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Regularity of Initial Ideal of Binomial Edge Ideals in Degree 2 and Their Powers

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Abstract: In this paper, we study the Castelnuovo-Mumford regularity of the initial ideal of binomial edge ideals in degree 2 $\left([in_<(J_G)]_2\right)$, and their powers for some classes of graphs. Also, we obtain a lower bound for the regularity of powers of binomial edge ideal associated to complete bipartite graphs.

Keywords: edge ideals, binomial edge ideals, Castelnuovo-Mumford regularity, polynomials ring

MSC: 13D13, 05E40, 13D02

1. Introduction

In the present paper, we are interested in the Castelnuovo-Mumford regularity of powers of homogeneous ideals in the polynomial rings. Let $R = K[x_1, ..., x_n]$ be such a standard graded polynomial ring over a field K and I be a homogeneous ideal in R. It is well-known that the regularity (R/I^s), as a function in s is asymptotically linear for $s \gg 0$ (cf. [1-2]). In general, it is difficult to decide when this function starts to be linear. Finding the linear function's exact form is also not easy (cf. [3]). In the following, we will call them the *linearization-of-regularity* problems, by abuse of terminology.

For these problems, the simplest case is when I is a quadratic squarefree monomial ideal. Whence, it can be recognized as the edge ideal of suitable graphs. Researchers have obtained some results for a few classes of simple graphs (e.g., forest graphs, cycle graphs, bipartite graphs) (cf. [4-5]). In contrast, little is known about the related binomial ideals. Recently, Jayanthan et al. in [6] provided the exact formulas for the regularity of powers of binomial edge ideals of several simple graphs such as cycle graphs, star graphs and balloon graphs. The main result of Jayanthan et al. in [6] is an upper bound for the regularity of powers of an ideal generated by a quadratic sequence. Inspired by their work, in this paper, we will provide the exact formulas for the regularity of powers of I, when I is an initial ideal of a binomial edge ideal of some graphs. As applications, we will study the linearization-of-regularity problems when the ideal I is an initial ideal of binomial edge ideal of some undirected simple graph G.

Given such a graph G, suppose that it has the vertex set $V(G) = [n] = \{1, \cdots, n\}$ and edge set $E(G)$. The *binomial edge ideal* of G is defined as

$$J_G = (x_i y_j - x_j y_i : \{i, j\} \in E(G), i < j) \subset S = K[x_i, y_i : i \in V(G)],$$

where K denotes an arbitrary field. Herzog et al. in [7] and Ohtani in [8] independently introduced the notion of binomial edge ideal, and after that, the research on "binomial edge ideal" becomes a trend topic in combinatorial commutative algebra. Recently, there have been many results relating to the combinatorial data of graphs with the algebraic properties of the corresponding binomial edge ideals, see [5-6, 9-11]. In particular, there has been active research connecting algebraic invariants of the binomial edge ideals such as Castelnuovo-Mumford regularity, depth, and Betti numbers, with combinatorial invariants associated with graphs such as length of maximal induced path, number of maximal cliques, matching number.

Let the polynomial ring $S = K[x_i, y_i : i \in V(G)]$ endowed with the lexicographic order induced by $x_1 > x_2 > \cdots > x_n > y_1 > y_2 > \cdots > y_n$. The initial ideal of binomial edge ideal in degree 2 is defined as $[in_<(J_G)]_2 = (x_i y_j : \{i, j\} \in E(G), i < j$, and it is the monomial edge ideal of a bipartite graph on $2n$ vertices. Herzog et al. showed in [12], that powers of edge ideals with linear resolution have linear resolution themselves. A classic result by Fröberg [13] says that an edge ideal has linear resolution if and only if there is no induced cycle of length greater than three in its complement. Similar attempts have been made for powers of edge ideals of bipartite graphs in [5, 14].

