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Research Article

More on Externally q-Hyperconvex Subsets of T_0-Quasi-Metric Spaces

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Abstract: We continue earlier research on T_0-quasi-metric spaces which are externally q-hyperconvex. We focus on external q-hyperconvex subsets of T_0-quasi-metric spaces in particular. We demonstrate that a countable family of pairwise intersecting externally q-hyperconvex subsets has a non-empty intersection that is external q-hyperconvex under specific requirements on the underlying space (see Proposition 22). Last but not least, we demonstrate that if A is a subset of a supseparable and externally q-hyperconvex space Y, where $Y \subseteq X$, then A is also externally q-hyperconvex in X (Proposition 25).

Keywords: quasi-metric space, q-hyperconvexity, external q-hyperconvexity, q-admissible subset.

MSC: 54D35, 54E15, 54E35, 54E50, 54E55

1. Introduction

A quasi-pseudometric space (X, d) is called q-hyperconvex (or Isbell-convex) if every collection $[(C_d(x_i, r_i))_{i\in I}; (C_{d^{-1}}(x_i, s_i))_{i\in I}]$ of double balls, with $x_i \in X$ and $r_i, s_i \geq 0$ whenever $i \in I$ and $d(x_i, x_j) \leq r_i + s_j$ implies that

$$\emptyset \neq \bigcap_{i\in I} C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i).$$

There have been previous studies on q-hyperconvexity in T_0-quasi-metric spaces (the reader is advised to consult [1–4]). Later, Agyingi et al. [1] studied the same concept in a slightly different direction. It has been shown that q-hyperconvex T_0-quasi-metric spaces are the same as the di-injective spaces [4, 5].

A subset E of a quasi-pseudometric space (X, d) is said to be external q-hyperconvex (relative to X) if given any family $(x_i)_{i\in I}$ of points in X and families of non-negative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ the following condition holds:

$$\text{If } d(x_i, x_j) \leq r_i + s_j \text{ whenever } i, \ j \in I, \ \ \text{dist}(x_i, E) \leq r_i \text{ and } \text{dist}(E, x_i) \leq s_i$$

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whenever $i \in I$, then

$$\bigcap_{i \in I} C_d(x_i,\ r_i) \cap C_{d^{-1}}(x_i,\ s_i) \cap E \neq \emptyset.$$

Recently, Künzi et al. in [6] studied a concept of external q-hyperconvexity, which is appropriate for the category of quasi-pseudometric spaces and non-expansive mappings. The intersection of a descending family of nonempty externally q-hyperconvex subspaces of a bounded q-hyperconvex T_0-quasi-metric space is one of the things they demonstrated, among other things.

In a recent work [7], the authors studied weak external q-hyperconvexity in quasi-pseudometric spaces by generalizing results from the metric setting. They extended results from [8] that state that a subset D of a hyperconvex metric space $(X,\ d)$ is weakly hyperconvex if and only if D is a proximinal retract of $D \cup \{z\}$ for any $z \in X \setminus D$.

In this study, we will demonstrate that double balls in q-hyperconvex T_0-quasi-metric spaces have several fundamental features with bounded externally q-hyperconvex subsets. We shall specifically demonstrate that any family $\{A_i\}_{i \in I}$ of pairwise intersecting externally q-hyperconvex subsets such that at least one of them is bounded has a non-empty intersection in a q-hyperconvex T_0-quasi-metric space $(X,\ d)$ (see Corollary 24). Finally, we will demonstrate that in the case where Y is an externally q-hyperconvex subset of $(X,\ d)$ and A is externally q-hyperconvex (relative to Y), we have that A is externally q-hyperconvex (relative to X).

The results of the current investigation will be very crucial in a future investigation by the authors on the characterization of all q-hyperconvex subsets of the space of all bounded real-valued functions defined on a set X, equipped with the usual quasi-pseudometric. Moreover, we found out that many classical results about weak external hyperconvexity do not make use of the symmetry condition and, with a slight modification, still hold for quasi-pseudometric spaces.

2. Preliminaries

Most of the material in this section has been taken from [5].

