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Research Article

Solving Nonlinear Fractional Differential Equations by Using Shehu Transform and Adomian Polynomials

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Abstract: The current article provides a detailed analysis of the solution of non-linear ordinary differential equations of fractional and non-fractional order in series forms using the Shehu transform (ST) and the Adomian decomposition method (ADM), also known as the Shehu transform Adomian decomposition method (STADM). Previously, these methods were used to solve differential equations of integer order as well as a very small number of ordinary and fractional differential equations (FDEs). The Caputo's operator is used in a number of well-known FDEs, including the logistic equation, the Van der Pole equation, and other non-fractional order differential equations like the non-linear Bratu type equation. It is noted that all of the example issues had series solutions thanks to STADM. Plotting the graph for several series terms of the series solution demonstrates how the approximate solution tends to the closed form solution. In some example problems the impact of α is shown.

***Keywords*:** ST, ADM, Caputo FDE, ordinary differential equations

MSC: 34A99

Nomenclature

ADM	Adomian decomposition method
FC	Fractional calculus
FDE	Fractional differential equations
ST	Shehu transform
$\mathcal{D}_t^n$	nth order derivative operator
${}_0^C\mathcal{D}_t^\beta$	Caputo fractional differential operator
${}_0^{RL}\mathcal{D}_t^\beta$	Riemann-Liouville's fractional differential operator
$\mathcal{E}_{\alpha,\beta}(z)$	Mittag-Leffler function
$\mathcal{H}$	Shehu transform symbol
$\mathcal{H}^{-1}$	Inverse Shehu transform symbol

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1. Introduction

Non-linearity is a significant challenge in the solution of fractional differential equations (FDEs) that frequently arises while addressing many models, such as chaotic models [1, 2], epidemiological model for computer viruses [3] and global population growth model [4], because there is no one approach that can be used to solve all non-linear FDEs. Thanks to the ground-breaking discipline of fractional calculus, which has a history spanning more than 300 years [5-8] and is concerned with the study of the integrals and derivatives of fractional-order functions (orders may be real or complex). Due to its many applications the study of fractional calculus received a lot of scientific attention, particularly to procedures needing memory effects [9], problems in rheology [10, 11], in electrochemistry [12-14], in epidemiology [15, 16], in chemical physics [17], in fluid mechanics [18], etc.

This certainty makes it more logical to look for an analytical solution given that many situations typically lack one. So, it is preferred to use numerical [19-21] or semi-analytical methodologies such as: Adomian decomposition method (ADM) [22, 23], new decomposition method (NDM) [24], Sumudu transform and homotopy perturbation method (STHPM) [25], Sumudu transform and variation iteration method (STVIM) [26], Sumudu tranform and new iterative method (NIM) [27].

One of the greatest methods for solving differential equations is the integral transformation. The main advantage of this approach is the transformation of a differential issue into an algebraic one. Researchers have employed a variety of integral transforms (such as the Sumudu transform and Laplace transform) as well as other decomposition strategies to deal with these kinds of non-linear FDEs, as has already been discussed. The Shehu transform (ST) is another integral transformation that was recently presented by Maitama et al. [28], which is a generalization of the Laplace and Sumudu integral transforms [29]. Shehu and Zhao [28] used ST to resolve a large number of non-linear ordinary as well as partial differential equations of integer order. After this, [30] explained additional ST features and used this trampoline to propose fractional order ordinary differential equations' solutions. Later, ST is utilized to solve numerous non-linear FDEs together with many other analytical approaches, such as NIM [31] and ADM [32, 33]. In order to handle integral and integro-differential equations that are both linear and nonlinear, Poltem et al. [32] and Yisa et al. [33] recently devised the Shehu transform Adomian decomposition method (STADM). A review of the literature reveals that the ST has already been integrated with ADM, also known as STADM, and it has been utilized to handle differential equations of integer order as well as of fractional order [32, 33]. With the same methodology, we solve nonlinear fractional and non-fractional differential equations once more in this paper using STADM and offer the solutions in more detail.

In the end, this work provides an overview of fractional calculus and its applications, emphasises on the challenges caused by fractional calculus (FC) nonlinearity, and provides a novel approach to solve non-linear non-integer differential equations, in which ST and Adomian polynomials are combined in this algorithm.

2. Preliminaries

2.1 *ST*

Let the set $\mathcal{X}$ be the collection of functions as:

$$\mathcal{X} = \left\{ f(t) : \exists M, \nu_1, \nu_2 > 0, |f(t)| < Me^{\frac{|t|}{\nu_i}}, if, t \in (-1)^i \times [0,\infty) \right\}. \tag{1}$$

Then, ST of function, $f(t)$ defined over the set of functions $\mathcal{X}$ written as:

$$\mathcal{H}[f(t)] = F(s,u) = \int_0^\infty f(t) e^{\frac{-st}{u}} dt. \tag{2}$$

2.2 *Inverse Shehu transform (IST)*

IST can be defined as:

$$\mathcal{H}^{-1}[F(s,u)] = f(t), \quad \text{for } t \geq 0, \tag{3}$$

or

$$f(t) = \mathcal{H}^{-1}[F(s,u)] = \frac{1}{2\pi\iota}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{1}{u} e^{\frac{st}{u}} F(s,u)ds. \tag{4}$$

This integral is taken along $s = \alpha$ in the complex plane, i.e., ($s = x + iy$). Here, α is real constants and s and u are the ST variables.

2.3 *Basic STs*

The ST of function $f(t) = t^n$ is defined as:

$$\mathcal{H}\left(\frac{t^n}{n!}\right) = \left(\frac{u}{s}\right)^{n+1}, n = 0, 1, 2, ... \tag{5}$$

$$\mathcal{H}\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right) = \left(\frac{u}{s}\right)^{\alpha+1}, \alpha > -1 \tag{6}$$

$\Gamma(.)$ is the value of gamma at $(\alpha + 1)$. While, ST of $\mathcal{D}^n f(t),\ n \geq 1$, can be written as:

$$\mathcal{H}[\mathcal{D}^n f(t)] = \left(\frac{s}{u}\right)^n F(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k-1} \mathcal{D}^k f(t)|_{t=0}\ . \tag{7}$$

Also, ST of Caputo's fractional derivative of order $\beta > 0$ is given below:

$$\mathcal{H}[{}_0^C\mathcal{D}_t^\beta f(t)] = \left(\frac{s}{u}\right)^\beta F(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{\beta-k-1} D^k f(t)|_{t=0}, \tag{8}$$

where $F(s, u)$ is ST of $f(t)$. Also, ST of the Mittag-Leffler function $\mathcal{E}_{\alpha,\beta}(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, z, \beta \in C, Real(\alpha) > 0$ is given below:

$$\mathcal{H}[t^{\beta-1}\mathcal{E}_{\alpha,\beta}(\omega t^\alpha)] = \left(\frac{u}{s}\right)^\beta\left(1-\omega\left(\frac{u}{s}\right)^\alpha\right)^{-1}. \tag{9}$$

2.4 *Fractional derivative (FD)*

FD of any function $f(t)$ with order β in Caputo's sense can be construed as:

$${}_0^C\mathcal{D}_t^\beta f(t) = \frac{1}{\Gamma(n-\beta)}\int_0^t (t-p)^{n-\beta-1} f^n(p)dp,\ \ n-1 < \beta \leq n,\ \ n \in N, \tag{10}$$

and the same in Riemann-Liouville's sense can be written as:

$${}_0^{RL}D_t^\beta f(t) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\left(\int_0^t (t-p)^{n-\beta-1} f(p)dp\right), n-1 < \beta \leq n, n \in N. \tag{11}$$

