Contemporary Mathematics
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UNIVERSAL WISER PUBLISHER

Research Article

Ramanujan Summation for Pascal's Triangle

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Received: 2 June 2023; **Revised:** 28 June 2023; **Accepted:** 31 July 2023

Abstract: The concept of Pascal's triangle has fascinated not only professional mathematicians but also everyone interested in exploring science. Similarly, the idea of Ramanujan summation has made a revolution in mathematical research after it was introduced by Srinivasa Ramanujan. In this paper, we will provide the Ramanujan summation methods for numbers located in slant diagonals of Pascal's triangle and derive a generalized formula for such summations.

***Keywords*:** Pascal's triangle, Ramanujan summation, Pascal's identity, hockey stick identity, Stirling's numbers of first kind

MSC: 05A18, 05A19, 11B73, 11Y60

1. Introduction

Though the concept of Pascal's triangle is known in the name of French mathematician and scientist Blaise Pascal, it has been known to ancient Indians and Chinese mathematicians who had explored several properties related to it. The beauty about Pascal's triangle is that it is so simple and possesses enormously rich mathematical properties. Great Indian mathematician Srinivasa Ramanujan described a novel way of summing up divergent series in the concept similar to Cesaro summation. Using the work related to Ramanujan summation as presented in [1], in this paper, we wish to derive curious results concerned with determining Ramanujan summation for numbers located in slant diagonals of Pascal's triangle and derive a general formula. For knowing about sum of powers of natural numbers and summation of infinite series, see [2-4].

2. Definition

Let $\sum_{n=1}^{\infty} a_n$ represent any divergent series of real numbers. The Ramanujan summation (RS) of $\sum_{n=1}^{\infty} a_n$ (see [1]) is defined by

DOI: https://doi.org/10.37256/cm.5120243150

$$(RS)\left(\sum_{n=0}^{\infty} a_n\right) = \int_{n=-1}^{0}\left(\sum_{r=0}^{n-1} a_r\right) dn. \tag{1}$$

For knowing more about Ramanujan summation and other related ideas see [5-8].

3. Pascal's triangle

Pascal's triangle is a triangular array of numbers whose entries are coefficients of the binomial expansion of the form $(a + b)^n$ where n is a non-negative integer. The triangle is displayed in Figure 1.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Figure 1. Pascal's triangle

We notice from Figure 1, that every row begins and ends with 1 and each entry of the triangle between the extreme 1's is obtained by adding two successive entries from previous row. In general, the rth element in nth row of the Pascal Triangle in Figure 1 is given by the binomial coefficient

$$\binom{n}{r} = \frac{n!}{r! \times (n-r)!},\ 0 \le r \le n. \tag{2}$$

Since

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \tag{3}$$

and the sum of entries in nth row of the Pascal triangle in Figure is given by

$$\sum_{r=0}^{n} \binom{n}{r} = (1+1)^n = 2^n \tag{4}$$

We will now prove an interesting and well-known property involved with Pascal's triangle.

3.1 *Hockey stick identity*

Theorem 1. If k is any non-negative integer then

$$\sum_{r=0}^{n-1} \binom{k+r}{k} = \binom{n+k}{k+1} \tag{5}$$

Proof. Using the fact that $\binom{k}{k} = \binom{k+1}{k+1} = 1$ and Pascal's identity $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$, we get

$$\begin{aligned}
\sum_{r=0}^{n-1} \binom{k+r}{k} &= \binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{k+n-2}{k} + \binom{k+n-1}{k} \\
&= \binom{k+1}{k+1} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{k+n-2}{k} + \binom{k+n-1}{k} \\
&= \binom{k+2}{k+1} + \binom{k+2}{k} + \cdots + \binom{k+n-2}{k} + \binom{k+n-1}{k} \\
&= \binom{k+3}{k+1} + \cdots + \binom{k+n-2}{k} + \binom{k+n-1}{k} \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&= \binom{k+n-2}{k+1} + \binom{k+n-2}{k} + \binom{k+n-1}{k} \\
&= \binom{k+n-1}{k+1} + \binom{k+n-1}{k} = \binom{n+k}{k+1}.
\end{aligned}$$

This completes the proof.