Definition 1 (Compare [4, Page 3]) Suppose $X \neq \emptyset$ and let $d : X \times X \to [0,\ +\infty)$. Then d is a quasi-pseudo-metric on X if

(a) $d(x,\ x) = 0$ for every $x \in X$, and

(b) $d(x,\ z) \leq d(x,\ y) + d(y,\ z)$ whenever $x,\ y,\ z \in X$.

The pair $(X,\ d)$ will be said to be a quasi-pseudo-metric space.

If d meets the additional requirement that for each $x,\ y \in X$ $d(x,\ y) = 0 = d(y,\ x)$ implies that $x = y$, we will say that d is a T_0-quasi-metric on X. A T_0-quasi-metric space is a pair $(X,\ d)$, where X is a set and d is a T_0-quasi-metric on X.

Example 2 For any $a,\ b \in \mathbb{R}$, by $a \dot{-} b$ we shall mean $\max\{a-b,\ 0\}$, which, in lattice-theoretic terms, can also be denoted by $(a-b) \vee 0$. If we define $u(x,\ y) = x \dot{-} y$ whenever $x,\ y \in [0,\ +\infty)$, then u turns out to be the standard T_0-quasi-metric on $[0,\ +\infty)$.

Example 3 (Compare [1, Remark 2]) (The general quasi-metric "segment I_{ab}") Let $X = [0,\ 1]$. Choose $a,\ b \in [0,\ +\infty)$ such that $a + b \neq 0$. Set $d_{ab}(x,\ y) = (x-y)a$ if $x > y$ and $d_{ab}(x,\ y) = (y-x)b$ if $y \geq x$. Then $([0,\ 1],\ d_{ab})$ is a T_0-quasi-metric space as it is readily checked, by considering the various cases for the underlying asymmetric norm n_{ab} on $\mathbb{R}$ defined by $n_{ab}(x) = xa$ if $x > 0$ and $n_{ab}(x) = -xb$ if $x \leq 0$.

Remark 4 If d is a quasi-pseudometric on a set X, then $d^{-1}(x,\ y) = d(y,\ x)$ whenever $x,\ y \in X$ is the conjugate quasi-pseudometric on X. If the quasi-pseudometric d satisfies $d = d^{-1}$, then it is known as a pseudo-metric. For example $d^s = d \vee d^{-1}$ is a pseudo-metric. For a T_0-quasi-metric d, we get that d^s is a metric.

For a quasi-pseudometric space $(X,\ d)$, we denote by $C_d(x,\ \varepsilon) = \{y \in X : d(x,\ y) \leq \varepsilon\}$ the $\tau(d^{-1})$-closed ball (compare [9, Proposition 1.5(1)]) with radius ε and center at x. We will write $B_d(x,\ \varepsilon) = \{y \in X : d(x,\ y) < \varepsilon\}$ to denote the"open ball" with radius ε and center x.

Definition 5 ([4, page 4]) If the map $f : (X,\ p) \to (Y,\ q)$ between two quasi-pseudometric spaces is called nonexpansive if the condition $q(f(x),\ f(y)) \le p(x,\ y)$ holds every time we have $x,\ y \in X$.

If instead the map f satisfies the condition that $q(f(x),\ f(y)) = p(x,\ y)$ whenever $x,\ y \in X$, then we call it an isometry. Two quasi-pseudometric spaces $(X,\ p)$ and $(Y,\ q)$ will be said to be isometric provided there exists a bijective isometry between them. It is easy to see that if X is a T_0-quasi-metric space, then the map f is one-one.

For every $x \in X$ and non-negative reals $r,\ s$, the pair $(C_d(x,\ r);\ C_{d^{-1}}(x,\ s))$ is said to be a double ball at x. The family $[(C_d(x_i,\ r_i))_{i\in I}; (C_{d^{-1}}(x_i,\ s_i))_{i\in I}]$ will be called a family of double balls, with $x_i \in X$ and $r_i,\ s_i \ge 0$ whenever $i \in I$.

Let $\mathscr{P}_0(X)$ denote the set of all non-empty subsets of a quasi-pseudometric space $(X,\ d)$. For $A \in \mathscr{P}_0(X)$, we define

$$\operatorname{dist}(x,\ A) = \inf\{d(x,\ a) : a \in A\}$$

and

$$\operatorname{dist}(A,\ x) = \inf\{d(a,\ x) : a \in A\}$$

whenever $x \in X$.

