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Research Article

Modeling and Analysis of Fractional Order Logistic Equation Incorporating Additive Allee Effect

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Abstract: This study investigates the logistic model of a single species incorporating the additive Allee effect using Caputo fractional order differential equations. The Allee effect describes a positive correlation between individual fitness and population density at low densities. Populations subjected to the strong Allee effect can move towards extinction when their population is below a critical level. This study calculates the threshold level of the population suffering from the strong Allee effect. Various published studies are showing that fractional order models are more appropriate for explaining real-world phenomena than ordinary integer-order systems; therefore, this study involves the use of the Caputo fractional order derivative. Single-species models have been extensively used in mathematical biology, such as insect control, optimal biological resource planning, epidemic avoidance and control, and cell growth regulation. This study can help save vulnerable species from extinction and eliminate unwanted species by subjecting them to a strong Allee effect using artificial strategies.

Keywords: logistic growth, single species, additive Allee effect, Caputo derivative, fractional order differential equation

MSC: 92-10, 37M05

1. Introduction

It has been recognized that individuals of many species can benefit from the presence of conspecifics [1]. This is simply contrary to the classical law of population dynamics, which states that a higher population density causes competition for resources, whereas a lower population density reduces competition and thus promotes growth. It has been observed that when population density is critically low, per capita growth rates in many populations decline. This may lead to a zero-growth rate or even a negative growth rate. The primary reason behind this may be the possibility of not finding a mate for reproduction when the population is quite low. This results in low reproductivity, which can lead to the extinction of the population. It is well observed that for a population to grow, a minimum density of that population is required, which is known as the critical density population or threshold level, below which that population cannot exist [2]. An American biologist, Eugene Odum, named this phenomenon Allee's principle, which is now known as the Allee effect [3]. The Allee effect can be caused not only by disability in finding mates at low population density but also by social dysfunction at low population density, inbreeding depression, swamping of enemies, allelic diversity, and food exploitation [4]. The Allee effect can be strong as well as weak [5]. The Allee effect is said to be strong when

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the birth rate becomes negative and weak when the birth rate continues to decrease but remains positive. A population experiencing a strong Allee effect requires a minimum population threshold level to survive; below this level, the population will become extinct [6].

To study biological interactions, mathematical modeling is a better tool as it saves time, energy, and resources. Mathematical models are designed to represent biological interactions realistically by carefully designing the variables, identifying the governing laws, and making some assumptions to make the model tractable easily. Many researchers revealed that the single-species modeling approach can help in ecosystem modeling to provide practical ecosystem-based management in fisheries, pest management, epidemic avoidance and management, optimal biological resource planning, cellular growth regulation, and many more [7]. There are many case studies in wildlife management where the single-species conservation approach helped to save biodiversity [8-10]. In population dynamics, there are three popular single-species models: the exponential growth model, the logistic growth model, and the delay model. Many researchers have used quantitative analysis to obtain some advantageous characteristics of these models, and the findings help us anticipate and control actual production [11, 12]. Various studies on the single-species logistic growth model for optimal harvesting [13, 14], the extinction and permanence of single species by considering the logistic growth of the population in a polluted environment [15], the dynamical behavior of stochastic single species with the Allee effect [16], Hopf bifurcation with delay [17], and the stochastic growth of single species with limited resources [18] are witnessed in the literature.

During the past few years, the scientific community has shifted its focus to explaining real-world problems using fractional calculus. Fractional calculus is inextricably linked to the memory systems seen in numerous real-world systems [19, 20]. Various published studies have proved that fractional order models are more appropriate for explaining real-world phenomena than ordinary integer-order systems [21-23]. Hereditary properties due to memory preservation and flexibility in degrees of freedom in fractional calculus have piqued the interest of the scientific community, particularly in the fields of physics, viscoelasticity, engineering, signal processing, etc. [24-30]. In the field of epidemiology, fractional order models are widely used nowadays, as results obtained using fractional order models fit better with the real world [31-38]. Another important reason for using fractional calculus is that it is defined globally rather than locally. Caputo, Riemann-Liouville, Grunwald-Letnikov, Hardamard, and many other methods are used to define fractional derivatives. However, these derivatives have advantages and disadvantages when used to solve real-world problems. Particularly if talking about the Riemann-Liouville derivative, it is not zero for a constant. The Caputo fractional derivative demands that the function be differentiable, and functions with no first-order derivative may have fractional derivatives of all orders less than one in the Riemann-Liouville sense. Still, it is preferred for studying real-world problems because it allows the formulation of the problem to include traditional boundary conditions, calculates the derivative of a constant zero, and is very simple to apply [39, 40]. Researchers have shown that while studying co-infection models through different fractional derivatives, Caputo derivatives give a better fit with real-world data [41, 42]. Various studies published on population dynamics using fractional-order differential equations have found that fractional derivative models provide flexible stability regions, which is helpful in finding better solutions than those given by traditional integer-order models. Since 2007, a lot of work has been done on population dynamics using fractional-order differential equations. Using fractional-order differential equations, the numerical solution of the prey-predator and rabies models was investigated by the authors of [43]. Various studies have been reported in the literature on fractional order prey predatory interactions incorporating delay and Holling type II functional response [44], Holling type III functional response [45], harvesting of the populations [46-50], prey refuge [51-53], and group defense [54-57]. The fractional-order eco-epidemiological models of diseases in populations were also studied [58-63]. Using ABC fractional derivatives, Nisar et al. [63] studied the fractional order food chain model with Holling Type II functional response and additive Allee effect on the prey population. El-Sayed et al. [64] established the conditions for the existence and uniqueness of the solution and the stability of the equilibrium points of a fractional-order single-species logistic equation. The authors used fractional derivatives in the Caputo sense. Gao and Zhao [65] used the Caputo fractional derivative-based model to investigate the single-species model with species dispersion in n patches as a coupled system on a network. Li et al. [66] used Caputo fractional differential equations to study the single species model incorporating population dispersion in several patches. Despite being of crucial importance, the Allee effect is not addressed in single-species fractional-order population dynamics. The Allee effect can be studied multiplicatively as well as additively. The first single-species logistic growth model considering the multiplicative Allee effect was proposed by Bazykin [67]. That equation was given by

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)(x - m). \tag{1}$$

In this model, x is the population density at any time, and r is the positive constant depicting the per capita intrinsic growth rate of the population. K is the carrying capacity of the environment for the population. It is the maximum population that can be supported by the ecosystem in which the population is interacting, and $m \geq 0$ is the threshold level of the population below which the population suffering from a strong Allee effect will not survive. According to this model, if $0 < m < K$ the population is considered to have a strong Allee effect and to have a weak Allee effect if $m \leq 0$. Abbas et al. [68] used the Riemann-Liouville definition of fractional order derivatives and integrals to analyze the model (1). The author proved numerically that when the initial population is greater than the threshold parameter m, the population will converge to the carrying capacity; otherwise, it will go extinct. He also observed the effectiveness of fractional derivatives over integer-order. Dennis [69] introduced the Allee effect additively as a negative factor in the logistic equation as

$$\frac{dx}{dt} = \left[r\left(1 - \frac{x}{K}\right) - \frac{\alpha a}{x + a}\right]x. \tag{2}$$

where x is population density at any time t, r, K, a, and α are all positive constants defined by r as the per capita intrinsic growth rate of the population, K is the carrying capacity of the environment, and a is the population density at which fitness is half of its maximum value. Here, fitness is considered in terms of reproduction efficiency. The factor $\frac{a}{x+a}$ measures the relative fitness of the population due to the Allee effect, so the greater the value of a the greater will be the reduction in the fitness of the population due to the Allee effect. The term α, which is the constant of proportionality, denotes the severity of the Allee effect. By taking $\alpha a = m$, equation (2) can be written as

$$\frac{dx}{dt} = \left[r\left(1 - \frac{x}{K}\right) - \frac{m}{x + a}\right]x. \tag{3}$$

Here, m and a depict the degree of the Allee effect. According to this model, if $0 < m < ar$, then the model has a weak Allee effect, and if $m > ar$, then the model has a strong Allee effect.