Also, Riemann-Liouville's and Caputo derivative are co-related with each other in following manner:

$$ {}_{0}^{RL}\mathcal{D}_t^{\beta} f(t) = {}_{0}^{C}\mathcal{D}_t^{\beta} f(t) + \sum_{k=0}^{n-1} \frac{t^{k-\beta}}{\Gamma(k-\beta+1)} f^k(0^+). \tag{12} $$

3. Proposed algorithm

We will explain how to handle nonlinear differential equations in this section. Let's take a look at the supplied differential equation in the following form:

$$ {}_{0}^{C}\mathcal{D}_t^{\beta} f(t) + \mathcal{R}f(t) + \mathcal{N}f(t) = h(t), \ \ \beta > 0. \tag{13} $$

$$ f^k(t)|_{t=0} = c_k, \ \ k = 0, 1, 2, ..., n-1, \ \ n-1 < \beta \le n. \tag{14} $$

Where $\mathcal{D}^{\beta}f(t)$, $\mathcal{N}f(t)$, $\mathcal{R}f(t)$ and $h(t)$ are considered as Caputo derivative of function $f(t)$ of fractional order (β), a nonlinear part of given FDE and remaining part containing linear order operator terms, respectively. Along with these terms there is one more function of t belongs to set $\mathcal{X}$.

Now in order to solve above equation, first take ST as:

$$ \begin{gathered} \mathcal{H}[{}_{0}^{C}\mathcal{D}_t^{\beta} f(t) + \mathcal{R}f(t) + \mathcal{N}f(t)] = \mathcal{H}[h(t)], \\ \mathcal{H}[{}_{0}^{C}\mathcal{D}_t^{\beta} f(t)] + \mathcal{H}[Rf(t)] + \mathcal{H}[\mathcal{N}f(t)] = \mathcal{H}[h(t)], \\ \left(\left(\frac{s}{u}\right)^{\beta} F(s,u) - c \right) + \mathcal{H}[Rf(t)] + \mathcal{H}[\mathcal{N}f(t)] = \mathcal{H}[h(t)], \end{gathered} $$

or

$$ F(s,u) = \left(\frac{u}{s}\right)^{\beta} c + \left(\frac{u}{s}\right)^{\beta} \mathcal{H}[h(t)] - \left(\frac{u}{s}\right)^{\beta} \mathcal{H}[Rf(t)] - \left(\frac{u}{s}\right)^{\beta} \mathcal{H}[\mathcal{N}f(t)]. \tag{15} $$

Where,

$$ c = \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{\beta-k-1} D^k f(t)|_{t=0} \ . \tag{16} $$

Applying IST to equation (15).

$$ f(t) = \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\beta} c\right] + \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\beta} \mathcal{H}[h(t)]\right] - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\beta} \mathcal{H}[Rf(t)]\right] - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\beta} \mathcal{H}[\mathcal{N}f(t)]\right]. \tag{17} $$

Suppose, infinite components of a series $f(t) = \sum_{i=0}^{\infty} f_i(t)$ should represent the solution and decompose the term $\mathcal{N}f(t)$ by Adomian polynomials [22, 23] as:

$$ A_k(t) = \frac{1}{k!} \frac{d^k}{d\lambda^n} N\left(\sum_{i=0}^{k} \lambda^i f_i(t)\right)|_{\lambda=0} \ . \tag{18} $$

Hence, equation (17) in terms of series solution can be written down as:

$$ \sum_{i=0}^{\infty} f_i(t) = \mathcal{H}^{-1}\left[\left[\left(\frac{u}{s}\right)^{\beta} c\right] + \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\beta} \mathcal{H}[h(t)]\right]\right] - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\beta} \mathcal{H}\left[\sum_{i=0}^{\infty} f_i(t)\right]\right] - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\beta} \mathcal{H}\left[\sum_{i=0}^{\infty} \mathcal{A}_i(t)\right]\right]. \tag{19} $$

Recursively, the following relationship was found.

$$
\begin{aligned}
&f_0(t) = \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\beta} c\right] \\
&f_n(t) = \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\beta} \mathcal{H}[h(t)]\right] - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\beta} \mathcal{H}[f_{n-1}(t)]\right] - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\beta} \mathcal{H}[\mathcal{A}_{n-1}(t)]\right], n = 1, 2, \ldots.
\end{aligned} \tag{20}
$$

By using above relation, the solution is obtained as sum of series components, i.e.,

$$
f(t) = f_0(t) + f_1(t) + f_2(t) + \ldots = \lim_{n\to\infty} \sum_{i=0}^{n} f_i(t).
$$

4. Application to FDE numerical problems

In this section, we use the aforementioned technique to solve differential equations, both integer and fractional order.

4.1 *Example 1*

Consider the following fractional initial value problem.

$$
{}_0^C\mathcal{D}_t^{\alpha} y(t) = (1-y)^4, \quad 0 < \alpha \le 1, \quad y(0) = 0. \tag{21}
$$

Now, taking ST of equation (21).

$$
\begin{aligned}
\mathcal{H}[{}_0^C\mathcal{D}_t^{\alpha} y(t)] &= \mathcal{H}[(1-y)^4], \\
\left(\frac{s}{u}\right)^{\alpha} F(s,u) - \left(\frac{s}{u}\right)^{\alpha-1} y(0) &= \mathcal{H}[\mathcal{N}(y(t))], \\
F(s,u) &= \left(\frac{u}{s}\right)^{\alpha} \mathcal{H}[\mathcal{N}(y(t))].
\end{aligned} \tag{22}
$$

Here, $\mathcal{H}[\mathcal{N}(y(t))]$ is the non-linear term of above expression. Now, applying IST on above equation as:

$$
y(t) = \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} \mathcal{H}[\mathcal{N}(y(t))]\right]. \tag{23}
$$

By dissecting the response into series components and nonlinear terms that may be understood using Adomian polynomials, the response is examined.

$$
\sum_{i=0}^{\infty} y_i(t) = \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} \mathcal{H}\left[\sum_{i=0}^{\infty} A_i(t)\right]\right] \tag{24}
$$

$$
\begin{aligned}
y_0 &= 0, \\
y_n &= \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} \mathcal{H}[A_{n-1}(t)]\right], \quad n = 1, 2, \ldots
\end{aligned}
$$

or

$$y_1 = \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} \mathcal{H}[A_0]\right] = \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} \mathcal{H}[1]\right] = \frac{t^{\alpha}}{\Gamma(1+\alpha)},$$

$$y_2 = \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} \mathcal{H}[A_1]\right] = \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} \mathcal{H}[-4y_1]\right] = \frac{-4t^{2\alpha}}{\Gamma(1+2\alpha)},$$

$$y_3 = \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} \mathcal{H}[A_2]\right] = \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} \mathcal{H}[6y_1^2 - 4y_2]\right],$$

$$= \left(\frac{6\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} + \frac{16}{\Gamma(1+3\alpha)}\right) t^{3\alpha}.$$

Hence,

$$\begin{aligned} y(t) &= y_0(t) + y_1(t) + y_2(t) + y_3(t) + \ldots.. \\ &= \frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{4t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(\frac{6\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} + \frac{16}{\Gamma(1+3\alpha)}\right) t^{3\alpha} - \ldots \end{aligned} \tag{25}$$

Put $\alpha = 1$ in above equation,

$$y(t) = t - 2t^2 + \frac{14}{3}t^3 - \frac{35}{3}t^4 + \ldots = \left(1 - \frac{(1+6t+9t^2)^{1/3}}{1+3t}\right). \tag{26}$$

This is the exact solution to the integer-ordered differential equation considered. The series solution can be expressed in the form of a closed form expression or an exact solution, as is seen from the equation above, which is adequate to show the suggested algorithm's validation. The convergence of the suggested algorithms can be verified by plotting a figure, therefore in order to take this into consideration, we have plotted Figure 1, for three series terms. However, assume one is unable to write down such a closed-form expression of the series solution. This image clearly shows how the estimated solution tends to quickly approach the precise answer as we increase the number of iterations or components of a series solution.

