$ such that $\overline{x}$ is one of its angular distance limit points. Describe the first step of constructing such trajectory. Let

$$y_0 = x,$$

$$y_t = \lambda y_{t-1} \text{ for every } \overline{1,\ m_1 - 2}.$$

Obviously, $y_t \in a\left(y_{t-1}\right)$, because $(x,\ \lambda x) \in Z$. Further, let

$$y_{m_1 - 1} = \theta_1 \lambda^{m_1 - 1} \overline{x}$$

$$y_{m_1} = \theta_2 \lambda^{m_1} x.$$

In view of (2), we have

$$y_{m_1 - 1} \in a\left(y_{m_1 - 2}\right) \text{ and } y_{m_1} \in a\left(y_{m_1 - 1}\right).$$

Describe the k-th step of construction. Let

$$y_t = \lambda y_{t-1} \text{ for } \sum_1^{k-1} m_l + 1 \le \sum_1^k m_l - 2,$$

$$y_t = \theta_1 \theta_2^{k-1} \lambda^t \overline{x} \text{ for } t = \sum_1^k m_l - 1,$$

$$y_t = \theta_2^k \lambda^t x \text{ for } t = \sum_1^k m_l.$$

As the point y_t is proportional to x for $t = \sum_1^{k-1} m_l$, it is obvious that $y_t \in a(y_{t-1})$ for

$$\sum_1^{k-1} m_l + 1 \le t \le \sum_1^k m_l - 2$$

Then, as (2) is satisfied, we have $y_t \in a(y_{t-1})$ for $t = \sum_1^{k-1} m_l - 1$ and $y_t \in a(y_{t-1})$ for $t = \sum_1^{k-1} m_l$.

Thus, we have constructed the trajectory X, which (it is not difficult to see) has the point x among its angular distance limit points.

Now let $X_2 = \left(\overline{y}_t\right)$ be an arbitrary trajectory with average growth rate. As shown in [17], the sequence $X_3 = \left(\check{y}_t\right)$ with

$$\check{y}_t^i = 0 \text{ for every } i \in \bigcup_1^{j-1} I_\mu,$$

$$y_t = \tilde{y}_t^i \text{ for every } i \in \bigcup_j^N I_\mu$$

is also a trajectory of the model with average growth rate a_j.

Construct the trajectory $X = (xt)$ letting

$$x_t^i = \check{y}_t^i \text{ for } i \in \bigcup_j^N I_\mu,$$

$$x_t^i = y_t^i \text{ for } i \in \bigcup_1^{j-1} I_\mu,$$

or, which is the same thing,

$$x_t = \check{y}_t + y_t$$

Obviously, X has an average growth rate a_j. As $\overline{\lambda} > \alpha_j$, it follows from Lemma 1 that

$$\lim \overline{\lambda}^{-t} x_t^i = 0 \text{ for every } i \in U_j^N I_\mu.$$

On the other hand, for $t = \sum_1^k m_l - 1$ we have

$$\bar{\lambda}^t x_t^i = \theta_1 \theta_2^{k-1} \left(\frac{\lambda}{\bar{\lambda}}\right)^t \bar{x} = \frac{\theta_1}{\theta_2} \frac{\bar{\lambda}}{\lambda} \prod_{l=1}^{k} \left[\left(\frac{\lambda}{\bar{\lambda}}\right)^{m_l} \theta_2\right] x > \frac{\theta_1}{\theta_2} \frac{\bar{\lambda}}{\lambda} \beta_k \bar{x}.$$

Then, in view of $\beta_k \to +\infty$ and $\overline{\lim} \bar{\lambda}^{-t} x_t^i = +\infty$ for $i \in U_1^{j-1} I_\mu$ for $\overline{\lim} \bar{\lambda}^{-t} x_t^i = +\infty$ for $i \in U_1^{j-1} I_\mu$, x is an angular distance limit point for the trajectory. The lemma is proved.

Theorem 1 If the turnpikes $M_{\alpha j}$ and $M_{\alpha\mu}$ are non-empty and $j < \mu$, then

$$M_{\alpha_j} \subset M_{\alpha_\mu}.$$

Proof. Let $X = (x_t)$ be an arbitrary trajectory with average growth rate α_j. Also let $X_1 = (\bar{x}_t)$, where

$$\bar{\chi}_t^i = 0 \text{ for every } i \in \bigcup_1^{u-1} I_l,$$

$$\bar{\chi}_t^i = x_t^i \text{ for every } i \in \bigcup_\mu^N I_l.$$

Obviously, X_1 is also a trajectory with average growth rate α_j

Let x be an arbitrary point in $M_{\alpha j}$. By the definition of $M_{\alpha j}$, there exists a trajectory $X_3 = (y_t)$ with average growth rate α_j such that the point x is among its angular distance limit points.