In the extensive literature review, no work is found on the single-species logistic growth model incorporating the additive Allee effect via the fractional order differential equation. Abbas et al. [68] solve the fractional order logistic equation with multiplicative Allee effect using the Riemann-Liouville definition of derivative. In the study, the authors have shown the impact of fractional derivative on the time taken by the population to reach the equilibrium points and proved numerically and mathematically that all populations below the value of m in equation (1) move to the extinction and all initial populations more than m move towards the carrying capacity. In this present study, the authors aim to study the fractional order single species logistic equation with the additive Allee effect, where the fractional derivative is considered in the Caputo sense. We aim to analyze the model under the influence of the strong Allee effect as well as the weak Allee effect and to find a threshold level for a population suffering from a strong Allee effect mathematically and numerically. To the best of our knowledge, no study has been published on fractional-order logistic equations with additive Allee effects. The organization of the paper is as follows: Section 2 describes a mathematical model that will be analyzed. Section 3 presents the mathematical analysis of the model in the form of establishing the existence and uniqueness of the solutions, as well as their positivity and boundedness. Section 4 presents a dynamic study of the model by estimating the equilibrium points and checking their local, asymptotical, and global stability. Section 5 discusses the numerical simulation results and their implications, and in Section 6, concluding remarks are given.

2. Mathematical model

In this study, we are considering the single species logistic growth model with the additive Allee effect given by equation (3) using the Caputo fractional order derivative. The model equation is defined as

$$D^{\alpha}x(t)=\left[r\left(1-\frac{x}{K}\right)-\frac{m}{x+a}\right]x, \tag{4}$$

subject to starting condition $x(t=t_0)=x_0>0$. Here x, r, K, a, and m are, respectively, the number of individuals in the population, the intrinsic growth rate, the carrying capacity of the environment, the population density of the species whose fitness is half of its maximum value, and the level of the Allee effect at any time, t. D^{α} is the Caputo fractional order derivative, and $\alpha\in(0,1]$ is the fractional order.

3. Mathematical analysis of the model

3.1 *Existence and uniqueness of solution*

Lemma 3.1.1. [61] Consider the fractional order differential equations of the order $\alpha\in(0,1]$, $D^{\alpha}x(t)=f(t,x)$, with initial condition $x(t=t_0)=x_0>0$. Here, $f:[t_0,\infty)\times D\to\mathbb{R}$ is a function. If $f(x,t)$ satisfies the Lipschitz condition w.r.t. variable x in $[t_0,\infty)\times D$, if there exists some real constant $L>0$ independent of t, X, and Y, such that $\left|f(t,X(t))-f(t,Y(t))\right|\le L\left|X(t)-Y(t)\right|$, where $D=\{x\in\mathbb{R}:|x|\le M\}$ and M is a positive finite real constant.

Theorem 3.1.2. Consider the interval $I=[t_0,T]$, $T<\infty$ and the region $D=\{x\in\mathbb{R}:|x|\le M\}$. Let $C(I)$ be the class of all real-valued functions defined on I, which have continuous first-order derivatives on $C(I)$. Then, the initial value problem $D^{\alpha}x(t)=f(t,x)$, where $f(t,x)=\left[r\left(1-\frac{x}{k}\right)-\frac{m}{x+a}\right]x$, with initial condition $x(t=t_0)=x_0$ and $f:I\times D\to\mathbb{R}$; $\alpha\in(0,1]$ satisfies the Lipschitz condition w.r.t. second variable x.

Proof. Consider a mapping $F:D\to\mathbb{R}$ by $F(x)=\left[r\left(1-\frac{x}{k}\right)-\frac{m}{x+a}\right]x$. Define $\|x\|=\sup_t\left|e^{-St}x\right|$, $S>0$. $t\in[t_0,T]$, $T<\infty$. Clearly, $\|x\|=\sup_t|x|$. Let $x,y\in D$. Consider

$$\begin{aligned}
\|F(x)-F(y)\| &=|F(x)-F(y)|,\\
&=\left|rx-\frac{rx^2}{K}-\frac{mx}{x+a}-ry+\frac{ry^2}{K}+\frac{my}{y+a}\right|,\\
&=\left|r(x-y)-\frac{r}{K}\left(x^2-y^2\right)-m\left(\frac{x}{x+a}-\frac{y}{y+a}\right)\right|,\\
&\le\left|r(x-y)\right|+\left|\frac{r}{K}\left(x^2-y^2\right)\right|+\left|m\left(\frac{x}{x+a}-\frac{y}{y+a}\right)\right|,\\
&=r|(x-y)|+\frac{r}{K}|(x-y)||(x+y)|+m\left|\frac{x(y+a)-y(x+a)}{(x+a)(y+a)}\right|,\\
&\le r|(x-y)|+2\frac{Mr}{K}|(x-y)|+ma|(x-y)|,\\
&=|(x-y)|\left(r+2\frac{Mr}{K}+ma\right),\\
&=L|(x-y)|,\ \text{where } L=r+2\frac{Mr}{K}+ma.
\end{aligned}$$

Therefore, F satisfies the Lipschitz condition.