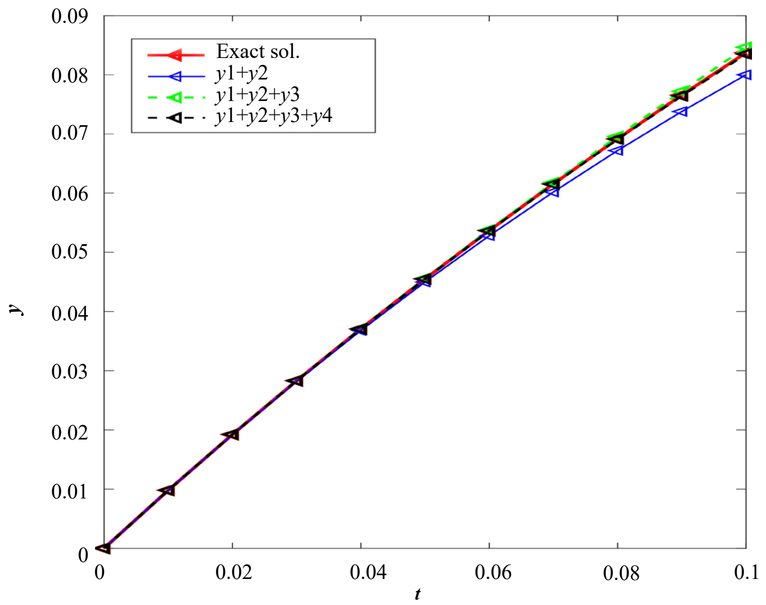

Figure 1. The closeness of a solution series' elements to the precise solution by taking three series terms for Example 1

4.2 *Example 2*

$$\mathcal{D}^3 y(t) + \mathcal{D}^{\frac{5}{2}} y(t) + y^2(t) = t^4, \ \ y(0) = y'(0) = 0 \text{ and } y''(0) = 2. \tag{27}$$

Applying ST on both sides on given differential equation.

$$\mathcal{H}[\mathcal{D}^3 y(t)] + \mathcal{H}[\mathcal{D}^{\frac{5}{2}} y(t)] + \mathcal{H}[y^2(t)] = \mathcal{H}[t^4],$$

$$\mathcal{H}[\mathcal{D}^{\frac{5}{2}} y(t)] = \mathcal{H}[t^4] - \mathcal{H}[\mathcal{D}^3 y(t)] - \mathcal{H}[y^2(t)],$$

$$\left(\frac{s}{u}\right)^{\frac{5}{2}} F(s,u) - \left(\frac{s}{u}\right)^{\frac{3}{2}} y(0) - \left(\frac{s}{u}\right)^{\frac{1}{2}} y'(0) - \left(\frac{s}{u}\right)^{\frac{-1}{2}} y''(0) = \left(\frac{u}{s}\right)^{5} 4! - \mathcal{H}[\mathcal{D}^3 y(t)] - \mathcal{H}[y^2(t)],$$

$$\left(\frac{s}{u}\right)^{\frac{5}{2}} F(s,u) = 2\left(\frac{s}{u}\right)^{\frac{-1}{2}} + \left(\frac{u}{s}\right)^{5} 4! - \mathcal{H}[\mathcal{D}^3 y(t)] - \mathcal{H}[y^2(t)],$$

$$F(s,u) = 2\left(\frac{u}{s}\right)^{3} + \left(\frac{u}{s}\right)^{\frac{15}{2}} 4! - \mathcal{H}[\mathcal{D}^3 y(t)] - \mathcal{H}[y^2(t)].$$

Applying IST to above equation.

$$y(t) = t^2 + 4!\frac{t^{\frac{13}{2}}}{\Gamma\left(\frac{15}{2}\right)} - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\frac{5}{2}} \mathcal{H}[\mathcal{D}^3 y(t)]\right] - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\frac{5}{2}} \mathcal{H}[\mathcal{N}(y(t))]\right]. \tag{28}$$

Analyzing the answer by breaking it down into series components and non-linear terms that can be understood using Adomian polynomials.

$$y(t) = \sum_{i=0}^{\infty} y_i(t) = t^2 + 4!\frac{t^{\frac{13}{2}}}{\Gamma\left(\frac{15}{2}\right)} - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\frac{5}{2}} \mathcal{H}\left[\mathcal{D}^3\left(\sum_{i=0}^{\infty} y_i(t)\right)\right]\right] - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\frac{5}{2}} \mathcal{H}\left[\sum_{i=0}^{\infty} A_i(t)\right]\right].$$

Let $y_0 = t^2$ and the recurrence formula for terms is as follows.

$$y_{n+1} = 4!\frac{t^{\frac{13}{2}}}{\Gamma\left(\frac{15}{2}\right)} - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\frac{5}{2}} \mathcal{H}[\mathcal{D}^3 y_n(t)]\right] - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\frac{5}{2}} \mathcal{H}[A_n(t)]\right], \ n = 0, 1, 2... \tag{29}$$

Now, we can simply extract the terms of the series as:

$$y_1 = 4!\frac{t^{\frac{13}{2}}}{\Gamma\left(\frac{15}{2}\right)} - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\frac{5}{2}} \mathcal{H}[\mathcal{D}^3 y_0(t)]\right] - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\frac{5}{2}} \mathcal{H}[A_0]\right],$$

$$= 4!\frac{t^{\frac{13}{2}}}{\Gamma\left(\frac{15}{2}\right)} - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\frac{5}{2}} \mathcal{H}[\mathcal{D}^3 y(t^2)]\right] - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\frac{5}{2}} \mathcal{H}[t^4]\right],$$

$$= 4!\frac{t^{\frac{13}{2}}}{\Gamma\left(\frac{15}{2}\right)} - 4!\frac{t^{\frac{13}{2}}}{\Gamma\left(\frac{15}{2}\right)} = 0.$$

$$y_2 = y_3 = y_4 = ... = 0. \tag{30}$$

Here, $\Gamma\left(\frac{15}{2}\right)$ is the value of gamma at $\frac{15}{2}$. Hence,

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + ...$$
$$= t^2. \tag{31}$$

Therefore, same as previous we also obtained an exact solution in this problem as well. Now, Figure 2 shows the plotting of this solution in t, $y(t)$-plane.

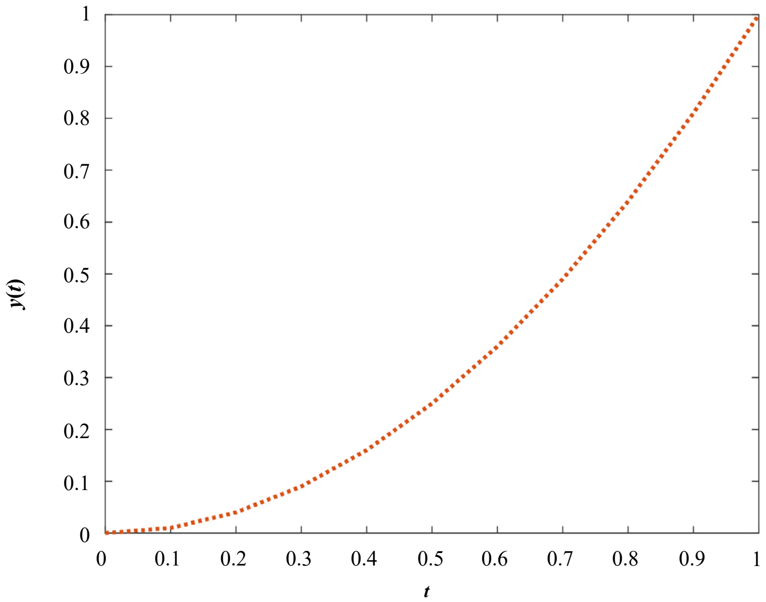

Figure 2. The exact solution for considered (Example 2) fractional order differential equation