Definition 6 (see [4, Definition 2]) We call a(n extended) quasi-pseudometric space $(X,\ d)$ q-hyperconvex if for every family $(x_i)_{i\in I}$ of points in X and families $(\alpha_i)_{i\in I}$ and $(\beta_i)_{i\in I}$ of non-negative real numbers the following conditions hold:

$$\text{If } d(x_i,\ x_j) \le \alpha_i + \beta_j \text{ whenever } i,\ j \in I, \text{ then } \bigcap_{i\in I}(C_d(x_i,\ \alpha_i) \cap C_{d^{-1}}(x_i,\ \beta_i)) \neq \emptyset.$$

Given a quasi-pseudometric space $(X,\ d)$ with $x \in X$ and non-negative real numbers α and β, we shall use the following notation

$$C_x(\alpha,\ \beta) := C_d(x,\ \alpha) \cap C_{d^{-1}}(x,\ \beta).$$

Let $(X,\ d)$ be a quasi-pseudometric space and $D \subseteq X$. We say that D is bounded if there exists $M > 0$ such that $d(x,\ y) < M$ for all $x,\ y \in D$. Notice that a subset D of a quasi-pseudometric space $(X,\ d)$ is bounded if and only if there exists $x \in X$ and $\alpha,\ \beta \in [0,\ \infty)$ such that $D \subseteq C_x(\alpha,\ \beta)$.

Definition 7 A non-empty bounded subset D of a quasi-pseudometric space $(X,\ d)$ that can be written as the intersection of a non-empty family of sets of the form $C_x(\alpha,\ \beta)$ where $\alpha,\ \beta \in [0,\ +\infty)$ and $x \in X$ will be called q-admissible. We will denote by $\mathscr{A}_q(X)$ the set of q-admissible subsets of X.

Definition 8 A subset D of a quasi-pseudometric space $(X,\ \rho)$ is said to be q-proximinal (with respect to X) if

$$D \cap \bigcap [C_\rho(x,\ \operatorname{dist}(x,\ D)) C_{\rho^{-1}}(x,\ \operatorname{dist}(D,\ x))] \neq \emptyset$$

for every $x \in X$.

Proposition 9 Every q-admissible subset and externally q-hyperconvex subset of a q-hyperconvex T_0-quasi-metric space is q-proximinal.

Proof. We prove the case of a q-admissible subset. The case of an externally q-hyperconvex subset is similar and will be left out.

Let $(X,\ d)$ be a q-hyperconvex T_0-quasi-metric space and

$$D = \bigcap_{x\in X} C_x(\alpha,\ \beta)$$

where $x \in X$ and $\alpha,\ \beta \geq 0$, be a q-admissible subset of X. Since $\text{dist}(x,\ D) = \inf_{a\in D} d(x,\ a)$, we have that for every $\varepsilon > 0$, there exists $a_\varepsilon \in D$ such that $d(x,\ a_\varepsilon) \leq \text{dist}(x,\ D)+\varepsilon$. Similarly, $d(a_\varepsilon,\ x) \leq \text{dist}(D,\ x)+\varepsilon$. This implies that

$$\bigcap_{x\in X} C_x(\alpha,\ \beta)\bigcap[C_d(x,\ \text{dist}(x,\ D)+\varepsilon)\cap C_{d^{-1}}(x,\ \text{dist}(D,\ x)+\varepsilon)] \neq \emptyset.$$

By q-hyperconvexity of X, we have

$$D\cap C_x(\text{dist}(x,\ D),\ \text{dist}(D,\ x)) = \bigcap_{x\in X} C_x(\alpha,\ \beta)\bigcap_{\varepsilon>0} C_x(\text{dist}(x,\ D)+\varepsilon,\ \text{dist}(D,\ x)+\varepsilon) \neq \emptyset$$

and hence D is q-proximinal. □

3. Externally q-hyperconvex subsets

In the following, we shall define an external q-hyperconvex subset of a quasi-pseudometric space $(X,\ d)$ which will be in analogy to [10, Definition 3.5]. We note that this will strengthen the concept of a q-hyperconvex subset of $(X,\ d)$ (see for instance, [11, Definition 3]).