Theorem 3.1.3. [68] The given fractional order system (4) have unique solution provided

$$\frac{\left(r+2\dfrac{Mr}{K}+ma\right)}{p^{\alpha-1}}<1. \tag{5}$$

Proof. The model given by equation (4) can be rewritten as

$$\begin{cases} D^{\alpha}x(t)=F(x(t)) \text{ when } t\in[t_0,T],\ T<\infty \\ x(t=t_0)=x_0. \end{cases}$$

Let $H(x)$ be the solution of system (4). Then,

$$H(x)=x-x_0=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(x(s))ds.$$

Consider

$$\begin{aligned}
H(x)-H(y) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{f(x(s))-f(y(s))\}ds. \\
\left|H(x)-H(y)\right| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{f(x(s))-f(y(s))\}ds\right|, \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(x(s))-f(y(s))\right|ds, \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|(x-y)\right|\left(r+2\frac{Mr}{K}+ma\right)ds, \\
&= \frac{\left(r+2\dfrac{Mr}{K}+ma\right)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|(x-y)\right|ds.
\end{aligned}$$

Now,

$$\begin{aligned}
e^{-pt}\ (H(x)-H(y)) &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ (x-y)\left(r+2\frac{Mr}{K}+ma\right)e^{-pt}ds. \\
\left\|H(x)-H(y)\right\| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ (x-y)\left(r+2\frac{Mr}{K}+ma\right)e^{-pt}ds, \\
&= \frac{\left(r+2\dfrac{Mr}{K}+ma\right)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} e^{-p(t-s)}\left(x-y\right)e^{-ps}\,ds, \\
&= \frac{\left(r+2\dfrac{Mr}{K}+ma\right)}{\Gamma(\alpha)}\left\|x-y\right\|\int_0^t (t-s)^{\alpha-1} e^{-p(t-s)}\,ds, \\
&\le \frac{\left(r+2\dfrac{Mr}{K}+ma\right)}{\Gamma(\alpha)}\left\|x-y\right\|\frac{\Gamma(\alpha)}{p^{\alpha-1}}, \\
&= \frac{\left(r+2\dfrac{Mr}{K}+ma\right)}{p^{\alpha-1}}\left\|x-y\right\|.
\end{aligned} \tag{6}$$

Choosing p sufficiently large so that $\frac{\left(r+2\frac{Mr}{K}+ma\right)}{p^{\alpha-1}}<1$, then the operator $H(x)$ becomes a contraction and it has a unique point. Hence, the given fractional order differential equation has a unique solution.

3.2 *Positivity and boundedness of the solution*

Lemma 3.2.1. [57] Let us assume that $\alpha\in(0,1]$ and consider that the function $f(t)$ and $D^{\alpha}f(t)\in C[a,b]$ for all $t\in[a,b]$, where $C[a,b]$ is the class of continuous functions on $[a,b]$ and D^{α} stands for fractional order Caputo derivative. The function $f(t)$ is said to be non-decreasing on $[a,b]$ if $D^{\alpha}f(t)\geq 0$, and the function $f(t)$ is said to be non-increasing on $[a,b]$ and if $D^{\alpha}f(t)\leq 0$.

Theorem 3.2.2. [57] Let $\alpha>0$, $n-1<\alpha<n$, where $n\in\mathbb{N}$. Assume $U(t)$ is n times continuously differentiable function and $D^{\alpha}U(t)$ is piecewise continuous on $[t_0,\infty)$, then $\mathcal{L}\{D^{\alpha}U(t)\}=s^{\alpha}\mathcal{F}(s)-\sum_{j=0}^{n-1}s^{\alpha-j-1}U^{j}(t_0)$, where $\mathcal{F}(s)=\mathcal{L}\{U(t)\}$, i.e., Laplace transform of $U(t)$, D^{α} is the Caputo fractional order derivative.

Theorem 3.2.3. [57] Let $\mathcal{C}$ be the complex plane. For each $m>0$, $p>0$, $k\in\mathcal{C}^{n\times n}$. The Laplace transform of $t^{p-1}E_{m,p}(kt^{m})$ is defined as

$$\mathcal{L}\{t^{p-1}E_{m,p}(kt^{m})\}=\frac{s^{m-p}}{s^{m}-K},\Re(s)>\|K\|^{1\backslash m}$$

where $\Re(s)$ is the real part of the complex number s, and $E_{m,p}$ is the Mittag-Leffler function defined as $E_{m,p}(z)=\sum_{n=0}^{\infty}\frac{z^{m}}{\Gamma(mn+p)}$, and Γ is the Gamma function.

Theorem 3.2.4. All solutions of the fractional differential equation (4) beginning in $\mathbb{R}^{+}$ are non-negative, where $\mathbb{R}^{+}$ is the set of positive reals including zero.

Proof. Let $x(t_0)=x_0\in\mathbb{R}^{+}$ be the starting solution of the given fractional order system (4). Let $t>t_0$ and we are to show that $x(t)\geq 0$ for all $t\geq t_0$. Let us suppose that it does not hold. It means that there exists some t_1, such that $t_1>t_0$ but

$$\left\{\begin{array}{c} x(t)>0 \text{ when } t_0\leq t<t_1, \\ x(t_1)=0, \\ x(t)<0 \text{ when } t_1\leq t<t^{*}. \end{array}\right\}. \tag{7}$$

where t^{*} is sufficiently close to t_1. Now, $x(t_1)=0$ gives $D^{\alpha}x(t_1)=0$.

Case 1. If $D^{\alpha}x(t)\geq 0$, for all $t\in(t_1,t^*]$. Now, from (4), $D^{\alpha}x(t)>r\,x(t)$. Taking Laplace transform on both sides and using Theorem 3.2.2, we get

$$s^{\alpha}X(s)-s^{\alpha-1}x(t_0)\geq r\,X(s), \text{ where } X(s)=\mathcal{L}\{x(t)\}$$
$$(s^{\alpha}-r)X(s)\geq s^{\alpha-1}x(t_0),$$
$$X(s)\geq s^{\alpha-1}\frac{x(t_0)}{(s^{\alpha}-r)}.$$

Taking inverse Laplace transform on both sides and using Theorem 3.2.3,

$$x(t) \geq \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{\left(s^{\alpha}-r\right)}x\left(t_0\right)\right\} = x\left(t_0\right)E_{\alpha,1}\left\{r\ (t-t_0)^{\alpha}\right\}$$

We have $x(t) \geq x(t_0)E_{\alpha,1}\left\{r\ (t-t_0)^{\alpha}\right\} \geq 0$. Therefore, we have $x(t) \geq 0$, which contradicts the assumption $x(t) < 0$ for all $t \in (t_1, t^*]$.

Case 2. If $D^{\alpha}x(t) < 0$, for all $t \in (t_1, t^*]$. This implies $x(t)$ is a non-increasing function for all $t \in (t_1, t^*]$. Consider

$$D^{\alpha}x(t) = \left[r\left(1-\frac{x}{k}\right)-\frac{m}{x+a}\right]x = \left(r-\frac{m}{x+a}\right)x-\frac{x^2r}{K}$$

$$D^{\alpha}x(t) \geq (r-\frac{m}{a})x-\frac{x^2r}{K} \geq (r-M)\ x - Nx^2, \text{where } M = \frac{m}{a} \text{ and } N = \frac{r}{K}.$$

Let t^* be chosen close to t_1 so that $x(t) < -x^2(t)$, for $t_1 < t < t^*$. We have $D^{\alpha}x \geq (r+N-M)\ x = P\,x$ where $P = r-M+N$. This implies that for $t_1 < t < t^*, D^{\alpha}x \geq P\,x$.

Taking Laplace transform on both sides, we have

$$X(s) \geq s^{\alpha-1}\frac{x(t_0)}{(s^{\alpha}-P)}, \text{ where } X(s) = \mathcal{L}\{x(t)\}.$$

By taking the inverse Laplace transform, $x(t) \geq x(t_0)E_{\alpha,1}\left\{P(t-t_0)^{\alpha}\right\} \geq 0$. Therefore, again we have $x(t) \geq 0$ for all $t_1 < t < t^*$, which again contradicts the assumption $x(t) < 0$ for all $t \in (t_1, t^*)$. Hence, all solutions begin in $\mathbb{R}^+$ are non-negative.