4.3 *Example 3: Logistic equation [34]*

The fractional logistic equation has shown a comparable phenomenon, see [26].

$${}_0^C\mathcal{D}_t^{\alpha} y(t) = ry(t)(1 - y(t)), \;\; y(0) = \frac{1}{2}, \;\; t > 0, \;\; r > 0, \;\; 0 < \alpha \le 1. \tag{32}$$

Applying the ST on both sides of the equation (32):

$$\mathcal{H}[{}_0^C\mathcal{D}_t^{\alpha} y(t)] = \mathcal{H}[ry(t)] - \mathcal{H}[ry^2(t)]$$

$$\left(\frac{s}{u}\right)^{\alpha} F(s,u) - \left(\frac{s}{u}\right)^{\alpha-1}, y(0) = r\mathcal{H}[y(t)] - r\mathcal{H}[\mathcal{N}(y(t))],$$

$$F(s,u) = \frac{1}{2}\left(\frac{u}{s}\right) + r\left(\frac{u}{s}\right)^{\alpha} \mathcal{H}[y(t)] - r\left(\frac{u}{s}\right)^{\alpha} \mathcal{H}[\mathcal{N}(y(t))].$$

Using the IST to solve the above equation on both sides as:

$$y(t)=\frac{1}{2}+r\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[y(t)]\right]-r\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[\mathcal{N}(y(t))]\right]. \tag{33}$$

Dividing the solution into series components and nonlinear terms with Adomian polynomials as decomposers:

$$y(t)=\sum_{i=0}^{\infty}y_i(t)=\frac{1}{2}+r\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}\left[\sum_{i=0}^{\infty}y_i(t)\right]\right]-r\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}\left[\sum_{i=0}^{\infty}A_i(t)\right]\right].$$

Let $y_0 \quad \frac{1}{2}$. The following is the recurrence formula for other terms in the series solution.

$$y_{n+1}=r\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[y_{n-1}]\right]-r\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[A_{n-1}(t)]\right],\ n=0,1,2,...$$

Hence, other series terms can be written as:

$$\begin{aligned}
&y_1=r\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[y_0]\right]-r\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[A_0(t)]\right],\\
&A_0=\mathcal{N}(y_0(t))=y_0^2,\\
&y_1=\frac{r}{4}\frac{t^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned} \tag{34}$$

$$\begin{aligned}
&y_2=r\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[y_1]\right]-r\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[A_1(t)]\right],\\
&A_1=2y_0y_1,\\
&y_2=0.
\end{aligned}$$

$$\begin{aligned}
&y_3=r\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[y_2]\right]-r\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[A_2(t)]\right],\\
&A_2=2y_0y_2+y_1^2,\\
&y_3=-\frac{r^3}{16}\frac{\Gamma(2\alpha+1)}{(\Gamma+1)^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\\
&...=.....
\end{aligned} \tag{35}$$

Therefore, the series solution can be written as:

$$\begin{aligned}
y(t)&=y_0(t)+y_1(t)+y_2(t)+y_3(t)+...\\
&=\frac{1}{2}+\frac{r}{4}\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{r^3}{16}\frac{\Gamma(2\alpha+1)}{(\Gamma+1)^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+...
\end{aligned} \tag{36}$$

Put $r=\frac{1}{2}$ and $\alpha=1$ in above equation, then

$$y(t) = \lim_{n\to\infty}\sum_{i=0}^{n} y_i(t) = \frac{e^{\frac{t}{2}}}{1+e^{\frac{t}{2}}}, \tag{37}$$

which represent exact solution of considered example. Similar to Figure 1, Figure 3 illustrates a comparison between an executable solution and a series solution by using three series terms. This image clearly shows how the estimated solution tends to quickly approach the precise answer as we increase the number of iterations or components of a series solution.

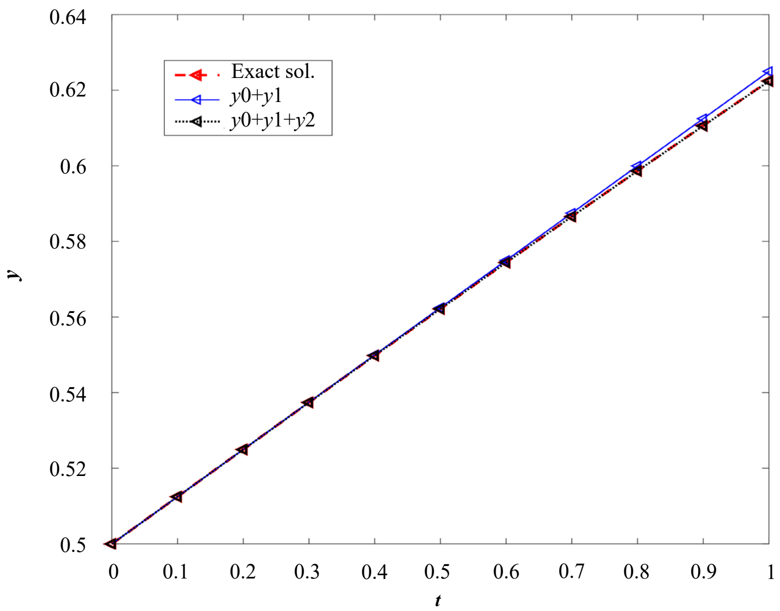

Figure 3. The proximity of solution series components toward the exact solution (Example 3)

4.4 *Example 4: Van der Pole equation [35]*

$$ {}_0^C\mathcal{D}_t^{\alpha} y(t) = -y(t) + \mu(1-y^2)y', \ \ t \geq 0, \ \ 0 < \alpha \leq 2, \ \ y(0) = 0, \ \ y'(0) = 1. \tag{38}$$

Similar to before, we must first take ST on both sides of the equation.

$$\mathcal{H}[{}_0^C\mathcal{D}_t^{\alpha} y(t)] = -\mathcal{H}[y(t)] + \mathcal{H}[\mu\mathcal{D}y] - \mathcal{H}[\mu(y'y^2)],$$

$$\left(\frac{s}{u}\right)^{\alpha} F(s,u) - \left(\frac{s}{u}\right)^{\alpha-1} y(0) - \left(\frac{s}{u}\right)^{\alpha-2} y'(0) = -\mathcal{H}[y(t)] + \mu\mathcal{H}[\mathcal{D}y] - \mu\mathcal{H}[\mathcal{N}(y(t))],$$

$$\left(\frac{s}{u}\right)^{\alpha} F(s,u) - \left(\frac{s}{u}\right)^{\alpha-1} y(0) - \left(\frac{s}{u}\right)^{\alpha-2} y'(0) = -\mathcal{H}[y(t)] + \mu\mathcal{H}[\mathcal{D}y] - \mu\mathcal{H}[\mathcal{N}(y(t))],$$

$$F(s,u) = \left(\frac{u}{s}\right)^{2} - \left(\frac{u}{s}\right)^{\alpha} \mathcal{H}[y(t)] + \left(\frac{u}{s}\right)^{\alpha} \mu\mathcal{H}[\mathcal{D}y] - \mu\left(\frac{u}{s}\right)^{\alpha} \mathcal{H}[\mathcal{N}(y(t))].$$

Applying IST on both sides to above equation.