In [10], the authors conjectured that the extremal Betti numbers; a Betti number $\beta_{i,j} = 0$ will be called extremal if $\beta_{l,r} = 0$ for all $l \geq i$, $r \geq j+1$ and $r - l \geq j - i$, of J_G and $in_<(J_G)$ coincide for any graph G that it is still open. In [9], the author gave a positive answer to this conjecture when graph G is a complete bipartite graph or a cycle graph. Also, the conjecture had been stated for every closed graph.

In this paper, we study the regularity of the initial ideal of binomial edge ideal in degree 2 $([in_<(J_G)]_2)$ for complete bipartite, cycle, and tree graphs. Furthermore, we obtain a lower bound for the regularity of powers of the binomial edge ideal of complete bipartite graphs.

2. Preliminaries

In this section, we review notation and fundamental results on the regularity of edge ideals and power of edge ideals that will be used in the next sections.

All graphs discussed in this paper are simple graphs. We assume basic familiarity with standard graph theory definition as in, e.g., [15], but in finding combinatorial bounds for the regularity of edge ideals, matching numbers and induced matching numbers have been used often. We recall their definitions.

Let G be a simple graph with the vertex set $V(G) = [n]$ and edge set $E(G)$. A *complete graph* on $[n]$, denoted by K_n, is the graph with the edge set $E(G) = \{i, j : 1 \leq i < j \leq n\}$. For $A \subset V(G)$, $G[A]$ denotes the *induced subgraph* of G on the vertex set A, that is, for $i, j \in A$, $\{i, j\} \in E(G[A])$ if and only if $\{i, j\} \in E(G)$. A subset U of $V(G)$ is said to be a *clique* if $G[U]$ is a complete graph. A vertex v of G is said to be a *simplicial vertex* if v is contained in only one maximal clique otherwise it is called an *internal vertex*. We denote the number of internal vertices of G by $iv(G)$.

A collection of edges $\{e_1, \cdots, e_s\}$ is said to be a *matching* if $e_i \cap e_j = \varnothing$ for all $i \neq j$ and this is said to be an *induced matching* if the induced subgraph on the vertices of $\{e_1, \cdots, e_s\}$ has an edge set, $\{e_1, \cdots, e_s\}$. For a graph G, the *induced matching number* of G is the largest size of an induced matching, and it is denoted by $v(G)$.

The *complement* of a graph G, denoted by G^c is the graph on the same vertex set in which $\{u, v\}$ is an edge of G^c if and only if it is not an edge in G.

A graph G is called *bipartite* if there are two disjoint independent subsets V_1, V_2 of $V(G)$ such that $V_1 \cup V_2 = V(G)$. Let J_G be the binomial edge ideal of a graph G. Then the *initial ideal* of binomial edge ideal in degree 2 is defined as, $[in_<(J_G)]_2 = (x_i y_j, \{i, j\} \in E(G), i < j)$. It can easily be seen that the ideal $[in_<(J_G)]_2$ associated to the graph G is the monomial edge ideal of a bipartite graph on $2n$ vertices with vertex sets $V_1 = \{x_1, x_2, \cdots, x_n\}$ and $V_2 = \{y_1, y_2, \cdots, y_n\}$.

Definition 1 [5] A graph G is *chordal graph* if every induced cycle in G of a length greater than three has a chord, and it is called *co-chordal* if the complement of the graph G^c is chordal. The co-chordal cover number, denoted *co-chord* (G), is the minimum number m such that there exist co-chordal subgraphs $H_1, \cdots, H_m$ of G with $E(G) = \cup_{i=1}^m E(H_i)$.