Definition 10 ([6, Definition 6.1]) A subspace E of a quasi-pseudometric space$(X,\ d)$ is said to be externally q-hyperconvex (relative to X) if for any family $(x_i)_{i\in I}$ of points in X and families $(\alpha_i)_{i\in I}$ and $(\beta_i)_{i\in I}$ in $[0,\ +\infty)$, the following condition holds:

$$\text{If } d(x_i,\ x_j) \leq \alpha_i + \beta_j,\ \ \text{dist}(x_i,\ E) \leq \alpha_i,\ \ \text{and dist}(E,\ x_i) \leq \beta_i$$

whenever $i \in I$, then

$$\bigcap_{i\in I} C_{x_i}(\alpha_i,\ \beta_i)\cap E \neq \emptyset.$$

We will denote by $\mathscr{E}_q(X,\ d)$ the set of all nonempty externally q-hyperconvex subsets of a quasi-pseudometric space $(X,\ d)$.

Example 11 [6, Example 6.2] Let $(X,\ d)$ be a quasi-pseudometric space and $A \subset X$ be externally q-hyperconvex. For any $x \in X$, set $\text{dist}(x,\ A) = \alpha$ and $\text{dist}(A,\ x) = \beta$. then by applying external q-hyperconvexity to $(C_d(x,\ \alpha);\ C_{d^{-1}}(x,\ \beta))$, we have that there is $p \in C_d(x,\ \alpha)\cap C_{d^{-1}}(x,\ \beta)$. Therefore $d(x,\ p) = \text{dist}(x,\ A)$ and $d(p,\ x) = \text{dist}(A,\ x)$.

Proposition 12 Let $(X,\ d)$ be a quasi-pseudometric space and $E \subseteq X$.

(a) If $E \in \mathscr{E}_q(X,\ d)$, then $E \in \mathscr{E}_q(X,\ d^{-1})$.

(b) If $E \in \mathscr{E}_q(X,\ d)$, then E is an externally hyperconvex subspace of $(X,\ d^s)$.

Proof. (a) This follows directly from the definitions.

(b) Let E be an externally q-hyperconvex subspace of X. Suppose that $d^s(x_i,\ x_j) \leq \alpha_i + \beta_j$ whenever $i,\ j \in I$, $d^s(x_i,\ E) \leq \alpha_i$ whenever $i \in I$. Since E is externally q-hyperconvex, we get

$$\emptyset \neq \bigcap_{i\in I} C_{x_i}(\alpha_i,\ \alpha_i) \cap E$$

$$= \bigcap_{i\in I} C_{d^s}(x_i,\ \alpha_i) \cap E.$$

This shows that E is an externally hyperconvex subspace of $(X,\ d^s)$. □

Let us recall the following lemma.

Lemma 13 ([6, Lemma 6.4]) Let $(X,\ d)$ be a T_0-quasi-metric space and $E \subseteq X$ be such that $E \in \mathscr{E}_q(X,\ d)$ and suppose that $A \in \mathscr{A}_q(X)$ satisfies $A \cap E \neq \emptyset$. Then $E \cap A \in \mathscr{E}_q(X)$. Moreover, if X is q-hyperconvex, we have that $\mathscr{A}_q(X) \subset \mathscr{E}_q(X)$, that is, every q-admissible subset of a q-hyperconvex space $(X,\ d)$ is externally q-hyperconvex (relative to X).

Proof. For the first part of the proof, see the proof of [6, Lemma 6.4].

For the moreover part, let $D \in \mathscr{A}_q(X)$, $(x_i)_{i\in I}$ be a family of points in X, $(\alpha_i)_{i\in I}$, $(\beta_i)_{i\in I}$ be families of points in $[0,\ \infty)$ such that $\rho(x_i,\ x_j) \leq \alpha_i + \beta_j$, $\text{dist}(x_i,\ D) \leq \alpha_i$ and $\text{dist}(D,\ x_i) \leq \beta_i$. By Proposition 9, we have that for every $i \in I$, $\exists\, d_i \in D$ such that $\rho(x_i,\ d_i) = \text{dist}(x_i,\ D)$ which then implies that $D \cap C_\rho(x_i,\ \alpha_i) \cap C_{\rho^{-1}}(x_i,\ \beta_i) \neq \emptyset$. By q-hyperconvexity of X, we get

$$\bigcap_{i\in I} [C_\rho(x_i,\ \alpha_i) \cap C_{\rho^{-1}}(x_i,\ \beta_i)] \neq \emptyset.$$