Theorem 3.2.5. All solutions of the fractional order system (4) starting in $\mathbb{R}^+$ are uniformly bounded.

Proof. Consider $D^{\alpha}F(t)+\frac{F(t)}{m}$, where $F(t) = x(t)$.

$$D^{\alpha}F(t)+\frac{F(t)}{m} = rx-\frac{rx^2}{K}-\frac{mx}{x+a}+\frac{x}{m} \leq x\left(r+\frac{1}{m}\right)-\frac{r}{K}x^2,$$

$$= -\frac{r}{K}x^2+\left(r+\frac{1}{m}\right)x = -\frac{r}{K}\left(x^2-\frac{K}{r}\left(r+\frac{1}{m}\right)x\right),$$

$$= -\frac{r}{K}\left[x^2-\left(K+\frac{K}{rm}\right)x\right] = -\frac{r}{K}\left[x^2-Lx\right] = -\frac{r}{K}\left[\left(x-\frac{L}{2}\right)^2-\frac{L^2}{4}\right]$$

$$= \frac{rL^2}{4K}-\frac{r}{K}\left(x-\frac{L}{2}\right)^2 \leq \frac{rL^2}{4K} = R, \text{where } L = \left(K+\frac{K}{rm}\right).$$

Therefore, have $D^{\alpha}F(t)+\frac{F(t)}{m} \leq R$.

Applying Laplace transform on both sides

$$s^{\alpha}G(s)-s^{\alpha-1}F(0)+\frac{1}{m}G(s) \leq \frac{R}{s}, \text{where } G(s) = \mathcal{L}\{F(t)\},$$

$$\left(s^{\alpha}+\frac{1}{m}\right)G(s) \leq \frac{R}{s}+s^{\alpha-1}F(0),$$

$$G(s) \le \frac{R}{s\left(s^{\alpha}+\frac{1}{m}\right)}+\frac{s^{\alpha-1}F(0)}{\left(s^{\alpha}+\frac{1}{m}\right)}=\frac{s^{\alpha-1}F(0)}{\left(s^{\alpha}+\frac{1}{m}\right)}+\frac{s^{\alpha-(\alpha+1)}}{\left(s^{\alpha}+\frac{1}{m}\right)}R.$$

Taking inverse Laplace transform

$$F(t)\le F(0)\ \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{\left(s^{\alpha}+\frac{1}{m}\right)}\right\}+R\,\mathcal{L}^{-1}\left\{\frac{s^{\alpha-(\alpha+1)}}{\left(s^{\alpha}+\frac{1}{m}\right)}\right\},$$

$$=F(0)E_{\alpha,1}\left\{-\frac{t^{\alpha}}{m}\right\}+Rt^{\alpha}E_{\alpha,\alpha+1}\left\{-\frac{t^{\alpha}}{m}\right\}.$$

Now, $E_{\alpha,\beta}\{z\}=z\,E_{\alpha,\alpha+\beta}\{z\}+\frac{1}{\Gamma(\beta)}$ gives $E_{\alpha,1}\left\{-\frac{t^{\alpha}}{m}\right\}=-\frac{t^{\alpha}}{m}E_{\alpha,\alpha+1}\left\{-\frac{t^{\alpha}}{m}\right\}+\frac{1}{\Gamma(1)}$, which further gives $t^{\alpha}E_{\alpha,\alpha+1}\left\{-\frac{t^{\alpha}}{m}\right\}=-m\left[E_{\alpha,1}\left\{-\frac{t^{\alpha}}{m}\right\}-1\right]$.

$$F(t)\le F(0)E_{\alpha,1}\left\{-\frac{t^{\alpha}}{m}\right\}-Rm\left[E_{\alpha,1}\left\{-\frac{t^{\alpha}}{m}\right\}-1\right]=\left(F(0)-Rm\right)E_{\alpha,1}\left\{-\frac{t^{\alpha}}{m}\right\}+Rm.$$

Now, as $t\to\infty, E_{\alpha,1}\left\{-\frac{t^{\alpha}}{m}\right\}\to 0$. Hence, we see that $F(t)\le Rm=\frac{rL^2m}{4K}$.

$\therefore$ All solutions of the given system which starts in $\mathbb{R}^+$ lies in the region

$$\left\{\bar{x}\in\mathbb{R}^+:\bar{x}\le\frac{rL^2m}{4K}+v, v>0\right\}. \tag{8}$$

4. Dynamical analysis of the model

4.1 *Equilibrium points*

The equilibrium points of the given fractional order system (3) are given by $x_e = 0$, this point exists always.

$$x_s=\frac{K}{2}\left[\left(1-\frac{a}{K}\right)+\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right],\ \text{provided } m<\frac{Kr}{4}\left(1+\frac{a}{K}\right)^2.$$

$$x_t=\frac{K}{2}\left[\left(1-\frac{a}{K}\right)-\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right],\ \text{provided } ar<m<\frac{Kr}{4}\left(1+\frac{a}{K}\right)^2.$$

4.2 *Local asymptotical stability analysis of the equilibrium points*

Firstly, we will explain the stability criteria to be used for the local stability of the equilibrium points:
Let $x = x_*$ be the equilibrium point of the fractional order system $D^{\alpha}x(t)=f(x(t))$. Then, $f\left(x_*\right)=0$.

To evaluate the asymptotical stability, let us perturb the equilibrium point by adding $\epsilon(t)$. Let $x(t)=x_*+\epsilon$, then we have $D^{\alpha}x(t) = D^{\alpha}(x_*+ \epsilon) = D^{\alpha}(x_*) = f(x_* + \epsilon)$, which implies $D^{\alpha}(x_*) = f(x_* + \epsilon)$.

By using Taylor's formula,

$$f(x_* + \epsilon) \approx f(x_*) + f'(x_*)\epsilon + \ldots$$
$$f(x_* + \epsilon) \approx f'(x_*)\epsilon, \text{where } f(x_*) = 0.$$

Again, $D^{\alpha}(x_*+\epsilon) = D^{\alpha}(x_*) + D^{\alpha}(\epsilon) = D^{\alpha}(\epsilon)$. We have $D^{\alpha}(\epsilon) = f'(x_*)\epsilon$. Solution of the above equation is given by $\varepsilon(t) = \epsilon(t_0)$ $\mathbb{E}_{\alpha,1}(f'(x_*)t^{\alpha})$. Therefore, $D^{\alpha}x(t) = D^{\alpha}(\epsilon(t)) = f'(x_*)\epsilon(t_0)\mathbb{E}_{\alpha,1}(f'(x_*)t^{\alpha})$.

If $\epsilon(t)$ increases, then the equilibrium point becomes unstable and if $\epsilon(t)$ decreases the equilibrium point become stable. Therefore, stability or unstability depends upon the sign of $f'(x_*)$. If $f'(x_*) > 0$, then the equilibrium point becomes unstable and if $f'(x_*) > 0$, the equilibrium point becomes unstable [68].