$$y(t) = t - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} \mathcal{H}[y(t)]\right] + \mathcal{H}^{-1}\left[\mu\left(\frac{u}{s}\right)^{\alpha} \mathcal{H}[\mathcal{D}y]\right] - \mathcal{H}^{-1}\left[\mu\left(\frac{u}{s}\right)^{\alpha} \mathcal{H}[\mathcal{N}(y(t))]\right].$$

The solution is divided into series components and nonlinear terms that are decomposed by Adomian polynomials as:

$$y(t)=\sum_{i=0}^{\infty}y_i(t)=t-\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}\left[\sum_{i=0}^{\infty}y_i(t)\right]\right]+\mathcal{H}^{-1}\left[\mu\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}\left[\mathcal{D}\left(\sum_{i=0}^{\infty}y_i(t)\right)\right]\right]-\mathcal{H}^{-1}\left[\mu\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}\left[\sum_{i=0}^{\infty}A_i(t)\right]\right].$$

Let $y_0=t$. The recurrence formula for terms is as follows.

$$y_{n+1}=-\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[y_n(t)]\right]-\mathcal{H}^{-1}\left[\mu\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[\mathcal{D}(y_n(t))]\right]+\mathcal{H}^{-1}\left[\mu\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[A_n(t)]\right],\ n=0,1,2,...$$

Therefore, using the aforementioned method, it is possible to calculate the other series terms as:

$$\begin{aligned}
y_1&=-\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[y_0(t)]\right]+\mathcal{H}^{-1}\left[\mu\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[\mathcal{D}(y_0(t))]\right]-\mathcal{H}^{-1}\left[\mu\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[A_0(t)]\right]\\
&=-\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[t]\right]+\mathcal{H}^{-1}\left[\mu\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[\mathcal{D}(t)]\right]-\mathcal{H}^{-1}\left[\mu\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[\mathcal{N}(y_0)]\right]\\
&=-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\mu\frac{t^{\alpha}}{\Gamma(\alpha+1)}-2\mu\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}.
\end{aligned}$$

$$\begin{aligned}
y_2&=-\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[y_1(t)]\right]+\mathcal{H}^{-1}\left[\mu\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[\mathcal{D}(y_1(t))]\right]-\mathcal{H}^{-1}\left[\mu\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[A_1(t)]\right]\\
&=-\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[y_1(t)]\right]+\mathcal{H}^{-1}\left[\mu\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[\mathcal{D}(y_1(t))]\right]-\mathcal{H}^{-1}\left[\mu\left(\frac{u}{s}\right)^{\alpha}\mathcal{H}[2y_0y_1+t^2y_1']\right]\\
&=\frac{\mu^2\alpha\Gamma(\alpha)}{\Gamma(\alpha+1)\Gamma(2\alpha)}t^{2\alpha-1}-\mu\left(\frac{1}{\Gamma(2\alpha+1)}+\frac{(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha+2)\Gamma(2\alpha)}\right)t^{2\alpha}\\
&-\left(\frac{1}{\Gamma(2\alpha+2)}+\frac{2\mu^2(\alpha+2)\Gamma(\alpha+2)}{\Gamma(\alpha+3)\Gamma(2\alpha+2)}+\frac{2\mu^2\Gamma(\alpha+2)}{\Gamma(\alpha+1)\Gamma(2\alpha+2)}+\frac{\mu^2\alpha\Gamma(\alpha+2)}{\Gamma(\alpha+1)\Gamma(2\alpha+2)}\right)t^{2\alpha+1}\\
&+\left(\frac{2\mu}{\Gamma(2\alpha+3)}+\frac{2\mu\Gamma(\alpha+3)}{\Gamma(\alpha+2)\Gamma(2\alpha+3)}+\frac{\mu(\alpha+1)\Gamma(\alpha+3)}{\Gamma(\alpha+2)\Gamma(2\alpha+3)}\right)t^{2\alpha+2}\\
&+\left(\frac{4\mu\Gamma(\alpha+4)}{\Gamma(\alpha+3)\Gamma(2\alpha+4)}+\frac{2\mu^2(\alpha+2)\Gamma(\alpha+4)}{\Gamma(\alpha+3)\Gamma(2\alpha+4)}\right)t^{2\alpha+3}\\
...&=.....
\end{aligned}$$

Thus, the solution series has obtained as follows:

$$
\begin{aligned}
y(t) &= y_0(t) + y_1(t) + y_2(t) + \ldots \\
&= t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \mu \frac{t^{\alpha}}{\Gamma(\alpha+1)} - 2\mu \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{\mu^2 \alpha \Gamma(\alpha)}{\Gamma(\alpha+1)\Gamma(2\alpha)} t^{2\alpha-1} \\
&\quad - \mu \left(\frac{1}{\Gamma(2\alpha+1)} + \frac{(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha+2)\Gamma(2\alpha)} \right) t^{2\alpha} \\
&\quad + \left(\frac{1}{\Gamma(2\alpha+2)} - \frac{2\mu^2(\alpha+2)\Gamma(\alpha+2)}{\Gamma(\alpha+3)\Gamma(2\alpha+2)} - \frac{2\mu^2\Gamma(\alpha+2)}{\Gamma(\alpha+1)\Gamma(2\alpha+2)} - \frac{\mu^2\alpha\Gamma(\alpha+2)}{\Gamma(\alpha+1)\Gamma(2\alpha+2)} \right) t^{2\alpha+1} \\
&\quad + \left(\frac{2\mu}{\Gamma(2\alpha+3)} + \frac{2\mu\Gamma(\alpha+3)}{\Gamma(\alpha+2)\Gamma(2\alpha+3)} + \frac{\mu(\alpha+1)\Gamma(\alpha+3)}{\Gamma(\alpha+2)\Gamma(2\alpha+3)} \right) t^{2\alpha+2} \\
&\quad + \left(\frac{4\mu\Gamma(\alpha+4)}{\Gamma(\alpha+3)\Gamma(2\alpha+4)} + \frac{2\mu^2(\alpha+2)\Gamma(\alpha+4)}{\Gamma(\alpha+3)\Gamma(2\alpha+4)} \right) t^{2\alpha+3} + \ldots
\end{aligned}
$$

For $\alpha = 2$ and $\mu = 0$ in above series solution then series solution becomes:

$$
y(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots = \sin t \tag{39}
$$

When nine iterations are considered, the approximate solutions for several values of α, i.e., 0.5, 1, 1.5, and 2, have also been visually shown (see Figure 4). This graphic demonstrates how the approximate solution $y(t)$ grows as the value of α increases.

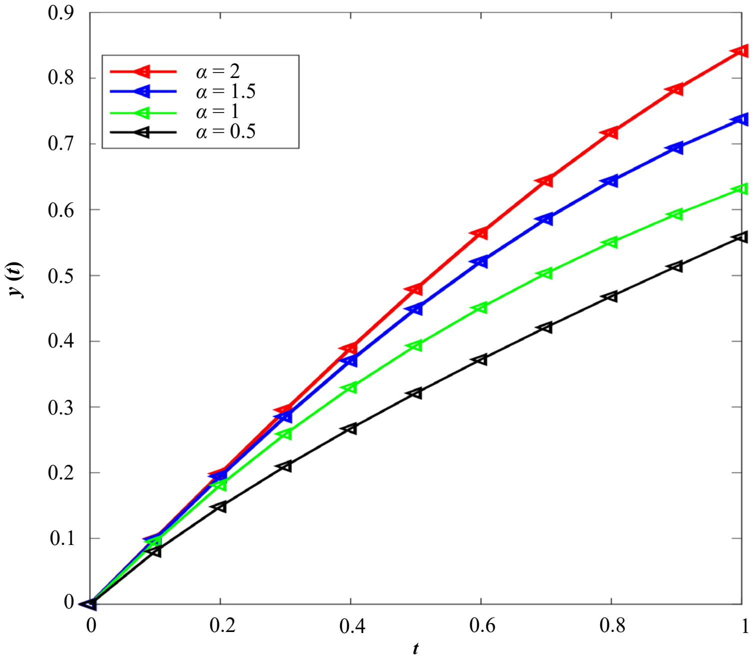

Figure 4. When $\mu = 0$, this figure illustrates the approximate solution obtained by considering nine iterations for various values of α