4. Ramanujan summation for slant diagonals of Pascal's triangle

In this section, we will determine the Ramanujan summation for slant diagonals of Pascal's triangle shown in Figure 1. We shall consider the slant diagonals read through the North-East direction.

4.1 *Ramanujan summation of the first slant diagonal*

Theorem 2.

$$(RS)(1+1+1+1+\cdots) = -\frac{1}{2}. \tag{6}$$

Proof. The first slant diagonal numbers (read through the North-East direction) in Figure 1 are 1, 1, 1, 1, 1, ….

Taking $k = 0$ in (5), we get $f_1(n) = 1+1+1+\cdots+1 = \sum_{r=0}^{n-1} \binom{r}{0} = \binom{n}{1} = n.$

By (1), we have $(RS)(1+1+1+\cdots) = \int_{n=-1}^{0} \left(\sum_{r=0}^{n-1} \binom{r}{0} \right) dn = \int_{n=-1}^{0} n \, dn = -\frac{1}{2}.$

This proves (6) and completes the proof.

4.1.1 *Geometric meaning of Theorem 2*

In Figure 2, we had provided the geometric meaning of the result (6). We notice that $f_1(n) = n$ forms a right triangle with respect to [−1, 0] and x-axis. Hence, its area is exactly half of the unit square below x-axis. This explains the answer we obtained in (6).

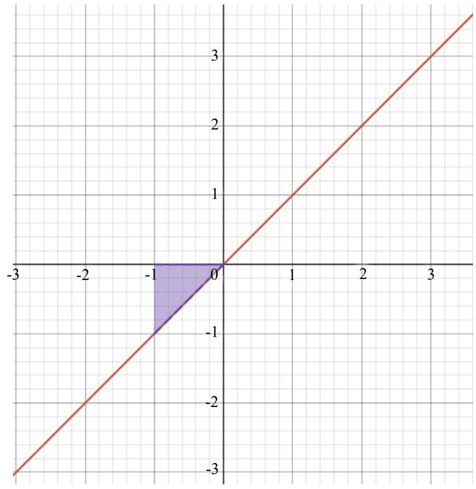

Figure 2. Area of $f_1(n) = n$ over $[-1, 0]$

4.2 *Ramanujan summation of the second slant diagonal*

Theorem 3.

$$(RS)(1+2+3+4+\cdots) = -\frac{1}{12}. \tag{7}$$

Proof. The second slant diagonal numbers (read through the North-East direction) in Figure 1 are 1, 2, 3, 4, 5, ….
Taking $k = 1$ in (5), we get $f_2(n) = 1+2+3+\cdots+n = \sum_{r=0}^{n-1}\binom{r+1}{1} = \binom{n+1}{2} = \frac{n^2+n}{2}$.

By (1), we have $(RS)(1+2+3+4+\cdots) = \int_{n=-1}^{0}\left(\sum_{r=0}^{n-1}\binom{r+1}{1}\right)dn = \int_{n=-1}^{0}\left(\frac{n^2+n}{2}\right)dn = -\frac{1}{12}$.

This proves (7) and completes the proof.

4.2.1 *Geometric meaning of Theorem 3*

In Figure 3, we had provided the geometric meaning of the result (7). We notice that the area bounded by the function $f_2(n) = \frac{n^2+n}{2}$ representing a parabola and the x-axis in $[-1, 0]$ forms a region below the x-axis as shown in the shaded portion. Hence, the resulting area is negative, explaining the answer we obtained in (7).