Now, we recall the necessary notation from commutative algebra. Let $R = K[x_1, \cdots, x_n]$ be a standard graded polynomial ring over a field K and M be a finitely generated graded R-module. Let

$$0 \rightarrow \oplus_{j \in Z} R(-j)^{\beta_{pj}(M)} \xrightarrow{\varphi_p} \cdots \rightarrow \oplus_{j \in Z} R(-j)^{\beta_{0j}(M)} \xrightarrow{\varphi_0} M \rightarrow 0,$$

be the minimal graded free resolution M, where $R(-j)$ is the free R-module of *rank* 1 generated in degree j. The numbers $\beta_{ij}(M)$ are called the (i, j)-th graded *Betti numbers* of M. The regularity of M, denoted by reg(M) is defined as,

$$\mathrm{reg}(M) = \max\{j - i, \; \beta_{ij}(M) \neq 0, \;\; \text{for some } i \text{ and } j\}.$$

When edge ideal $I(G)$ has a linear resolution, all the nonzero entries in its Betti diagram are located on the first row. The next theorem, due to Fröberg is used repeatedly throughout this paper.

Theorem 1 [13] The minimal free resolution of edge ideal $I(G)$ is linear if and only if the complement graph G^c is chordal, that is, does not have induced cycles of *length* ≥ 4.

In the following theorem, a lower bound for the regularity of powers of edge ideals is given.

Theorem 2 Theorem 4.5 in [4]. Let G be a graph with edge ideal $I = I(G)$ and let $v(G)$ denote its induced matching number. Then for all $s \geq 1$, we have $(I^s) \geq 2s + v(G) - 1$.

3. Regularity of powers of initial ideal of binomial edge ideal in degree 2 associate to cycle graphs

The following theorem was proved by Jayanthan et al. in [6]. In this section, we prove it for initial ideals of binomial edge ideals in degree 2 associated to cycle graphs. A connected simple graph G of n vertices whose vertices are all of degree two is called a *n-cycle* and denoted by C_n.

Theorem 3 Theorem 3.6 in [6]. Let C_n be a cycle graph on vertex set $[n]$ and J_{C_n} be the binomial edge ideal of C_n. Then, $\mathrm{reg}(J_{C_n}^s) = 2s + n - 3$, for all $s \geq 1$.

Observation 1 Let C_n be a cycle graph on vertex set $[n]$ and J_{C_n} be the binomial edge ideal of C_n. Dokuyucu in [9] (Corollary 1.4), gave a following presentation for $in_<(J_{C_n})$:

$$in_<(J_{C_n}) = (x_1 y_2, \ldots, x_{n-1} y_n, x_1 y_n, \{x_i x_{j+1} \ldots x_n y_1 \ldots y_{i-1} y_j\}_{2 \leq j - i \leq n-2}).$$

Therefore, $[in_<(J_{C_n})]_2 = (x_1 y_2, \ldots, x_{n-1} y_n, x_1 y_n)$ and the graph associated with $[in_<(J_{C_n})]_2$ is a bipartite graph of the form as shown in Figure 1.

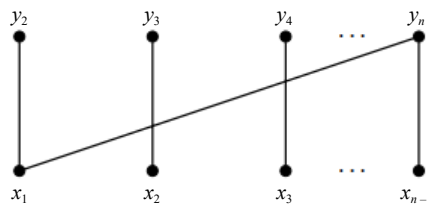**Figure 1.** Graph associated to $[in_<(J_{C_n})]_2$

The induced matching number $v(C_n)$ of this type of bipartite graph is associated to $[in_<(J_{C_n})]_2$ is equals to $n - 2$.

Theorem 4 Let C_n be a cycle graph on vertex set $[n]$ and J_{C_n} be the binomial edge ideal of C_n. Then $\mathrm{reg}\left(\left[in_<(J_{C_n})\right]_2^s\right) = 2s + n - 3$, for all $s \geq 1$.