Theorem 4.2.1. If $m > \frac{Kr}{4}\left(1 + \frac{a}{K}\right)^2$, then system (4) has only one equilibrium point $x = 0$, which is asymptotically stable.

Proof. For equilibrium points of fractional order system (4)

$$\left[r\left(1 - \frac{x}{k}\right) - \frac{m}{x + a}\right]x = 0.$$

To have the only solution $x = 0$, the condition to be satisfied is given by

$$\left(1 - \frac{a}{K}\right)^2 - \frac{4}{K}\left(\frac{m}{r} - a\right) = \left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr} < 0$$

This gives

$$m > \frac{Kr}{4}\left(1 + \frac{a}{K}\right)^2.$$

For stability analysis, considering the given system (4) as $D^{\alpha}x = f(x)$. We have $f(x) = \left[r\left(1 - \frac{x}{k}\right) - \frac{m}{x + a}\right]x.$

Its differential coefficient is given by

$$f'(x) = r - \frac{2rx}{K} - \frac{ma}{(x + a)^2}.$$

Now, $f'(0) = r - \frac{m}{a} < 0$ gives $m > ar$. Hence, if $= m > \max\left\{\frac{Kr}{4}\left(1 + \frac{a}{K}\right)^2, ar\right\} = \frac{Kr}{4}\left(1 + \frac{a}{K}\right)^2$, the only equilibrium points of (4) is $x_e = 0$, which is asymptotically stable.

Theorem 4.2.2. If $0 < m < ar$, or when there is a weak Allee effect, then the system (4) has two equilibrium points $x_e = 0, x_s = \frac{K}{2}\left[\left(1 - \frac{a}{K}\right) + \sqrt{\left(1 + \frac{a}{K}\right)^2 - \frac{4m}{Kr}}\right]$, where x_e is always unstable and x_s is asymptotically stable.

Proof. In order to have non-zero equilibrium points of the system (4), the condition to be satisfied is given by

$$\left(1-\frac{a}{K}\right)^2-\frac{4}{K}\left(\frac{m}{r}-a\right)=\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr} > 0.$$

This gives

$$m<\frac{Kr}{4}\left(1+\frac{a}{K}\right)^2.$$

The non-zero equilibrium points are given as

$$x=\frac{K}{2}\left[\left(1-\frac{a}{K}\right)\pm\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right].$$

Now, $\min\left\{\frac{Kr}{4}\left(1+\frac{a}{K}\right)^2, ar\right\}=ar$. By taking $m<ar$, it can be seen easily

$$\frac{K}{2}\left[\left(1-\frac{a}{K}\right)-\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right]<0.$$

As we are interested only in non-negative solutions so system (4) has two equilibrium points $x_e=0$ and $x_s=\frac{K}{2}\left[\left(1-\frac{a}{K}\right)+\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right]$. Following the last theorem, we can write $f'(x)=r-\frac{2rx}{K}-\frac{ma}{(x+a)^2}$. And for the stability analysis of $x_e=0$, it can be seen that $f'(0)=r-\frac{m}{a}<0$ gives $m>ar$, which proves that $x_e=0$, is unstable, and doing the calculations, it is found that $f'(x_s)<0$ gives $0<m<ar$. Hence, the stability of x_s is established when $0<m<ar$.

Theorem 4.2.3. If $ar<m<\frac{Kr}{4}\left(1+\frac{a}{K}\right)^2$, the system (4) has three equilibrium points given $x_e=0$, which is stable and $x_s=\frac{K}{2}\left[\left(1-\frac{a}{K}\right)+\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right]$, $x_t=\frac{K}{2}\left[\left(1-\frac{a}{K}\right)-\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right]$ and both are unstable.

Proof. Following last theorem if $m<\frac{Kr}{4}\left(1+\frac{a}{K}\right)^2$, system (4) has three equilibrium points $x_e=0$, $x_s=\frac{K}{2}\left[\left(1-\frac{a}{K}\right)+\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right]$, $x_t=\frac{K}{2}\left[\left(1-\frac{a}{K}\right)-\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right]$. From Theorem 4.2.1,

$$f(x)=\left[r\left(1-\frac{x}{k}\right)-\frac{m}{x+a}\right]x.$$

$$f'(x)=r-\frac{2rx}{K}-\frac{ma}{(x+a)^2}.$$

$$f'(0)=r-\frac{m}{a}<0 \text{ gives } m>ar.$$

$$f'(x_s)<0 \text{ gives } 0<m<ar.$$

$$f'(x_t)<0 \text{ gives } 0<m<ar.$$

Hence, $x_e = 0$ is stable but x_s and x_t are unstable.

4.3 *Global stability analysis of the equilibrium points*

In this section, we will discuss the global stability of the equilibrium point by using Lyapunov's theorem whose statement is given by:

Lemma 4.3.1. [57] If $D^{\alpha}x(t)=f(t,x),x(t_0)>0$ be a given non-autonomous system and if $x_*\in D\subset R$ is its equilibrium point. Let $F(t,x):[0,\infty)\times D\to R$ be a differentiable function, and there exists positive definite continuous functions $F_1(x)$, $F_2(x)$, and $F_3(x)$ on D, such that

$$F_1(x)\le F(t,x)\le F_2(x) \text{ and } F(t,x)\le -F_3\ (x),\ \forall\,\alpha\in(0,1),\forall x(t)\in D.$$

Then, the equilibrium point $x_*\in D$ is globally stable and $F(t,x)$ is said to be Lyapunov function.

Lemma 4.3.2. [57] If $x(t)\in\mathbb{R}^{+}$ be a continuous and differentiable function, then any time for

$$t>t_0, D^{\alpha}\left(x(t)-x_*-x_*\ln\frac{x(t)}{x_*}\right)\le\left(\frac{x-x_*}{x}\right)D^{\alpha}x(t).$$

Theorem 4.3.3. If $r-\frac{m}{M+a}+\frac{rx_*}{K}<0$ and $\frac{mx_*}{a}-rx_*<0,$ where $|x|\le M$, the equilibrium point $x_s=\frac{K}{2}\left[\left(1-\frac{a}{K}\right)+\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right]$ is uniformly globally stable where M is a positive finite real constant.

Proof. Consider the function

$$V(x)=\left(x-x_*-x_*\ln\frac{x}{x_*}\right),$$

where $x(t)=\left[r\left(1-\frac{x}{k}\right)-\frac{m}{x+a}\right]x$, and $x_*=x_s=\frac{K}{2}\left[\left(1-\frac{a}{K}\right)+\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right].$

Again, consider $D^{\alpha}V(x)=D^{\alpha}\left(x-x_*-x_*\ln\frac{x}{x_*}\right).$

Using the stated lemma, we have

$$D^{\alpha}V(x) \leq \left(\frac{x-x_*}{x}\right)D^{\alpha}x,$$
$$D^{\alpha}V(x) \leq \left(\frac{x-x_*}{x}\right)\left[r\left(1-\frac{x}{k}\right)-\frac{m}{x+a}\right]x,$$
$$D^{\alpha}V(x) \leq rx-\frac{mx}{x+a}-rx_*+\frac{rxx_*}{K}+\frac{mx_*}{x+a},$$
$$D^{\alpha}V(x) \leq x\left[r-\frac{m}{M+a}+\frac{rx_*}{K}\right]+\frac{mx_*}{a}-rx_*.$$

If $r-\frac{m}{M+a}+\frac{rx_*}{K}<0$ and $\frac{mx_*}{a}-rx_*<0$, then V becomes a positive definite function (Lyapunov function) and hence, the point $x_*=x_s$ is globally stable.