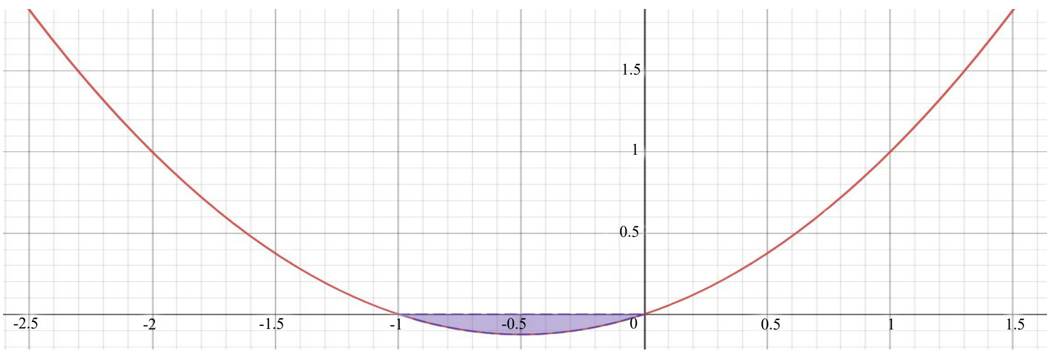

Figure 3. Area of $f_2(n) = \frac{n^2+n}{2}$ over $[-1, 0]$

4.3 *Ramanujan summation of the third slant diagonal*

Theorem 4.

$$(RS)(1+3+6+10+\cdots)=-\frac{1}{24} \tag{8}$$

Proof. The third slant diagonal numbers (read through the North-East direction) in Figure 1 are 1, 3, 6, 10, 15, …. Taking $k = 2$ in (5), we get $f_3(n)=1+3+6+10+15+\cdots+(n\,terms)=\sum_{r=0}^{n-1}\binom{r+2}{2}=\binom{n+2}{3}=\frac{n^3+3n^2+2n}{6}$.

By (1), we have $(RS)(1+3+6+10+\cdots)=\int_{n=-1}^{0}\left(\sum_{r=0}^{n-1}\binom{r+2}{2}\right)dn=\int_{n=-1}^{0}\left(\frac{n^3+3n^2+2n}{6}\right)dn=-\frac{1}{24}$.

This proves (8) and completes the proof.

4.3.1 *Geometric meaning of Theorem 4*

In Figure 4, we had provided the geometric meaning of the result (8). In particular, we notice that the area bounded by the function $f_3(n)=\frac{n^3+3n^2+2n}{6}$ and the x-axis in [–1, 0] forms a region below the x-axis as shown in the shaded portion. Hence, the resulting area is negative, explaining the answer we obtained in (8).

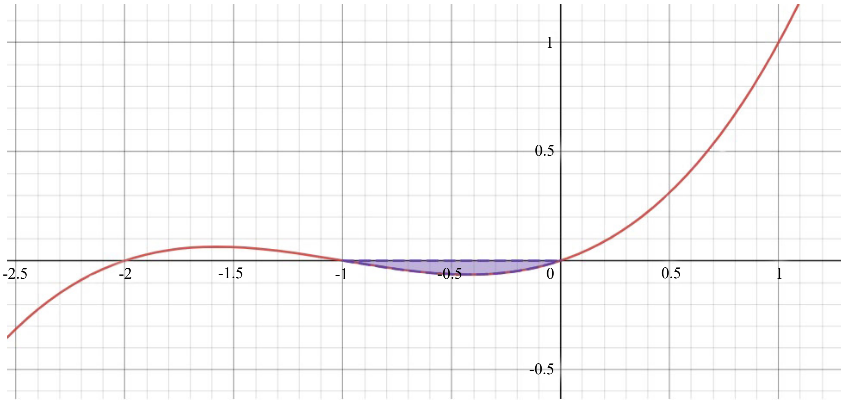

Figure 4. Area of $f_3(n)=\frac{n^3+3n^2+2n}{6}$ over [–1, 0]

4.4 *Ramanujan summation of the fourth slant diagonal*

Theorem 5.

$$(RS)(1+4+10+20+35+56+\cdots)=-\frac{19}{720} \tag{9}$$

Proof. The fourth slant diagonal numbers (read through the North-East direction) in Figure 1 are 1, 4, 10, 20, 35, 56, …. Taking $k = 3$ in (5), we get $f_4(n)=1+4+10+20+35+\cdots+(n\,terms)=\sum_{r=0}^{n-1}\binom{r+3}{3}=\binom{n+3}{4}=\frac{n^4+6n^3+11n^2+6n}{24}$.