Proof. Let C_n be a cycle graph. Then $[in_<(J_{C_n})]_2$ is an edge ideal and by Theorem 2 we have:

$$\mathrm{reg}\left(\left[in_<(J_{C_n})\right]_2^s\right) \geq 2s + v(C_n) - 1,$$

where $v(C_n)$ is the maximum size of an induced matching in C_n, and by Observation 1, $v(C_n) = n - 2$. Hence,

$$\text{reg}\left(\left[in_<(J_{C_n})\right]_2^s\right) \geq 2s + n - 2 - 1$$

or

$$\text{reg}\left(\left[in_<(J_{C_n})\right]_2^s\right) \geq 2s + n - 3.$$

On the other hand, since $[in_<(J_{C_n})]_2$ is an edge ideal by [11] (Theorem 5.3 (c)), we have $[in_<(J_{C_n})]_2 = n - 1$. Also, the graph associative to $[in_<(J_{C_n})]_2$ is a bipartite graph and by [16] (Theorem 1.1 (ii)), we get that,

$$\text{reg}\left([in_<(J_{C_n})]_2^s\right) \leq 2s + \text{reg}\left([in_<(J_{C_n})]_2\right) - 2$$

$$\leq 2s + n - 1 - 2$$

$$\leq 2s + n - 3.$$

Therefore, $\text{reg}\left([in_<(J_{C_n})]_2^s\right) = 2s + n - 3$, for all $s \geq 1$.

Corollary 1 Let C_n be a cycle graph. Then, $\text{reg}\left([in_<(J_{C_n})]_2^s\right) = \text{reg}(J_{C_n}^s) = 2s + n - 3$, for all $s \geq 1$.

4. Regularity of binomial edge ideal and initial of binomial edge ideal with their powers associated with bipartite graphs

In this section, we give a lower bound for the regularity of powers of binomial edge ideals of complete bipartite graphs and we give the exact value for the regularity of initial degree 2 of binomial edge ideals and their powers. Also, we obtain the regularity of certain bipartite graphs, and, in particular trees.

We proceed by recalling a few well-known results.

Lemma 1 [10] (Lemma 3.3). Let G be a finite bipartite graph on $\{x_1, \cdots, x_n\} \cup \{y_1, \cdots, y_n\}$ with the edges $\{x_i, y_j\}$ with $1 \leq i \leq j \leq n$. Then, the complementary graph G^c of G is a chordal graph.

Proposition 1 [17], (Proposition 2.18). If G is a bipartite graph, then its complement does not have any induced cycle of $length > 4$. In particular, edge ideal of a bipartite graph has linear presentation, if and only if all its powers have linear resolution.

Corollary 2 Let G be a bipartite graph and $I(G)$ be an edge ideal of the graph G. If $\text{reg}(I(G)) = 2$, then $I(G)$ have linear presentation.

Proof. Since $\text{reg}(I(G)) = 2$, then by Theorem 1, G^c does not have an induced cycle of $length \geq 4$ this means that G^c is a chordal graph, and by [18] (Proposition 1.3), we get that $I(G)$ has a linear presentation.

Corollary 3 Let G be a finite bipartite graph on $\{x_1, \cdots, x_n\} \cup \{y_1, \cdots, y_n\}$ with the edges $\{x_i, y_j\}$ with $1 \leq i \leq j \leq n$. Then $\text{reg}(I(G)) = 2$, where $I(G)$ is the edge ideal of graph G.

Proof. Let $I(G)$ be an edge ideal of a finite bipartite graph on $\{x_1, \cdots, x_n\} \cup \{y_1, \cdots, y_n\}$, by Lemma 1, G^c is a chordal graph, this means that G^c does not have induced cycles of $length \geq 4$, and by using Theorem 1, we get that $\text{reg}(I(G)) = 2$.

The next corollary immediately follows from Theorem 1, [18] (Proposition 1.3 Theorem 1.8), and [12] (Theorem 3.2).

Corollary 4 Let $G = K_{m, n}$ be a complete bipartite graph and $[in_<(J_G)]_2$ be the initial ideal of binomial edge ideal in degree 2 associated with G. Then, the following are equivalent:

- $\text{reg}([in_<(J_G)]_2) = 2$;
- G^c is chordal graph;
- $[in_<(J_G)]_2$ has a linear presentation;

- $([in_<(J_G)]_2)^s$ has a linear resolution for all $s \geq 1$.

