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Research Article

Rational Numbers and Metallic Ratios

J. Suganthi[1], **R. Sivaraman**[2*]

[1]Department of Mathematics, S.S.K.V. College of Arts and Science for Women, Kanchipuram, India
[2]Department of Mathematics, Dwaraka Doss Goverdhan Doss Vaishnav College, Chennai, India
E-mail: rsivaraman1729@yahoo.co.in

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Abstract: Metallic ratios are class of real numbers which are positive roots of specific quadratic equation. In this paper, we had proved results regarding expressing any integer as well as rational numbers in terms of metallic ratios in the form of a cubic polynomial with integer coefficients. Three theorems had been established to meet this objective. This novel idea adds one more dimension to existing known properties of metallic ratios. Detailed explanation is provided in the final section to verify the results obtained.

***Keywords*:** golden ratio, metallic ratio, linear combination, roots, cubic polynomial

MSC: 11B39, 11C08, 11R09, 12D05

1. Introduction

Right from the ancient times mathematicians has been obsessed with a special number called Golden Ratio. Greek mathematicians have explored this number so much that it has been used not only in mathematics but in fields like Art, Architecture and even in Painting. Apart from Golden Ratio, there are numbers like Silver Ratio, Bronze Ratio and in general Metallic Ratio which occurs as roots of a particular quadratic equation. Metallic Ratios of order k are basically irrational numbers. The well known real number, the Golden Ratio happens to be metallic ratio of order 1. Similarly, the Silver and Bronze Ratios are metallic ratios of orders 2 and 3 respectively. Several mathematicians spanning over centuries have investigated these wonderful ratios which have enabled us to understand them quite deeply. Expressing a given positive integer as linear combination of special class of numbers has been an interesting and useful concept in mathematics. The Figurate Number Theorem is one such example. The most famous Waring's Problem is another example. Zeckendorf's Theorem is about expressing any positive integer as sums of Fibonacci numbers. Extending this concept for metallic ratios, in this paper, we had proved three results regarding expressing any integer as well as rational number in terms of metallic ratios. This new idea will add more value to already known results regarding metallic ratios.

2. Definition

The metallic ratio of order k is defined as the positive root of the quadratic equation

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$$x^2 - kx - 1 = 0. \tag{1}$$

Let λ_k denote the metallic ratio of order k. Then from (1), we see that

$$\lambda_k = \frac{k + \sqrt{k^2 + 4}}{2} \tag{2}$$

If $k = 1$, then the metallic ratio of order 1 is given by

$$\lambda_1 = \frac{1 + \sqrt{5}}{2}. \tag{3}$$

The number λ_1 is called as "Golden Ratio". Thus the metallic ratio of order 1 is the golden ratio.
If $k = 2$, then the metallic ratio of order 2 is given by

$$\lambda_2 = \frac{2 + \sqrt{8}}{2} = 1 + \sqrt{2} \tag{4}$$

The number λ_2 is called as "Silver Ratio". Thus the metallic ratio of order 2 is the silver ratio.
If $k = 3$, then the metallic ratio of order 3 is given by

$$\lambda_3 = \frac{3 + \sqrt{13}}{2}. \tag{5}$$

The number λ_3 is called as "Bronze Ratio". Thus the metallic ratio of order 3 is the bronze ratio.
For knowing more about metallic ratios and its properties, see [1-7].

3. Theorem 1

If $\lambda_1 = \varphi$ is the golden ratio, then for any integer N, we have

$$\varphi^3 + (N - 1)\varphi^2 - (N + 1)\varphi = N \tag{6}$$

Proof. Since the golden ratio is the metallic ratio of order 1, from (1) we see that $\lambda_1 = \varphi$ satisfies the equation

$$\varphi^2 - \varphi - 1 = 0 \text{ giving } \varphi^2 = \varphi + 1. \tag{7}$$

Thus from (7), we have

$$\varphi^3 = \varphi^2 + \varphi = (\varphi + 1) + \varphi = 2\varphi + 1. \tag{8}$$

In fact, using (7), it is possible to express any higher power of φ as linear polynomial of φ whose coefficients are consecutive Fibonacci numbers.