Consider the trajectory $X_4 = X_1 + X_3$. Obviously, X_4 has an average growth rate α_μ. Let $\bar{y}_t = \bar{x}_t + y_t$. By the corollary of Lemma 1,

$$\lim \alpha_j^{-t} \bar{y}_t^i = 0 \text{ for every } i \in \bigcup_{j+1}^N I_l.$$

And, as $x \in M_{\alpha_j}$, we have

$$\lim \alpha_j^{-t} \bar{y}_t^i > 0 \text{ for every } i : x^i > 0.$$

Taking into account that

$$\bar{x}_t^i = 0 \forall i \in \bigcup_1^{\mu-1} I_l, \text{ for every } t \text{ and } j < \mu,$$

we obtain $x \in M_{\alpha_j}$. The theorem is proved.

Theorem 2 Let Z be a Neumann-Gale model, and let there exist a trajectory X and an infinite set of time moments τ such that the sets of indices $I = \{1, 2, ..., n\}$ can be divided into three subsets J_1, J_2, J_3 as follows:

$$x_t^i = 0 \text{ for every } i \in J_1,\ t \in \tau, \tag{3}$$

$$0 < c_1 \le x_t^i \le c_2 < \infty \text{ for every } i \in J_2,\ t \in \tau, \tag{4}$$

$$\lim x_t^i = +\infty \text{ for every } i \in J_3. \tag{5}$$

Then, for every point $x \in R_+^n$ satisfying the conditions

$$x^i = 0 \text{ for every } i \in J_1 \cup J_2,$$

$$x^i > 0 \text{ for every } i \in J_3, \tag{6}$$

there exists a trajectory X_1, which satisfies the conditions (3)-(5) and has the point x among its angular distance limit points.

Proof. Let x be an arbitrary point in $x \in R_+^n$ satisfying the conditions (6). To prove the theorem, it suffices to construct the trajectory X_1 with the properties (3)-(5), which has the point x among its angular distance limit points. Introduce the following notations:

$\check{x}$ is a projection of $x \in R_+^n$ onto R^{J1},

$\tilde{x}$ is a projection of $x \in R_+^n$ onto R^{J2},

$\overline{x}$ is a projection of $x \in R_+^n$ onto R^{J3}.

Consider the sequence of positive numbers $\psi_k < 1$ such that

$$\prod_{t=1}^{\infty} \psi_t = c > 0$$

Select the subset $\tau_1 = \{t_1, ..., t_m ...\} \subset \tau$ of the set of moments τ such that

$$\tilde{x}_{t_m+1} \geq \varphi_{t_m} \tilde{x}_{t_m} \text{ for every } t_m \in \tau \tag{7}$$

$$\forall t_m \in \tau_1 \exists \lambda_{t_m} : \overline{x}_{t_m} \leq \lambda_{t_m} \overline{x} \leq \overline{x}_{t_m+1} \tag{8}$$

The existence of such a set follows from (4) and (5). In fact, as $\lim \overline{x}_t^i = +\infty$ and the points $\overline{x}_{t_m}$ and $\overline{x}$ have the same non-zero coordinates, there exists $\tau' \subset \tau$ such that the relation (8) is true for every $t \in \tau'$. The condition of the theorem implies that the sequence $(\overline{x}_t)$, where $t \in \tau'$, has a thickening point in $R_+^{J_2}$. But then we can choose $\tau_1 \in \tau'$ such that the relation (7) is satisfied for every $t \in \tau'$.

To construct our trajectory, we start with the trajectory $X = (x_t)$ appearing in the theorem.

Step 1. Let

a) $y_t = x_t$ for every $t < t_2$;

b) $\check{y}_{t_2} = 0$, $\tilde{y}_{t_2} = \tilde{x}_{t_2}$, $\overline{y}_{t_2} = \lambda_1 \overline{x}$

Obviously, $y_{t_2} \in a\left(y_{t_{2-1}}\right)$, because $y_{t2} < x_{t2}$

Step 2. From the properties (6) and (7) of the trajectory X it follows that

$$\tilde{y}_{t_2} \geq \psi_1 \tilde{x}_{t_1}, \overline{y}_{t_2} \geq \overline{x}_{t_2}.$$

As $\check{y}_{t_2} = \check{x}_{t_1} = 0$ and $\psi_1 < 1$, we have $y_{t_2} \geq \psi_1 x_1$. But then $\psi_1 a\left(x_{t_1}\right) \subset a\left(y_{t_2}\right)$. This inclusion allows constructing the $t_2 + s_2$-th piece of X_1 letting

a) $y_{t_2+\tau} = \psi_1 x_{t_1+\tau}$ for $\tau \in [1, s_2 - 1]$;

b) $y_{t_2+s_2} = 0, \tilde{y}_{t_2+s_2} = \psi_1 \tilde{x}_t, \overline{y}_{t_2+s_2} = \lambda_2 \psi_1 \overline{x}$, where $s_2 = t_3 - t_1$.