Example 1 Let G be a bipartite graph on $\{x_1, x_2, x_3\} \cup \{y_1, y_2, y_3\}$ with the edges $\{x_i, y_j\}$ with $1 \leq i \leq j \leq 3$. Then $I(G) = (x_1 y_1, x_1 y_2, x_1 y_3, x_2 y_2, x_2 y_3, x_3 y_3)$ and $\text{reg}(I(G)) = 2$. Figure 2 shows the F_3 graph.
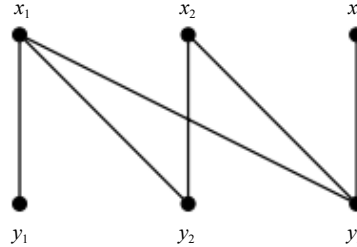

Figure 2. F_3 graph

Remark 1 Let $G = K_{m,n}$ be the complete bipartite graph on the vertex set $\{1, \cdots, m\} \cup \{m+1, \cdots, m+n\}$ with $m \geq n \geq 1$ and let J_G be its binomial edge ideal. J_G generated by all the binomials $f_{ij} = x_i y_j - x_j y_i$ where $1 \leq i \leq m$ and $m+1 \leq j \leq m+n$. Dokuyucu in [9] (Corollary 1.3) shows that:

$$in_<(J_G) = \left(\{x_i y_j\}_{\substack{1 \leq i \leq m \\ m+1 \leq j \leq m+n}}, \{x_i x_{m+k} y_j\}_{\substack{1 \leq i < j \leq m \\ 1 \leq k \leq n}}, \{x_{m+i} y_k y_{m+j}\}_{\substack{1 \leq i < j \leq n \\ 1 \leq k \leq m}} \right).$$

Therefore, $\left[in_<(J_G) \right]_2 = \left(\{x_i y_j\}_{\substack{1 \leq i \leq m \\ m+1 \leq j \leq m+n}} \right)$ and the graph associated to $\left[in_<(J_G) \right]_2$ is also complete bipartite graph G.

In the following proposition, we obtain the regularity of powers of $\left[in_<(J_G) \right]_2$, associate to the complete bipartite graphs.

Corollary 5 Let G be a complete bipartite graph and J_G be the binomial edge ideal of G. Then $\text{reg}\left(\left[in_<(J_G) \right]_2^s \right) = 2s$, for all $s \geq 1$.

Proof. Since the graph G is complete bipartite, then by Remark 1, the graph related to $\left[in_<(J_G) \right]_2$ is also a complete bipartite graph. So, it is a co-chordal graph and by [3] (Theorem 5.7), $\text{reg}\left(\left[in_<(J_G) \right]_2^s \right) = 2s$ for all $s \geq 1$.

Example 2 Let $G = K_{2,3}$ be a complete bipartite graph with binomial edge ideal

$$J_G = \left(x_1 y_3 - x_3 y_1, \ x_1 y_4 - x_4 y_1, \ x_1 y_5 - x_5 y_1, \ x_2 y_3 - x_3 y_2, \ x_2 y_4 - x_4 y_2, \ x_2 y_5 - x_5 y_2 \right).$$

Then,

$$[in_<(J_G)]_2 = (x_1 y_3, x_1 y_4, x_1 y_5, x_2 y_3, x_2 y_4, x_2 y_5)$$

Hence, $\text{reg}\left(\left[in_<(J_G) \right]_2 \right) = 2$, $\text{reg}\left(\left[in_<(J_G) \right]_2^2 \right) = 4$ and $\text{reg}\left(\left[in_<(J_G) \right]_2^3 \right) = 6$. In general $\text{reg}\left(\left[in_<(J_G) \right]_2^s \right) = 2s$. Figure 3 shows the $K_{2,3}$ graph.
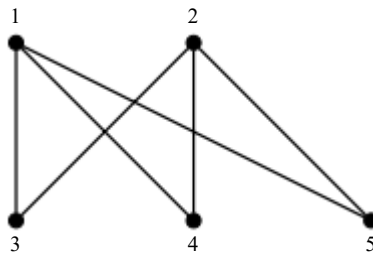
Figure 3. $K_{2,3}$ graph