Now, if N is any integer then using (7) and (8) we get

$$\varphi^3 + (N-1)\varphi^2 - (N+1)\varphi = (2\varphi+1) + (N-1)(\varphi+1) - (N+1)\varphi = 2\varphi + 1 + N\varphi + N - \varphi - 1 - N\varphi - \varphi = N$$

This proves (6) and completes the proof.
We now prove a similar result for metallic ratio of order k.

4. Theorem 2

If λ_k is the metallic ratio of order k and if N is any integer, then

$$\lambda_k^3 + (N-k)\lambda_k^2 - (kN+1)\lambda_k = N \tag{9}$$

Proof. From (1), we know that the metallic ratio of order k namely λ_k is a positive root of the quadratic equation $x^2 - kx - 1 = 0$. Hence, we have $\lambda_k^2 - k\lambda_k - 1 = 0$ giving

$$\lambda_k^2 = k\lambda_k + 1. \tag{10}$$

From this, we get

$$\lambda_k^3 = (k\lambda_k + 1)\lambda_k = k\lambda_k^2 + \lambda_k = k(k\lambda_k + 1) + \lambda_k = \left(k^2 + 1\right)\lambda_k + k. \tag{11}$$

Thus from (10) and (11), we get

$$\lambda_k^3 + (N-k)\lambda_k^2 - (kN+1)\lambda_k = \left[\left(k^2+1\right)\lambda_k + k\right] + (N-k)(k\lambda_k + 1) - (kN+1)\lambda_k = k^2\lambda_k + \lambda_k + k +$$

$$Nk\lambda_k + N - k^2\lambda_k - k - Nk\lambda_k - \lambda_k = N$$

This proves (9) and completes the proof.
We now prove a theorem relating metallic ratios with a rational number.

5. Theorem 3

If λ_k is the metallic ratio of order k and if $\frac{p}{q}$ is any rational number where p, q are integers and q is non-zero then

$$q\lambda_k^3 + (p - qk)\lambda_k^2 - (kp+q)\lambda_k = p \tag{12}$$

Proof. As in theorem 2, we see that if λ_k is the metallic ratio of order k, then

$$\lambda_k^2 = k\lambda_k + 1 \tag{13}$$

and

$$\lambda_k^3 = \left(k^2 + 1\right)\lambda_k + k. \tag{14}$$

Considering the left hand side of (12), we have

$$q\lambda_k^3 + (p - qk)\lambda_k^2 - (kp + q)\lambda_k = q\left[\left(k^2 + 1\right)\lambda_k + k\right] + (p - qk)\left(k\lambda_k + 1\right) - (kp + q)\lambda_k$$

$$= qk^2\lambda_k + q\lambda_k + qk + pk\lambda_k + p - qk^2\lambda_k - qk - pk\lambda_k - q\lambda_k$$

$$= p$$

This proves (12) and completes the proof.

6. Verification

In this section, we will verify the results obtained in Theorems 1, 2 and 3 of Sections 3, 4 and 5 respectively. If we consider the equation (6) of Theorem 1, we see that the metallic ratio of first order, namely, the golden ratio satisfies the cubic polynomial equation

$$x^3 + (N - 1)x^2 - (N + 1)x - N = 0. \tag{15}$$

In fact, $x^3 + (N - 1)x^2 - (N + 1)x - N = (x + N)\left(x^2 - x - 1\right)$. Thus for any integer N, the roots of the cubic polynomial obtained in Theorem 1 are $-N, \lambda_1 = \varphi, 1 - \lambda_1 = 1 - \varphi$ where $\lambda_1 = \varphi$ is the golden ratio or metallic ratio of order 1.

If we now consider the equation (9) of Theorem 2, we see the metallic ratio of order k, satisfies the cubic polynomial equation

$$x^3 + (N - k)x^2 - (kN + 1)x - N = 0. \tag{16}$$

In fact, $x^3 + (N - k)x^2 - (kN + 1)x - N = (x + N)\left(x^2 - kx - 1\right)$. Thus for any integer N, the roots of the cubic polynomial obtained in Theorem 2 are $-N$, λ_k, $k - \lambda_k$ where λ_k is the metallic ratio of order k.