5. Numerical analysis of the model

In this study, we investigated the fractional order, single species logistic equation with the additive Allee effect. To validate the theoretical results, we ran numerical simulations using the Adam-Bashforth-Moulton predictor-corrector method [70], and Roberto Garrappa developed the Predict, Evaluate, Correct, Evaluate (PECE) scheme for fractional differential equations in MATLAB [71]. This is a very effective method to solve fractional order differential equations, and it is applicable to both linear and non-linear fractional order differential equations. This method can be extended to solve the fractional differential equation with more than one differential operator. This method has error bound $O(h^q)$, where $q=\min=\{1+\alpha,\ 2\}=\begin{cases} 2 \text{ if } \alpha \geq 1 \\ (1+\alpha) \text{ if } \alpha<1 \end{cases}$ [72]. The stability analysis has been done in [73].

For a numerical example, we have taken the set of parameters as $K=100$, $a=20$, and $r=0.5$, this gives us the values of $ar=10, \frac{Kr}{4}\left(1+\frac{a}{K}\right)^2=18$. As the mathematical analysis of the model is done by dividing Allee's constant m, into three different classes, that are given by $m>\frac{Kr}{4}\left(1+\frac{a}{K}\right)^2$, $0<m<ar$, and $ar<m<\frac{Kr}{4}\left(1+\frac{a}{K}\right)^2$ and the same is done to verify the results numerically.

Figure 1 is drawn by taking $m=19>18=\frac{Kr}{4}\left(1+\frac{a}{K}\right)^2$ and initial population of 100, which is considered as carrying capacity. From Figure 1, it is clear that all populations are moving towards extinction. Therefore, the results obtained in Theorem 4.2.1 are verified. It has been proven mathematically as well as numerically that when a population is exposed to such a high degree of the Allee effect, even the strongest populations can move towards extinction. This result can be useful in controlling unwanted species. Regardless of the method used, if unwanted populations are exposed to such a high degree of the Allee effect, it can be controlled or removed locally. Simulations are run by taking fractional derivatives of the orders 0.9, 0.8, 0.7, 0.6, and the integer order 1. Populations die out in all models. This demonstrates that fractional derivatives are as reliable as ordinary integer-order derivatives.

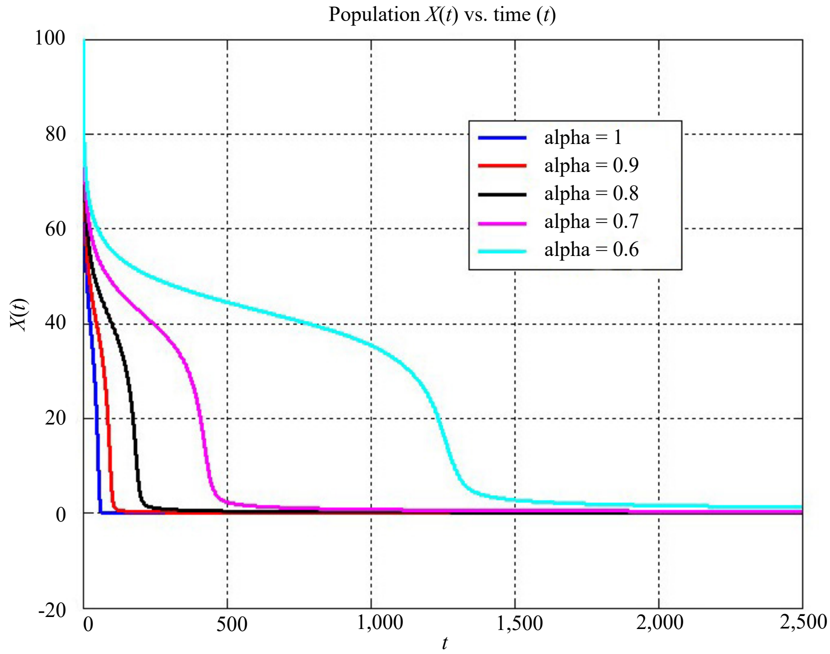

Figure 1. The stability of the extinction point when populations are subjected to the strong Allee effect $m > \frac{Kr}{4}\left(1+\frac{a}{K}\right)^2$

For the next simulation, selecting $m = 8$ $(0 < m < ar)$ populations are subjected to weak Allee effect and we got the value of the equilibrium point $x_s = \frac{K}{2}\left[\left(1-\frac{a}{K}\right)+\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right] = 84.72$. Figures 2 and 3 are plotted by taking $m = 8$, the weak Allee effect $(0 < m < ar)$ and an initial population of 15.

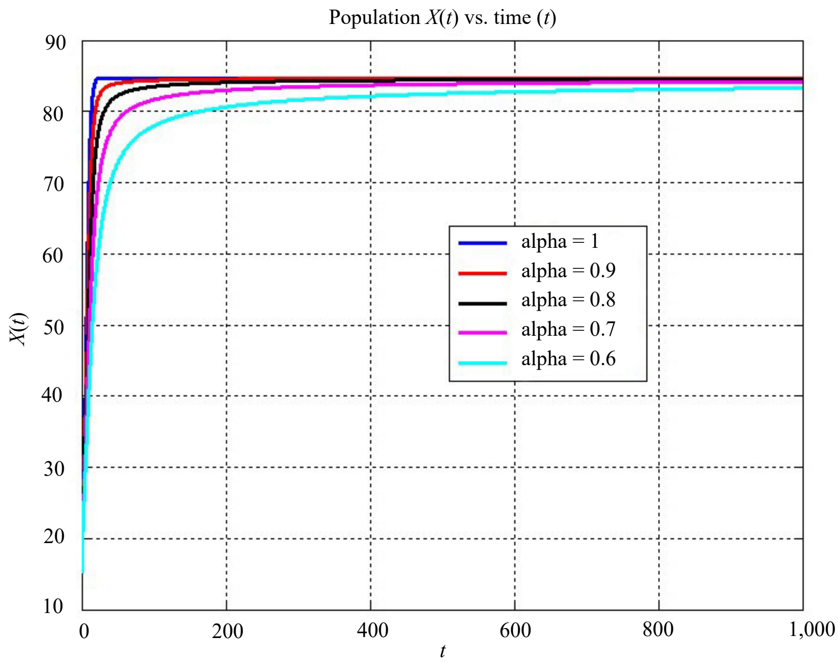

Figure 2. The stability of the equilibrium point $x_s = 84.72$ when populations are subjected to the weak Allee effect $(0 < m < ar)$

In Figure 2, the values of the fractional orders are considered. as 0.9, 0.8, 0.7, 0.6, and 1. It is clear from the figure that with all orders of the derivative considered, the population is converging to point 84.72.