4.5 *Example 5*

$$
{}_0^C\mathcal{D}_t^{\alpha} y(t) - \lambda y(t) = 0,\ t \geq 0,\ \ 0 < \alpha \leq 1,\ y(0) = 0. \tag{40}
$$

Applying ST to given equation.

$$\mathcal{H}[{}_0^C\mathcal{D}_t^\alpha y(t)] = \lambda\mathcal{H}[y(t)],$$
$$\left(\frac{s}{u}\right)^\alpha F(s,u) - \left(\frac{s}{u}\right)^{\alpha-1} y(0) = \lambda\mathcal{H}[y(t)],$$
$$F(s,u) = c\left(\frac{u}{s}\right) + \lambda\left(\frac{u}{s}\right)^\alpha \mathcal{H}[y(t)].$$

Now applying IST as:

$$y(t) = c + \lambda\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^\alpha \mathcal{H}[y(t)]\right]. \tag{41}$$

Decomposing the solution into series components and nonlinear terms with Adomian polynomials as decomposers.

$$\sum_{i=0}^{\infty} y_i(t) = c + \lambda\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^\alpha \mathcal{H}\left[\sum_{i=0}^{\infty} A_i(t)\right]\right].$$

Let $y_0 = c$. Additionally, the following is the formula for series terms' recurrence.

$$y_{n+1} = \lambda\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^\alpha \mathcal{H}[A_n(t)]\right],\ n = 0,1,2,... \tag{42}$$

Hence, the series terms can be written as:

$$y_1 = \lambda\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^\alpha \mathcal{H}[A_0(t)]\right] = \lambda\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^\alpha \mathcal{H}[c]\right] = c\lambda\frac{t^\alpha}{\Gamma(\alpha+1)}.$$
$$y_3 = c\lambda^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$
$$.... = ...$$

Hence,

$$y(t) = y_0(t) + y_1(t) + y_2(t) + ... = c + c\lambda\frac{t^\alpha}{\Gamma(\alpha+1)} + c\lambda^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + c\lambda^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ... \tag{43}$$

The solution series has obtained in terms of the Mittag-Leffler function by taking an infinite sum.

$$y(t) = c\mathcal{E}_{\alpha,\alpha}(\lambda t)^\alpha \tag{44}$$

5. Application to ordinary differential equation (ODE)

Let's restate the preceding approach to solve ODEs as follows.

$$\mathcal{D}^n f(t) + \mathcal{R}f(t) + \mathcal{N}f(t) = h(t),\ \ n \geq 1. \tag{45}$$

$$f^k(t)|_{t=0} = c_k,\ \ k = 0, 1, 2...n-1. \tag{46}$$

In order to demonstrate the better methodology, all example problems were solved using ST utilising the same steps as before.

$$f_0(t) = \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^n c\right]. \tag{47}$$

$$f_n(t) = \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^n \mathcal{H}[h(t)]\right] - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^n \mathcal{H}[f_{n-1}(t)]\right] - \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^n \mathcal{H}[\mathcal{A}_{n-1}(t)]\right],\ n \geq 1. \tag{48}$$

The solution is obtained by adding the elements of the series or by applying the relation shown above.

$$f(t) = f_0(t) + f_1(t) + f_2(t) + \ldots = \lim_{n\to\infty} \sum_{i=0}^{n} f_i(t). \tag{49}$$

5.1 *Example 6*

Consider ODE, whose exact solution is $\frac{1}{1-t}$.

$$\frac{dy}{dt} = y^2,\ \ y(0) = 1. \tag{50}$$

Applying ST to above ODE.

$$\mathcal{H}\left[\frac{dy}{dt}\right] = \mathcal{H}[y^2],$$

$$\left(\frac{s}{u}\right)F(s,u) - 1 = \mathcal{H}[\mathcal{N}(y(t))],$$

$$F(s,u) = \left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)\mathcal{H}[\mathcal{N}(y(t))].$$

Applying IST.

$$y(t) = 1 + \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)\mathcal{H}[\mathcal{N}(y(t))]\right]. \tag{51}$$

Dividing the solution into series components and nonlinear terms with Adomian polynomials as decomposers as:

$$\sum_{i=0}^{\infty} y_i(t) = 1 + \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)\mathcal{H}[A_i]\right]. \tag{52}$$

Let $y_0 = 1$. The recurrence formula for terms is as follows.

$$y_{n+1} = \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)\mathcal{H}[A_n]\right],\ \ n = 0, 1, 2, \ldots \tag{53}$$

By using above recurrence relation and Adomian polynomials, the series terms can be obtained as follows.

$$
\begin{aligned}
&y_1 = \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)\mathcal{H}[A_0]\right],\\
&A_0 = \mathcal{N}(y_0(t)) = y_0^2,\\
&\quad \text{hence,}\\
&y_1 = t.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&y_2 = \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)\mathcal{H}[A_1]\right],\\
&A_1 = 2y_0y_1,\\
&\quad \text{hence,}\\
&y_2 = t^2.
\end{aligned}
$$

$$
\begin{aligned}
&y_3 = \mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)\mathcal{H}[A_2]\right],\\
&A_2 = 2y_0y_2 + y_1^2,\\
&\quad \text{hence,}\\
&y_3 = t^3.
\end{aligned}
$$

Thus,

$$
y(t) = y_0(t) + y_1(t) + y_2(t) + \ldots = 1 + t + t^2 + \ldots + t^n = \frac{1}{1-t}. \tag{54}
$$

When the approximate solution is thought of as the sum of four terms, Figure 5 illustrates the approximate solution's progress toward the precise solution graphically.

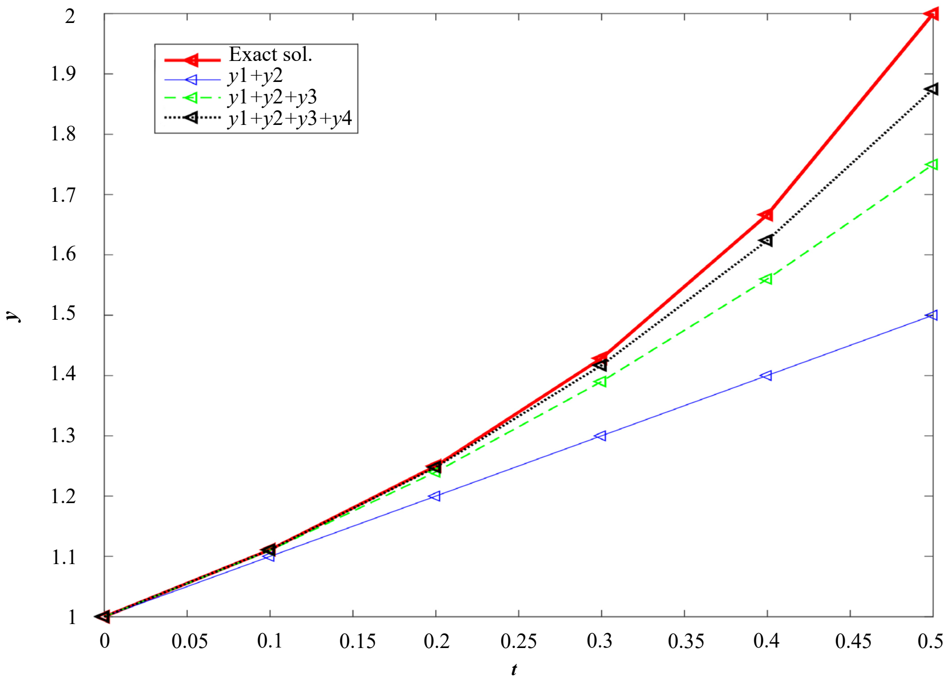

Figure 5. When four iterations are taken into consideration, the closeness of solution series elements to the precise solution (Example 6)