By (1), we have $(RS)(1+4+10+20+\cdots)=\int_{n=-1}^{0}\left(\sum_{r=0}^{n-1}\binom{r+3}{3}\right)dn=\int_{n=-1}^{0}\left(\frac{n^4+6n^3+11n^2+6n}{24}\right)dn=-\frac{19}{720}$.

This proves (9) and completes the proof.

4.4.1 *Geometric meaning of Theorem 5*

In Figure 5, we had provided the geometric meaning of the result (9). In particular, we notice that the area bounded by the function $f_4(n) = \dfrac{n^4 + 6n^3 + 11n^2 + 6n}{24}$ and the x-axis in [–1, 0] forms a region below the x-axis as shown in the shaded portion. Hence, the resulting area is negative, explaining the answer we obtained in (9).

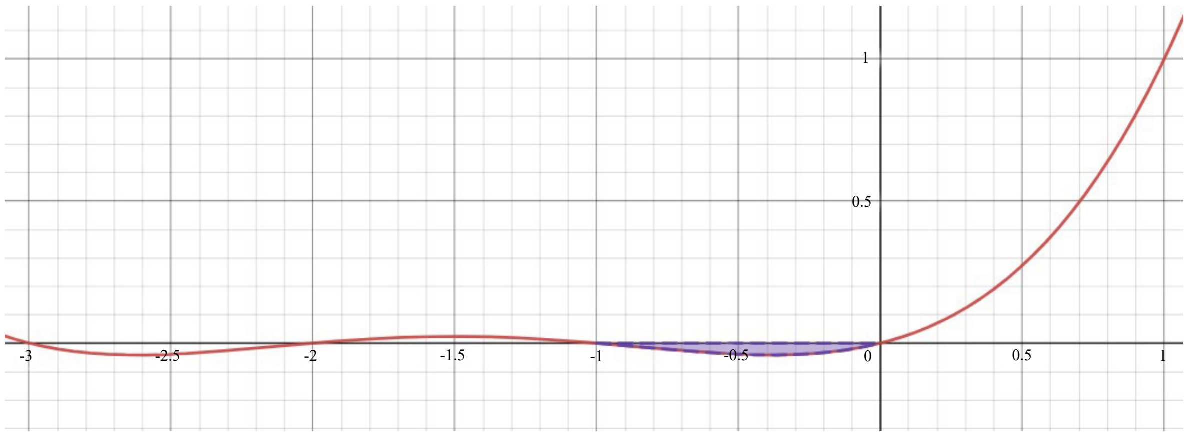

Figure 5. Area of $f_4(n) = \dfrac{n^4 + 6n^3 + 11n^2 + 6n}{24}$ over [–1, 0]

4.5 *Ramanujan summation of the fifth slant diagonal*

Theorem 6.

$$(RS)(1 + 5 + 15 + 35 + 70 + \cdots) = -\frac{3}{160} \tag{10}$$

Proof. The fifth slant diagonal numbers (read through the North-East direction) in Figure 1 are 1, 5, 15, 35, 70, … Taking $k = 4$ in (5), we get

$$f_5(n) = 1 + 5 + 15 + 35 + 70 + \cdots + (n\, terms) = \sum_{r=0}^{n-1} \binom{r+4}{4} = \binom{n+4}{5} = \frac{n^5 + 10n^4 + 35n^3 + 50n^2 + 24n}{120}.$$

By (1), we have

$$(RS)(1 + 5 + 15 + 35 + 70 + \cdots) = \int_{n=-1}^{0} \left(\sum_{r=0}^{n-1} \binom{r+4}{4} \right) dn = \int_{n=-1}^{0} \left(\frac{n^5 + 10n^4 + 35n^3 + 50n^2 + 24n}{120} \right) dn = -\frac{3}{160}.$$

This proves (10) and completes the proof.