Since D is q-admissible and $D \cap C_\rho(x_i,\ \alpha_i) \cap C_{\rho^{-1}}(x_i,\ \beta_i) \neq \emptyset$, it follows that

$$D \cap \bigcap_{i\in I} [C_\rho(x_i,\ \alpha_i) \cap C_{\rho^{-1}}(x_i,\ \beta_i)] \neq \emptyset.$$

□

Example 14 Consider the set of real numbers equipped with the T_0-quasi-metric defined in Example 2. Then we have a q-hyperconvex T_0-quasi-metric space (see [4, Example 1]). The subset $A = [-1,\ 2]$ is externally q-hyperconvex (relative to $\mathbb{R}$) by Lemma 13 since $A \in \mathscr{A}_q(X)$, i.e., $A = C_u(0,\ 1) \cap C_{u^{-1}}(0,\ 2)$.

Lemma 15 Let $A \in \mathscr{E}_q(X)$ where $(X,\ d)$ is a q-hyperconvex T_0-quasi-metric space. Then $C_A(\alpha,\ \beta) \in \mathscr{E}_q(X)$ for $\alpha,\ \beta \in [0,\ +\infty)$.

Proof. Let $[(C_d(x_i,\ \alpha_i))_{i\in I}; (C_{d^{-1}}(x_i,\ \beta_i))_{i\in I}]$ be a family of double balls, with $x_i \in X$ and $\alpha_i,\ \beta_i \geq 0$ whenever $i \in I$, $d(x_i,\ x_j) \leq \alpha_i + \beta_j$, $\text{dist}(x_i,\ A) \leq \alpha_i$ and $\text{dist}(A,\ x_i) \leq \beta_i$. Moreover, we have that $\text{dist}(x_i,\ A) \leq \alpha_i + \alpha$ and $\text{dist}(A,\ x_i) \leq \beta_i + \beta$. By external q-hyperconvexity of A, we get that

$$\bigcap_{i\in I} C_{x_i}(\alpha_i,\ \beta_i) \cap A \neq \emptyset.$$

Thus there is some point y in this intersection. With this point y, one sees that

$$d(x_i,\, y) \le \alpha_i + \alpha \text{ and } d(y,\, x_i) \le \beta_i + \beta.$$

By q-hyperconvexity of X, we get that

$$\emptyset \neq \bigcap_{i\in I} C_{x_i}(\alpha_i,\, \beta_i) \cap C_y(\alpha,\, \beta)$$

$$\subset \bigcap_{i\in I} C_{x_i}(\alpha_i,\, \beta_i) \cap C_A(\alpha,\, \beta).$$

□

Lemma 16 [12, Lemma 4.5]) Consider a q-hyperconvex T_0-quasi-metric space$(X,\, d)$ and let $A,\, B \in \mathscr{E}_q(X,\, d)$ with an element y belonging to both A and B and $x \in X$. Then we have that

$$A \cap B \cap C_x(\alpha,\, \alpha) \cap C_y(\beta,\, \beta) \neq \emptyset$$

for some non-negative real numbers α and β.

Proof. Since $(X,\, d)$ is q-hyperconvex, we have that the metric space $(X,\, d^s)$ is hyperconvex by [4, Proposition 2]. We also have, by part (b) of Proposition 12 that A and B are externally hyperconvex subspaces of the hyperconvex space $(X,\, d^s)$. We then apply Lemma 4.5 of [12] to conclude that

$$\emptyset \neq A \cap B \cap C_{d^s}(x,\, \alpha) \cap C_{d^s}(y,\, \beta)$$

$$= A \cap B \cap C_x(\alpha,\, \alpha) \cap C_y(\beta,\, \beta).$$

□

The next lemma demonstrates that a finite family of pairwise intersecting externally q-hyperconvex subspaces of a q-hyperconvex space has a nonempty intersection.