Similarly, if we consider the equation (5.1) of Theorem 3, then we notice that the polynomial can be written as $qx^3 + (p - qk)x^2 - (kp + q)x - p = (qx + p)\left(x^2 - kx - 1\right)$. Thus, for any rational number of the form $\frac{p}{q}$ where $q \neq 0$, the roots of the cubic polynomial obtained in Theorem 3 are $-\frac{p}{q}$, λ_k, $k - \lambda_k$ where λ_k is the metallic ratio of order k.

We thus observe that the metallic ratios of order k, indeed satisfies the cubic polynomials of the three theorems proved.

7. Illustration

If we consider $N = 36$ for example, then according to Theorem 1, we can write 36 as

$$\varphi^3 + 35\varphi^2 - 37\varphi = 36 \tag{17}$$

Since $\varphi^2 = \varphi + 1$ we notice that

$$\varphi^3 + 35\varphi^2 - 37\varphi = \varphi^2 + \varphi + 35\left(\varphi^2 - \varphi\right) - 2\varphi = 2\varphi + 1 + 35 - 2\varphi = 36$$

Similarly from Theorem 2, that we can express the number 36 in the form

$$\lambda_k^3 + (36 - k)\lambda_k^2 - (36k + 1)\lambda_k = 36 \quad (18)$$

Since $\lambda_k^2 = k\lambda_k + 1$, we notice that

$$\lambda_k^3 + (36 - k)\lambda_k^2 - (36k + 1)\lambda_k = k\lambda_k^2 + \lambda_k + 36\lambda_k^2 - k\lambda_k^2 - 36k\lambda_k - \lambda_k = 36\left(\lambda_k^2 - k\lambda_k\right) = 36$$

Similarly, if we consider $p = 22$, $q = 7$ then from Theorem 3, the rational number $\frac{p}{q} = \frac{22}{7}$ can be expressed as

$$7\lambda_k^3 + (22 - 7k)\lambda_k^2 - (22k + 7)\lambda_k = 22 \quad (19)$$

Since $\lambda_k^2 = k\lambda_k + 1$, we notice that

$$7\lambda_k^3 + (22 - 7k)\lambda_k^2 - (22k + 7)\lambda_k = 7k\lambda_k^2 + 7\lambda_k + 22\lambda_k^2 - 7k\lambda_k^2 - 22k\lambda_k - 7\lambda_k = 22\left(\lambda_k^2 - k\lambda_k\right) = 22$$

Thus the equations (17), (18) and (19) provide numerical illustrations for the three theorems obtained in this paper.

8. Conclusion

Expressing a given number as linear combination of some specific class of numbers, has always been performed by mathematicians of all ages. For example, in the famous polygonal number theorem, we observe that any number can be written as sum of at most k polygonal numbers of order k. Similarly, it is well known that any positive integer is expressible as sum of powers of two in a unique way. Similarly, we can express any positive integer as sum of Fibonacci numbers. These ideas not only amuse mathematicians but also provide interesting applications in other branches of Science. Keeping this in mind, we had tried to express any integer or rational number as linear combination of metallic ratios where the coefficients of the linear combination are integers.

In this paper, we had proved three theorems to meet this objective. In particular, in Theorem 1, we had shown that any integer N can be written as linear combination of first three powers of the golden ratio, which is viewed as metallic ratio of order 1. Similarly, in Theorem 2, we proved that any integer N can be written as linear combination of first three powers of the metallic ratios of order k. Finally, in Theorem 3, we had shown that any rational number can also be expressible as linear combination of first three powers of the metallic ratios of order k. We had also verified the results obtained in the final section by factorizing the polynomials. In fact, in proving these three theorems we have shown that the cubic polynomials through which we obtain the desired linear combination has three real roots which are $-N$ (negative of the given number), λ_k, $k - \lambda_k$ where λ_k is the metallic ratio of order k. Numerical illustration were provided to justify the results obtained in this paper. These results provide us a way to relate metallic ratios with any given rational number and opens up new understanding of the properties of metallic ratios.

Conflict of interest

The authors declare no competing financial interest.

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