4.5.1 *Geometric meaning of Theorem 6*

In Figure 6, we had provided the geometric meaning of the result (10). In particular, we notice that the area bounded by the function $f_5(n) = \dfrac{n^5 + 10n^4 + 35n^3 + 50n^2 + 24n}{120}$ and the x-axis in [–1, 0] forms a region below the x-axis as shown in the shaded portion. Hence, the resulting area is negative, explaining the answer we obtained in (10).

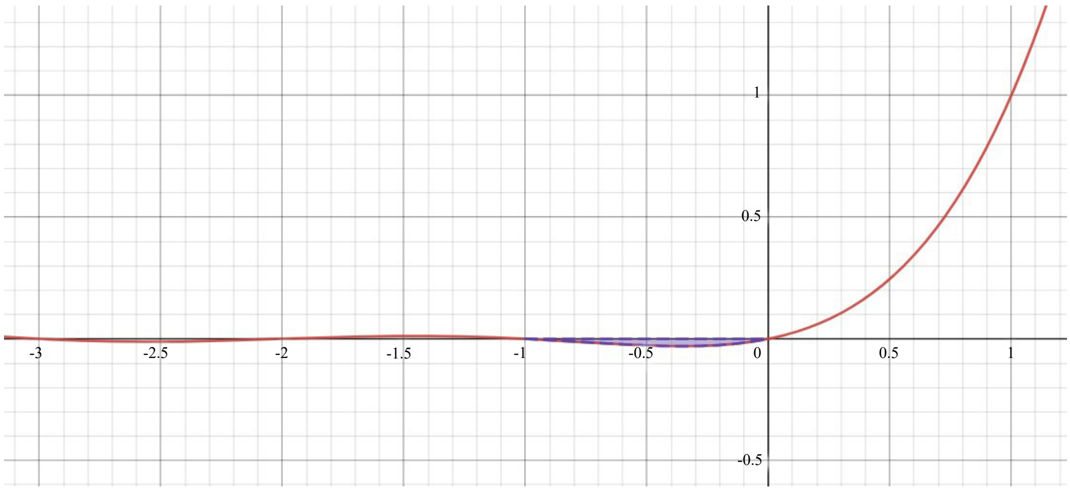

Figure 6. Area of $f_5(n) = \dfrac{n^5 + 10n^4 + 35n^3 + 50n^2 + 24n}{120}$ over [–1, 0]

5. General case

In this section, we will provide a compact formula for determining Ramanujan summation of numbers located in mth slant diagonal of Pascal's triangle displayed in Figure 1. First, we will define Stirling's numbers of the first kind.

5.1 *Stirling's numbers of first kind*

The number of permutations in a symmetric group with m elements namely S_m whose disjoint cyclic factorizations consists of exactly n cycles is defined as the Stirling's number of the first kind denoted by $s(m, n)$. Referring to [9, 10], we see that the values of Stirling's numbers of the first kind for $1 \le n \le m$ are given (Figure 7).

m \ n	1	2	3	4	5	6	7
1	1						
2	1	1					
3	2	3	1				
4	6	11	6	1			
5	24	50	35	10	1		
6	120	274	225	85	15	1	
7	720	1764	1624	735	175	21	1

...

Figure 7. List of Stirling's numbers of first kind

With respect to the results derived in Theorems 2 to 5, and comparing the entries of Stirling's numbers of the first kind from Figure 7, we notice the following equations.