5.2 *Example 7: Non-linear Bratu type equation*

$$y''(t) - 2e^{y} = 0,\ y(0) = y'(0) = 0. \tag{55}$$

Applying ST to given equation.

$$\mathcal{H}[y''(t)] = 2\mathcal{H}[e^{y}],$$

$$\left(\frac{s}{u}\right)^2 F(s,u) - \left(\frac{s}{u}\right) y(0) - y'(0) = 2\mathcal{H}[\mathcal{N}(y(t))],$$

$$F(s,u) = 2\left(\frac{u}{s}\right)^2 \mathcal{H}[\mathcal{N}(y(t))]. \tag{56}$$

Applying IST on both sides to above equation.

$$y(t) = 2\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^2 \mathcal{H}[\mathcal{N}(y(t))]\right]. \tag{57}$$

Decomposing the solution into series components and non-linear term decomposed by Adomian polynomials as:

$$\sum_{i=0}^{\infty} y_i(t) = 2\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^2 \mathcal{H}\left[\sum_{i=0}^{\infty} A_i\right]\right]. \tag{58}$$

Let $y_0 = 0$. The recurrence formula for terms is as follows.

$$y_{n+1} = 2\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^2 \mathcal{H}[A_n]\right], \quad n = 0,1,2... \tag{59}$$

Now, series terms can be obtained as:

$$y_1 = 2\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^2 \mathcal{H}[A_0]\right],$$

$$A_0 = \mathcal{N}(y_0(t)) = e^{y_0},$$

hence,

$$y_1 = t^2.$$

Similarly,

$$y_2 = 2\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^2 \mathcal{H}[A_1]\right],$$

$$A_1 = y_1 e^{y_0},$$

hence,

$$y_2 = \frac{t^4}{6}.$$

$$y_3 = 2\mathcal{H}^{-1}\left[\left(\frac{u}{s}\right)^2 \mathcal{H}[A_2]\right],$$

$$A_2 = \frac{1}{2}e^{y_0}(y_1^2 + 2y_2),$$

hence,

$$y_3 = \frac{2t^6}{45}.$$

$$.... = ...$$

Consequently,

$$\begin{aligned} y(t) &= y_0 + y_1 + y_2 + y_3 + ... \\ &= t^2 + \frac{t^4}{6} + \frac{2t^6}{45} + ... \\ &= -2\left(-\frac{t^2}{2} - \frac{t^4}{12} - \frac{t^6}{45} - ...\right) \\ &= \lim_{n\to\infty}\sum_{i=0}^{n} y_i(t) = -2\log(\cos t). \end{aligned}$$

Figure 6 shows the comparison of series solution with exact solution for three series terms and considered algorithms.

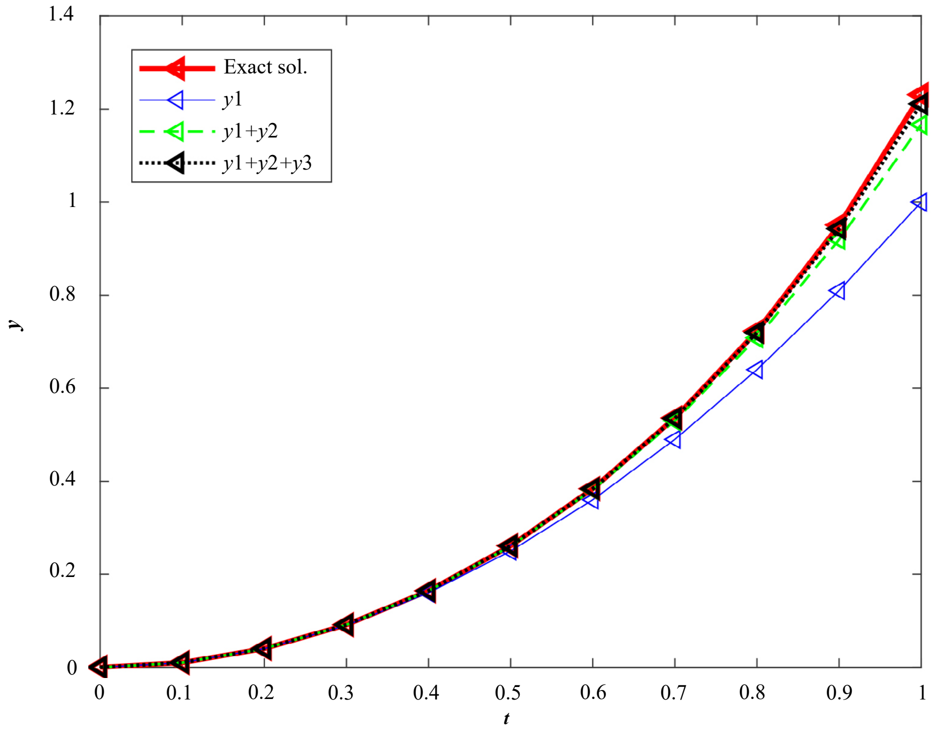

Figure 6. The proximity of solution series components toward the exact solution (Example 7)

6. Conclusion and future work

STADM is used in this article to provide a detailed analysis of the solution of non-linear ODEs of fractional and non-fractional order in series forms. These techniques have previously been used to resolve a very small number of ordinary and FDEs as well as some simple differential equations of integer order. The logistic equation, the Van der Pole equation, and other non-fractional order differential equations like the non-linear Bratu type equation are among the well-known FDEs that are taken into consideration by the Caputo's operator. The convergence of the approximative solution towards the closed form solution is shown by establishing the graph for various series terms of the series solution. In terms of future development, this technique may be utilized to resolve both integer and non-integer differential equations with ease. The suggested approach may also be used to resolve fractional models in a variety of domains, including physics, engineering, etc.

Conflict of interest

There was no relevant conflict of interest regarding this paper.

References

[1] Peng G. Synchronization of fractional order chaotic systems. *Physics Letters A*. 2007; 363(5-6): 426-432. Available from: https://doi.org/10.1016/j.physleta.2006.11.053.

[2] Baskonus HM, Mekkaoui T, Hammouch Z, Bulut H. Active control of a chaotic fractional order economic system. *Entropy*. 2015; 17(8): 5771-5783. Available from: https://doi.org/10.3390/e17085771.

[3] Singh J, Kumar D, Hammouch Z, Atangana A. A fractional epidemiological model for computer viruses pertaining to a new fractional derivative. *Applied Mathematics and Computation*. 2018; 316: 504-515. Available from: https://doi.org/10.1016/j.amc.2017.08.048.

[4] Almeida R, Bastos NR, Monteiro MT. Modeling some real phenomena by fractional differential equations. *Mathematical Methods in the Applied Sciences*. 2015; 39(16): 4846-4855. Available from: https://doi.org/10.1002/mma.3818.

[5] Oldham KB, Spanier J. (eds.) *The fractional calculus*. New York: Academic Press; 1974.

[6] Nagy AM, Sweilam NH. An efficient method for solving fractional Hodgkin-Huxley model. *Physics Letters A*. 2014; 378(30-31): 1980-1984. Available from: https://doi.org/10.1016/j.physleta.2014.06.012.