Schenzel et al. [14] (Theorem 4.1), proved that $\text{reg}(J_{K_{1,n}}) = 3$, where $K_{1,n}$ is a special class of complete bipartite graphs that called star graphs. Also, Jayanthan et al. [6] computed the regularity of their powers [6] (Theorem 3.5) and show that $\text{reg}\left(J_{K_{1,n}}^{s}\right) = 2s + 1$. In the following theorem, we extend this result for binomial edge ideal of complete bipartite graphs.

Theorem 5 Let G be a complete bipartite graph $K_{m,n}$ Then $\text{reg}(J_G^s) \geq 2s + 1$, for all $s \geq 1$.

Proof. Since G is a complete bipartite graph then it is connected and based on [6] (Corollary 3.4), we have

$$\text{reg}(J_G^s) \geq 2s + \iota - 1,$$

where ι is the length of the longest induced path of G. The length of the longest induced path of a complete bipartite graph is 2, that is $\iota = 2$. So, $\text{reg}(J_G^s) \geq 2s + 1$.

At present, we are unable to prove, but computational evidence indicates that the answer to the following question is affirmative:

Question 1 Let G be a complete bipartite graph. Is $\text{reg}(J_G^s) = 2s + 1$, for all $s \geq 1$?

Now, we deal with tree graphs. The tree is a graph in which any two vertices are connected by exactly one path. For the initial ideal of binomial edge ideal in degree 2 of a tree graph, we fix a labeling of the tree that we call grapes labeling. The labeling is as follows: First, we hang the tree just like a bunch of grapes by any of the vertex having a degree greater than 1. We labeled this vertex by 1 and named this vertex as a father vertex and all the vertices connected with the father vertex by an edge are called its children and labeled by 2, 3 and so on. Now in the next step consider all the children's vertices as father vertices and label their children's vertices as before and continue this process until the whole tree is labeled.

Example 3 Let's apply grapes labeling on the following tree shown in Figure 4. The father vertex is labeled as v_1 who has three children labeled as v_2, v_3, and v_4. Now in the next step, the father vertex labeled with v_2 has three children labeled as v_5, v_6, and v_7. The father vertex labeled with v_3 has no child and v_4 has one child labeled with v_8 as shown in Figure 4.
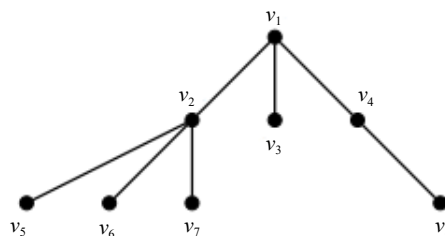

Figure 4. Tree graph

In the rest of this section, we study $[in_{<}(J_T)]_2$ with the above mentioned labeling. We compute the regularity of initial ideal of binomial edge ideal in degree two for an arbitrary tree graph.

Theorem 6 Let T be a tree graph on vertex set $[n]$ and J_T be the binomial edge ideal of graph T. Let $[in_{<}(J_T)]_2$ be the

initial ideal of binomial edge ideal in degree two associated with graph T. Then, $\text{reg}([in_<(J_T)]_2) = iv(T)$.