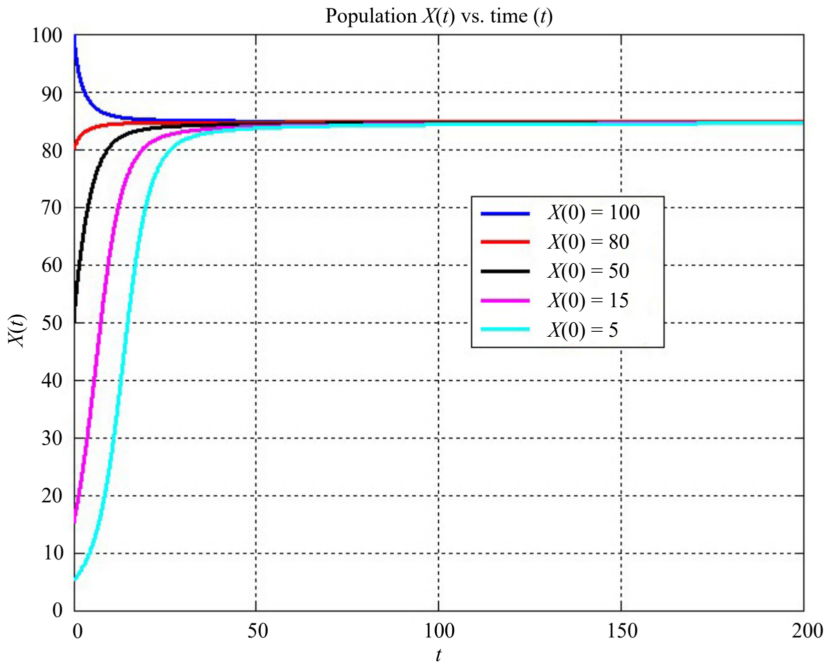

Figure 3. The stability of the equilibrium point x_s = 84.72 when populations are subjected to the weak Allee effect ($0 < m < ar$)

Figure 3 is plotted by taking fractional order 0.9 and with different initial populations as 5,155,080 and 100. Simulations have shown that all populations reach x_s = 84.72 with time and hence, x_s is asymptotically (locally as well as globally) stable when subjected to the weak Allee effect ($0 < m < ar$). The result found in Theorem 4.2.2 is also verified numerically. This simulation also established the fact that no minimum population is required for the survival population when subjected to the weak Allee effect.

For the next simulation, we have selected the value of m = 15 (strong Allee effect) is taken. The numerical value of the equilibrium points obtained as

$$x_s = \frac{K}{2}\left[\left(1-\frac{a}{K}\right)+\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right] = 64.5, \text{ and } x_t = \frac{K}{2}\left[\left(1-\frac{a}{K}\right)-\sqrt{\left(1+\frac{a}{K}\right)^2-\frac{4m}{Kr}}\right] = 15.5.$$

Figure 4 is plotted by taking an initial population of 15 below the value of x_t = 15.5 and by taking alpha = 0.6, 0.7, 0.8, 0.9, and 1. Simulations show that in each model, populations are getting extinct.

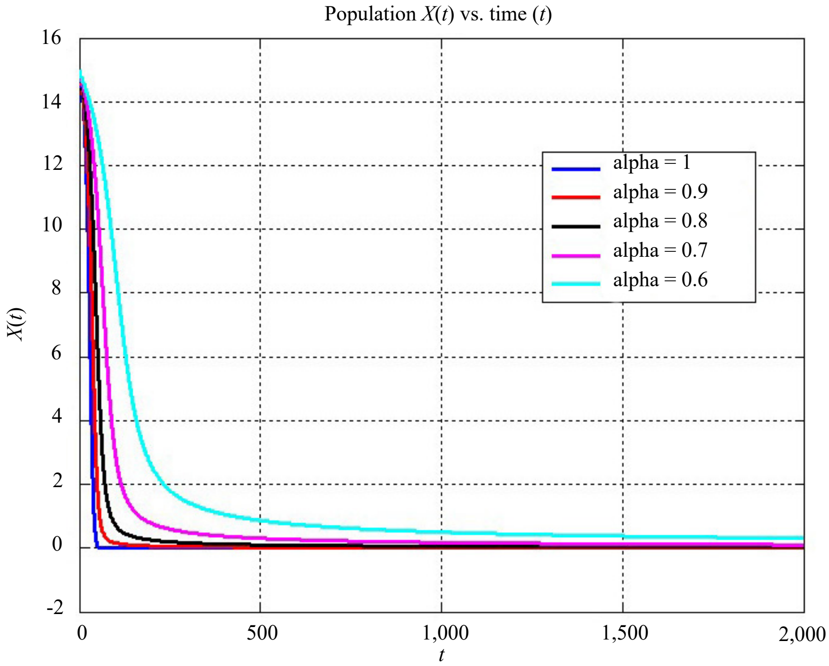

Figure 4. The stability of extinction point when initial population is less than $x_t = 15.5$ when the population is subjected to very strong Allee effect $ar < m < \frac{Kr}{4}\left(1+\frac{a}{K}\right)^2$

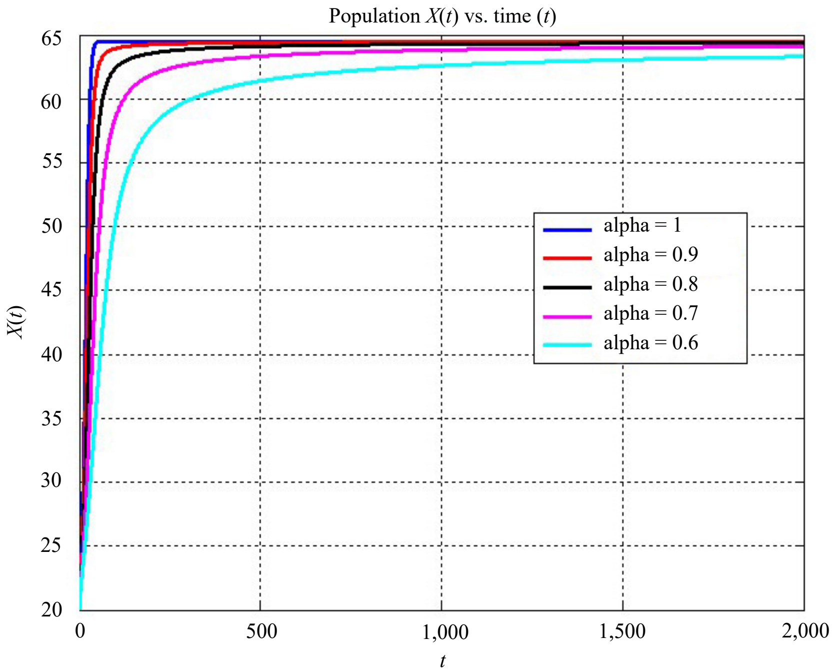

Figure 5. The stability of point $x_s = 64.5$ when initial population is more than $x_t = 15.5$ when the population is subjected to very strong Allee effect $ar < m < \frac{Kr}{4}\left(1+\frac{a}{K}\right)^2$