$$\sum_{r=0}^{n-1} \binom{r}{0} = \binom{n}{1} = n = \frac{s(1,1)n}{1!} \tag{11}$$

$$\sum_{r=0}^{n-1}\binom{r+1}{1}=\binom{n+1}{2}=\frac{n^2+n}{2}=\frac{s(2,2)n^2+s(2,1)n}{2!} \tag{12}$$

$$\sum_{r=0}^{n-1}\binom{r+2}{2}=\binom{n+2}{3}=\frac{n^3+3n^2+2n}{6}=\frac{s(3,3)n^3+s(3,2)n^2+s(3,1)n}{3!} \tag{13}$$

$$\sum_{r=0}^{n-1}\binom{r+3}{3}=\binom{n+3}{4}=\frac{n^4+6n^3+11n^2+6n}{24}=\frac{s(4,4)n^4+s(4,3)n^3+s(4,2)n^2+s(4,1)n}{4!} \tag{14}$$

$$\begin{aligned}\sum_{r=0}^{n-1}\binom{r+4}{4}=\binom{n+4}{5}&=\frac{n^5+10n^4+35n^3+50n^2+24n}{120}\\&=\frac{s(5,5)n^5+s(5,4)n^4+s(5,3)n^3+s(5,2)n^2+s(5,1)n}{5!}\end{aligned} \tag{15}$$

For details of proof, see [16].

5.2 *Ramanujan summation of the mth slant diagonal*

Theorem 7. The Ramanujan summation of the mth slant diagonal numbers located in Pascal's Triangle is given by

$$\frac{1}{m!}\sum_{q=1}^{m}\frac{(-1)^{q+2}s(m,q)}{q+1} \tag{16}$$

where $s(m, q)$ are Stirling's numbers of the first kind.

Proof. We first note that the sum of first n terms of numbers located in mth slant diagonal of Pascal's triangle (Figure 1) is given by $\sum_{r=0}^{n-1}\binom{r+m-1}{m-1}$.

Now, using (5) and observing the pattern of equations described in equations from (11) to (15), we have

$$\sum_{r=0}^{n-1}\binom{r+m-1}{m-1}=\binom{n+m-1}{m}=\frac{1}{m!}\sum_{q=1}^{m}s(m,q)\,n^q.$$

Now by (1), the Ramanujan summation of numbers located in mth slant diagonal of Pascal's triangle is given by

$$\begin{aligned}(RS)\left(\sum_{r=0}^{\infty}\binom{r+m-1}{m-1}\right)&=\int_{n=-1}^{0}\left(\sum_{r=0}^{n-1}\binom{r+m-1}{m-1}\right)dn=\int_{n=-1}^{0}\frac{1}{m!}\sum_{q=1}^{m}s(m,q)\,n^q\;dn\\&=\frac{1}{m!}\left(\sum_{q=1}^{m}s(m,q)\left(\int_{n=-1}^{0}n^q\,dn\right)\right)=\frac{1}{m!}\left(\sum_{q=1}^{m}s(m,q)\left(\frac{(-1)^{q+2}}{q+1}\right)\right)\\&=\frac{1}{m!}\sum_{q=1}^{m}\frac{(-1)^{q+2}s(m,q)}{q+1}\end{aligned}$$

This proves (16) and hence completes the proof.

6. Conclusion

In this paper, we had determined Ramanujan summation which is a special case of Cesaro type summations for slant diagonal numbers located in standard Pascal's triangle. After introducing the concept of Ramanujan summation as in (1), we had determined Ramanujan summation values for the first five slant diagonals (read through the North-East direction and shown in colors in Figure 1) in Section 4 through equations (6) to (10) respectively. We had provided Figures 2 to 6 to give a geometric understanding of the answers obtained in equations from (6) to (10).

Curiously, in Section 5, we observed that sum of first n terms of slant diagonal numbers of Pascal's triangle are related to Stirling's numbers of first kind. Using this connection, we had derived a more general and compact formula for obtaining Ramanujan summation for numbers located in mth slant diagonal for any natural number m through equation (16) of Theorem 7. The results corresponding to (6) and (7) which are special cases of (16) by considering m = 1 and 2 respectively, were mentioned by Ramanujan in his famous notebooks. Hence, we had obtained the same results as provided by Ramanujan as well as provided a new and compact formula for determining Ramanujan summation of numbers located in mth slant diagonal of Pascal's triangle. This result will be an additional feature among several existing properties related to Pascal's triangle as well as Ramanujan summation.

Conflict of interest

There is no conflict of interest among authors of this manuscript.

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