Step k. Repeating the method of Step 2, let

a) $y_{t_k+\tau} = \prod_1^{k-1} \psi_l x_{t_k+\tau}$ for $1 \le \tau \le s_k + k - 3$;

b) $y_{t_k+s_k} = 0$, $\tilde{y}_{t_k+s_k} = \prod_1^{k-1} \psi_l \tilde{x}_{t_k+1}$, $\bar{y}_{t_k+s_k} = \lambda_k \prod_1^{k-1} \psi_l \bar{x}$,

where $s_k = t_{k+1} - t_1 + k - 2$.

Thus, we have constructed the trajectory $X_1 = (y_t)$. From construction it follows that the point x is one of the angular distance limit points of X_1. It also follows that X_1 possesses the properties (3)-(5). The theorem is proved.

Corollary If there exists the trajectory X with average growth rate α_j satisfying the conditions

$$x_t^i = 0 \text{ for every } i \in J_1 \text{ and } t \in \tau, \tag{9}$$

$$0 < c_1 \le \alpha_j^{-t} x_t^i \le c_2 < \infty \text{ for every } i \in J_2 \text{ and } t \in \tau, \tag{10}$$

$$\overline{\lim} \alpha_j^{-t} x_t^i = +\infty \text{ for every } i \in J_3, \tag{11}$$

then $R_+^{J_3} \subset M_{\alpha_j}$.

Proof. Without loss of generality, by virtue of the homogeneity of Neumann-Gale model we can assume $\alpha_j = 1$. Then, applying Theorem 2, we obtain the desired result. The corollary is proved.

Let's make some remarks to clarify the meaning of the conditions (9)-(11). If there exists the trajectory with average growth rate α_j, then it is obvious that

$$\mathrm{U}_{j+1}^N I_\mu \subset J_1$$

$$J_3 \subset \cup_1^j I_\mu.$$

The set J_1 can be empty.

From the point of view of corollary, it doesn't matter at all whether α_j is a growth rate or not. The only thing that matters is the satisfaction of conditions (9)-(11). In case where $J_3 = U_1^r I_\mu$, the corollary to Theorem 2 gives us nothing new compared to the result of Lemma 2.

It is obvious that when α_j is a growth rate and there exist the equilibrium prices p:$p^i > 0$ for every $i \in I_r$, then the following relation is always true:

$$J_3 = \cup_1^{j-1} I_\mu.$$

It can be shown that when α_j is not a growth rate and there exists the trajectory with average growth rate α_j, then the following relation is always true:

$$J_3 \cap I_j \ne \varnothing. \tag{12}$$

In case where α_j is a growth rate with no corresponding positive pries in J_j, the satisfaction of (9) depends on the properties of the considered model. Let's mention a rather typical case, where the conditions (9)-(11) and (12) hold. For example, a two-dimensional model Z, consisting of two processes:

$$(1,\ 0) \to (1,\ 0) \text{ and } (1,\ 1) \to (2,\ 1)$$

So, we have obtained an upper estimate for a non-empty turnpike $M_{\alpha j}$

$$\operatorname{int} R_+^{J'} \supset \operatorname{ri} M_{\alpha_j}, \text{ where } J' = \cup_1^j I_\mu,$$

a lower estimate

$$R_+^{J''} \subset M_{\alpha_j}, \text{ where } J'' = \bigcup_1^{j-1} I_\mu,$$

and a more precise lower estimate under conditions (9)-(11):

$$R_+^{J_3} \subset M_{\alpha_j}$$

It is clear that the validity of the inclusion

$$R_+^{J_3} \subset A_z$$

requires some special properties of the model, for example, $J_3 \cap I_j \neq \varnothing$ or the condition that the intersection $I_j \cap J_3$ consists of one or the other indices, i.e. the turnpike in general can be "great", namely, it can be $M_{\alpha_j} \not\subset A_z$ regardless of whether is a growth rate or not.

3. Results

The cone set Z_i and the index set I^i are constructed.

The relationship between the set of turnpikes M_α for different values of α and the sets A_i of conic hulls of the sets of all angular distance limit points of optimal trajectories is examined.

Upper and lower estimates for a non-empty turnpike are obtained. Under some additional conditions, more accurate lower estimate is obtained.

Algorithm is presented for constructing a trajectory X, which has the point x among its angular distance limit points.

4. Conclusions

The main result of the study is the construction of an algorithm for constructing trajectories with an average growth rate having a point among all points limiting the angular distance. The obtained results can be used to construct highways in models of economic dynamics of the Neumann type with discrete time, which have an average growth rate, and can also be useful in studying the asymptotic properties of trajectories on an infinite time interval.

Conflict of interest

The author declares no competing financial interest.

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