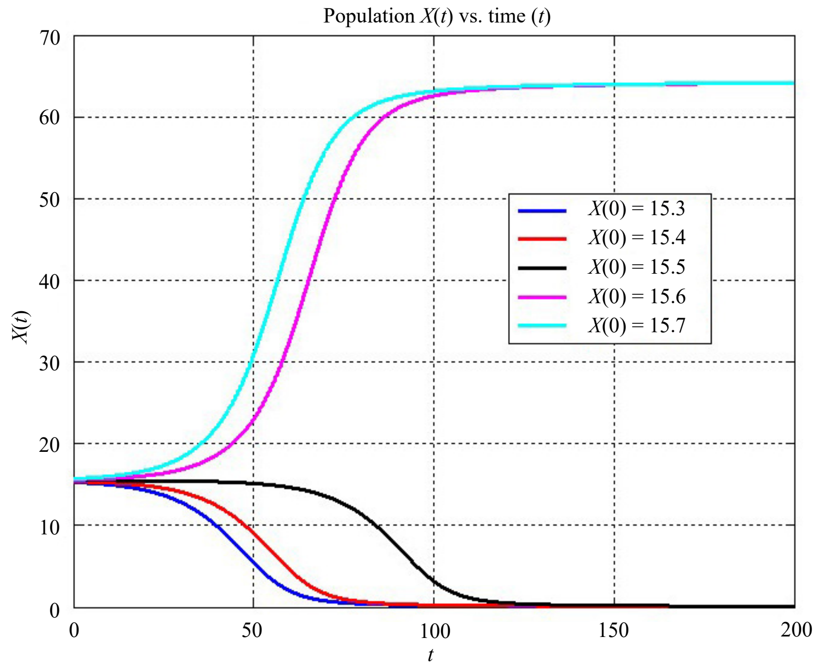

Figure 6. Showing x_t is the threshold level for the populations suffering from strong Allee effect

Figure 5 is plotted by taking the initial population of 20, which is above the value of $x_t = 15.5$ and by taking alpha = 0.6, 0.7, 0.8, 0.9, and 1. Simulations show that in each model populations are increasing and converging to point 64.5. Thus, it is proved numerically that when the initial populations are more than the value of x_t when population is subjected to the strong Allee effect that is when $ar < m < \frac{Kr}{4}\left(1+\frac{a}{K}\right)^2$, the population will converge to x_s and when initial population in this situation is less than x_t it will get extinct. Figure 6 is plotted by taking the fractional order as 0.9 and with different initial populations of 15.3, 15.4, 15.5, 15.6, and 15.7, and it shows that all initial populations greater than 15.5 are moving toward 64.5 and the initial populations below 15.5 are moving toward extinction. Hence, it is verified that populations experiencing a strong Allee effect require a threshold level for the population to grow.

6. Conclusions

In this study, while investigating the fractional order single species logistic equation with the additive Allee effect, the authors have studied the necessary and sufficient conditions for the existence and uniqueness of the solution, positivity, and uniform boundedness of the solutions. The authors have shown numerically that when the population is under the strong Allee effect or when $ar < m < \frac{Kr}{4}\left(1+\frac{a}{K}\right)^2$, the threshold level of the population is given by $x_t = \frac{K}{2}\left[\left(1-\frac{a}{K}\right) - \sqrt{\left(1+\frac{a}{K}\right)^2 - \frac{4m}{Kr}}\right]$. But, if the level of the Allee effect exceeds $\frac{Kr}{4}\left(1+\frac{a}{K}\right)^2$, all populations will die irrespective of the initial population. If the populations are experiencing a weak Allee effect ($0 < m < ar$), the population will not die whatever may be the initial population. Understanding the dynamics of low-density populations and the involvement of Allee effects is always a focus due to concerns about the extinction of endangered species. The Allee effect is a density-dependent phenomenon in which population growth rises with increasing population density. When population size or density is small, Allee effects are often characterized demographically as a positive connection between average individual fitness and population size or density. Various published studies have led to the accumulation of potential evidence for the presence of the Allee effect in various systems when a population is overharvested. Knowing the values of all parameters for a specific spatial distribution of a vulnerable species, a threshold level beyond which survival of that species is not conceivable can be calculated using the result for calculating the threshold level. Once we know the threshold level of the population prone to the strong Allee effect, timely strategies can help save that population from extinction. The strong Allee effect notion is frequently used in pest control. To eliminate the unwanted pest population, a strong Allee effect can be introduced in the pest population by reducing its number below the threshold level (independent of the method used), which can be computed. The behavior of the fractional-order derivatives is observed while studying population dynamics. It has been discovered that replacing the ordinary derivative with fractional derivatives has no effect on the equilibrium points and their stability; rather, it provides greater flexibility in the stability region of the equilibrium. Therefore, fractional derivatives are relatively better as compared with integer-order derivatives. Figures 1, 2, 4, and 5 show that the ordinary integer order model exhibits a sharp decrease or increase in population growth. In the fractional order model, however, as the order of fractional derivatives is reduced, the time required for populations to reach the equilibrium point increases. This demonstrates the memory-preserving nature of fractional derivatives. According to Du et al. [19], fractional derivatives have learning and forgetting stages, so populations take less time to learn the behavior and more time to forget the behavior as the order of the fractional derivative increases. As a result, higher-order fractional derivatives preserve higher levels of memory, whereas integer-order models do not exhibit such behaviors.

Thus, the use of fractional order models contributes to warning us of impending extinction at an early stage. Therefore, fractional order models provide opportunities to prevent population extinction by providing flexible stability regions. The proposed study's contribution is that once the threshold level of the population prone to the strong Allee effect is known, timely strategies can help save that population from extinction, and populations already below the threshold level can be saved by preventing them from adopting the isolation behavior. Artificial planting of trees or plants by sowing seeds can help in improving their density, and a small group of undercrowded populations can be saved by helping them mat by providing a safe environment with better carrying capacity and by controlling the harvesting pressures. As fractional calculus provides rich dynamics by showing the memory effect, it is suggested that it is better to address real-world problems using fractional calculus. As the present study makes use of only the Caputo derivative and there is a new proposed improved definition of fractional order derivatives like Caputo-Fabrizio and Atangana-Baleanu, in the near future, the study could be to solve equations by making use of the Caputo-Fabrizio and Atangana-Baleanu derivatives and with different numerical schemes to make a comparison so that an accurate model can be proposed with minimum error.

Conflict of interest

The authors declare no conflict of interest.

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Appendix

MATLAB Code

```
K= ;a= ;r= ;m= ;
fdefun=@(t,x)[x*r*(1-x/K)-m*x/(x+a)];
alpha=;
t0=0;tfinal=;x0=[];
h=2^(-6);

[t,x_fde12]=fde12(alpha,fdefun,t0,tfinal,x0,h);
figure(1)
plot(t,x_fde12(1,:));
```