Proof. The graph associated to the monomial edge ideal $[in_<(J_T)]_2$ is a bipartite graph on $[2n]$ vertices and each connected component of this bipartite graph is $K_{i,1}$-star graph. It can easily be seen that the number of internal vertices $iv(T)$ or (father vertices) of graph T equals the number of components of $K_{i,1}$-star graph associated to $[in_<(J_T)]_2$. Based on Corollary 2 in [3], we get that:

$$\text{reg}\left(S/[in_<(J_T)]_2\right) = \sum_{i=1}^{iv(T)} reg(S/K_{i,1}) = \sum_{i=1}^{iv(T)} 1 = iv(T).$$

Corollary 6 Let T be a tree graph on vertex set $[n]$. Then, $\text{reg}\left([in_<(J_T)]_2^s\right) = 2s + iv(T)) - 1$.

Proof. The upper bound is obtained from [11] (Theorem 5.3 (c)) and Theorem 6. The lower bound deduces from Theorem 2 and regarding that, the number of internal vertices of graph T is equal to the induced matching number of a bipartite graph associated with the monomial ideal $[in_<(J_T)]_2$.

5. Discussion

Since the early 1990s, binomial ideals have become gradually fashionable. They now appear in various areas of commutative algebra and combinatorics as well as statistics. A comprehensive analysis of the algebraic properties of binomial ideals, including their primary decomposition, regularity, and depth, was given.

The binomial edge ideal of a graph could be seen as the ideal generated by a collection of 2-minors of the generic $(2 \times n)$-matrix X. Therefore, the binomial edge ideal of a complete graph with n vertices is nothing but the well-studied determinantal ideal $I_2(X)$.

A big effort concerning the binomial edge ideals has been understanding their minimal graded free resolutions. In general, it is quite difficult to describe the resolution of a binomial ideal. The graded Betti numbers give the numerical data of the resolution and determine the regularity and projective dimension of the ideal. In [19], the graphs whose binomial edge ideals have a linear resolution were characterized. Indeed, it was shown that a graph with no isolated vertices J_G has a linear resolution if and only if G is a complete graph. Moreover, it was shown that the equivalent conditions hold if and only if J_G has linear relations. Another approach in studying the graded minimal free resolution of J_G has been determining its Castelnuovo-Mumford regularity. There have been two conjectures regarding some upper bounds for the regularity of J_G, one by Matsuda et al. [20], and the other one by Kiani et al. [21]. The first one was proved in [21], while the second one is still widely open. This conjecture was verified in [22] for a class of chordal graphs, namely block graphs. Several other algebraic properties and invariants of binomial edge ideals have also been studied in [7, 9-10].

In this work, we obtain an upper bound for the regularity of powers of the binomial edge ideal of complete bipartite graphs. Also, we compute the regularity of the initial ideal powers of binomial edge ideal associated with the cycle graphs, complete bipartite graphs, and trees.

6. Conclusion

The initial binomial edge ideal is the monomial edge ideal of a bipartite graph. In [10], the authors conjectured that the extremal Betti numbers of J_G and $in_<(J_G)$ coincide for any graph G that it is still open. In [9], the author gave a positive answer to this conjecture when the graph G is a complete bipartite graph or a cycle graph. Also, the conjecture had been stated for every closed graph.

In [23], authors obtained the regularity of binomial edge ideal powers for cycle graphs and in the present work, we compute the regularity of powers of initial ideals of binomial edge ideals in degree 2 associated with cycle graphs. This gives a partial answer to the conjecture of Ene et al. [10]. Schenzel et al. in [14] (Theorem 4.1), proved that $\text{reg}(J_{K_{1,n}}) = 3$, where $K_{1,n}$ is a special class of complete bipartite graphs that called star graphs. Also, Jayanthan et al. [6] (Theorem 3.5) computed the regularity of their powers and show that $\text{reg}\left(J_{K_{1,n}}^s\right) = 2s + 1$. In Theorem 5, we extend this result for the

binomial edge ideal of complete bipartite graphs.

Finally, we study the regularity of powers of the initial ideal of binomial edge ideal associated with trees.

Conflict of interest

The authors declare that they have no competing interests.

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