[7] Sweilam NH, Nagy AM, El-Sayed AA. Second kind shifted Chebyshev polynomials for solving space fractional order diffusion equation. *Chaos, Solitons & Fractals*. 2015; 73: 141-147. Available from: https://doi.org/10.1016/j.chaos.2015.01.010.

[8] Petráš I. *Fractional-order nonlinear systems: Modeling, analysis and simulation*. Berlin: Springer; 2011. Available from: https://doi.org/10.1007/978-3-642-18101-6.

[9] Tarasov VE. Generalized memory: Fractional calculus approach. *Fractal and Fractional*. 2018; 2(4): 23. Available from: https://doi.org/10.3390/fractalfract2040023.

[10] Blair GS, Veinoglou BC, Caffyn JE. Limitations of the Newtonian time scale in relation to non-equilibrium rheological states and a theory of quasi-properties. *Proceedings of the Royal Society of London. Series A*. 1947; 189(1016): 69-87. Available from: https://doi.org/10.1098/rspa.1947.0029.

[11] Graham A, Blair GWS, Withers RFJ. A methodological problem in rheology. *The British Journal for the Philosophy of Science*. 1961; 11(44): 265-288. Available from: http://www.jstor.org/stable/685129.

[12] Belavin VA, Nigmatullin RS, Miroshnikov AI, Lutskaya NK. Fractional differentiation of oscillographic polarograms by means of an electrochemical two-terminal network. *Trudy Kazan Aviatsion Institutel*. 1964; 5: 144-145.

[13] Oldham KB. Signal-independent electroanalytical method. *Analytical Chemistry*. 1972; 44(1): 196-198. Available from: https://doi.org/10.1021/ac60309a028.

[14] Grenness M, Oldham KB. Semiintegral electroanalysis. Theory and verification. *Analytical Chemistry*. 1972; 44(7): 1121-1129. Available from: https://doi.org/10.1021/ac60315a037.

[15] Yusuf A, Qureshi S, Inc M, Aliyu AI, Baleanu D, Shaikh AA. Two-strain epidemic model involving fractional

derivative with Mittag-Leffler kernel. *Chaos: An Interdisciplinary Journal of Nonlinear Science*. 2018; 28(12): 123121. Available from: https://doi.org/10.1063/1.5074084.

[16] Khan I. New idea of Atangana and Baleanu fractional derivatives to human blood flow in nanofluids. *Chaos: An Interdisciplinary Journal of Nonlinear Science*. 2019; 29(1): 013121. Available from: https://doi.org/10.1063/1.5078738.

[17] Somorjai RL, Bishop DM. Integral-transformation trial functions of the fractional-integral class. *Physical Review A*. 1970; 1(4): 1013-1018. Available from: https://doi.org/10.1103/PhysRevA.1.1013.

[18] Kulish VV, Lage JL. Application of fractional calculus to fluid mechanics. *Journal of Fluids Engineering*. 2002; 124(3): 803-806. Available from: https://doi.org/10.1115/1.1478062.

[19] Khan H, Khan A, Chen W, Shah K. Stability analysis and a numerical scheme for fractional Klein-Gordon equations. *Mathematical Methods in the Applied Sciences*. 2018; 42(2): 723-732. Available from: https://doi.org/10.1002/mma.5375.

[20] Khan A, Khan ZA, Abdeljawad T, Khan H. Analytical analysis of fractional-order sequential hybrid system with numerical application. *Advances in Continuous and Discrete Models*. 2022; 2022: 12. Available from: https://doi.org/10.1186/s13662-022-03685-w.

[21] Khan FM, Ali A, Abdullah, Shah K, Khan A, Mahariq I. Analytical approximation of Brusselator model via LADM. *Mathematical Problems in Engineering*. 2022; 2022: 8778805. Available from: https://doi.org/10.1155/2022/8778805.

[22] Wu L, Xie LD, Zhang JF. Adomian decomposition method for nonlinear differential-difference equations. *Communications in Nonlinear Science and Numerical Simulation*. 2009; 14(1): 12-18. Available from: https://doi.org/10.1016/j.cnsns.2007.01.007.

[23] Duan JS, Rach R, Baleanu D, Wazwaz AM. A review of the Adomian decomposition method and its applications to fractional differential equations. *Communications in Fractional Calculus*. 2012; 3(2): 73-99. Available from: https://www.academia.edu/26515316/A_review_of_the_Adomian_decomposition_method_and_its_applications_to_fractional_differential_equations.

[24] Elzaki TM, Chamekh M. Solving nonlinear fractional differential equations using a new decomposition method. *Universal Journal of Applied Mathematics and Computation*. 2018; 6: 27-35. Available from: https://www.papersciences.com/Chamekh-Univ-J-Appl-Math-Comp-Vol6-2018-3.pdf.

[25] Singh J, Kumar D, Kılıçman A. Homotopy perturbation method for fractional gas dynamics equation using Sumudu transform. *Abstract and Applied Analysis*. 2013; 2013: 934060. Available from: https://doi.org/10.1155/2013/934060.

[26] Goswami P, Alqahtani RT. Solutions of fractional differential equations by Sumudu transform and variational iteration method. *Journal of Nonlinear Sciences and Applications*. 2016; 9(4): 1944-1951. Available from: https://doi.org/10.22436/jnsa.009.04.48.

[27] Wang K, Liu S. A new Sumudu transform iterative method for time-fractional Cauchy reaction–diffusion equation. *SpringerPlus*. 2016; 5: 865. Available from: https://doi.org/10.1186/s40064-016-2426-8.

[28] Maitama S, Zhao W. New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations. *International Journal of Analysis and Applications*. 2019; 17(2): 167-190. Available from: https://doi.org/10.28924/2291-8639-17-2019-167.

[29] Watugala GK. Sumudu transform: A new integral transform to solve differential equations and control engineering problems. *International Journal of Mathematical Education in Science and Technology*. 1993; 24(1): 35-43. Available from: https://doi.org/10.1080/0020739930240105.

[30] Belgacem R, Baleanu D, Bokhari A. Shehu transform and applications to Caputo-fractional differential equations. *International Journal of Analysis and Applications*. 2019; 17(6): 917-927. Available from: https://doi.org/10.28924/2291-8639-17-2019-917.

[31] Akinyemi L, Iyiola OS. Exact and approximate solutions of time-fractional models arising from physics via Shehu transform. *Mathematical Methods in the Applied Sciences*. 2020; 43(12): 7442-7464. Available from: https://doi.org/10.1002/mma.6484.

[32] Poltem D, Srimongkol S. A note on computational method for Shehu transform by Adomian decomposition method. *Advances in Mathematics: Scientific Journal*. 2021; 10(2): 713-721. Available from: https://doi.org/10.37418/amsj.10.2.3.

[33] Yisa BM, Baruwa M. Shehu transform Adomain decomposition method for the solution of linear and nonlinear integral and integro-differential equations. *Journal of the Nigerian Mathematical Society*. 2022; 41(2): 105-128.

Available from: https://jnms.ictp.it/jnms/index.php/jnms/article/view/861.
[34] El-Sayed AM, El-Mesiry AE, El-Saka HA. On the fractional-order logistic equation. *Applied Mathematics Letters*. 2007; 20(7): 817-823. Available from: https://doi.org/10.1016/j.aml.2006.08.013.
[35] Mishra V, Das S, Jafari H, Ong SH. Study of fractional order Van der Pol equation. *Journal of King Saud University-Science*. 2016; 28(1): 55-60. Available from: https://doi.org/10.1016/j.jksus.2015.04.005.
[36] Ahmadabadi MN, Ghaini FMM. An Adomian decomposition method for solving Liénard equations in general form. *The ANZIAM Journal*. 2009; 51(2): 302-308. Available from: https://doi.org/10.1017/S1446181109000431.