Lemma 17 Let $(X,\, d)$ be a q-hyperconvex T_0-quasi-metric space and $A,\, B,\, C \in \mathscr{E}_q(X,\, d)$ be pairwise intersecting. Then $A \cap B \cap C \neq \emptyset$.

Proof. The proof follows from [12, Lemma 4.6], the fact that $(X,\, d^s)$ is hyperconvex $A,\, B,\, C$ are all externally hyperconvex subspaces of $(X,\, d^s)$. □

Lemma 18 Let $(X,\, d)$ be a q-hyperconvex T_0-quasi-metric space and $A_1,\, A_2 \in \mathscr{E}_q(X,\, d)$ with $A_1 \cap A_2 \neq \emptyset$. Then $A_1 \cap A_2 \in \mathscr{E}_q(X,\, d)$.

Proof. Let $[(C_d(x_i,\, \alpha_i))_{i\in I};\, (C_{d^{-1}}(x_i,\, \beta_i))_{i\in I}]$ be a family of double balls, with $x_i \in X$ and $\alpha_i,\, \beta_i \geq 0$ whenever $i \in I$, $d(x_i,\, x_j) \le \alpha_i + \beta_j$, $d(x_i,\, A_1 \cap A_2) \le \alpha_i$ and $d(A_1 \cap A_2,\, x_i) \le \beta_i$. Set

$$A := \bigcap_{i\in I} C_{x_i}(\alpha_i,\, \beta_i).$$

Since A_1 and A_2 are externally q-hyperconvex, we get that

$$A \cap A_1 = \bigcap_{i \in I} C_{x_i}(\alpha_i,\ \beta_i) \cap A_1 \neq \emptyset$$

and

$$A \cap A_2 = \bigcap_{i \in I} C_{x_i}(\alpha_i,\ \beta_i) \cap A_2 \neq \emptyset.$$

Moreover, since q-admissible subsets of a q-hyperconvex space are externally q-hyperconvex (see Lemma 13), we get by Lemma 17 that $A \cap A_1 \cap A_2 \neq \emptyset$. □

Taking into account Lemma 17 and Lemma 18, we have shown (by induction) that the following result holds.

Proposition 19 Any finite collection of pairwise intersecting externally q-hyperconvex subsets of a q-hyperconvex T_0-quasi-metric space has a nonempty intersection which is also externally q-hyperconvex

Remark 20 One now wonders if Proposition 19 still holds if we replace the finite collection with a countable collection. By putting an additional condition on the space, the answer is yes and we shall prove this next (see Proposition 22). Before that, let us recall the following result that we shall need.

Theorem 21 ([6, Theorem 6.5]) Let $(X,\ d)$ be a bounded q-hyperconvex T_0-quasi-metric space. Moreover, let $(H_i)_{i \in I}$ be a descending family of non-empty externally q-hyperconvex subsets of X, where we assume that I is a chain such that $i_1,\ i_2 \in I$ and $i_1 \leq i_2$ hold if and only if $H_{i_2} \subseteq H_{i_1}$. Then,

$$\emptyset \neq \bigcap_{i \in I} H_i \in \mathscr{E}_q(X,\ d).$$

Proposition 22 Consider a bounded T_0-quasi-metric space $(X,\ d)$ and let $\{A_i\}_{i \in \mathbb{N}}$ be a countable family of pairwise intersecting externally q-hyperconvex subsets of X. Then

$$\emptyset \neq \bigcap_{i \in \mathbb{N}} A_i \in \mathscr{E}_q(X,\ d).$$

Proof. Note that the sets $\{B_n\}_n$, where,

$$B_n = \bigcap_{i=1}^{n} A_i$$

is a descending chain of nonempty externally q-hyperconvex subsets (compare Proposition 19). Thus, by applying Theorem 21 we get

$$\bigcap_i A_i = \bigcap_n B_n \neq \emptyset,$$

and this intersection is externally q-hyperconvex as well. □

Proposition 23 Consider a bounded q-hyperconvex T_0-quasi-metric space $(X,\ d)$ and let $\{A_i\}_{i\in I}$ be any family of pairwise intersecting externally q-hyperconvex subsets of X. Then

$$\emptyset \neq \bigcap_{i\in I} A_i \in \mathscr{E}_q(X,\ d).$$

Proof. We proceed by applying Zorn's lemma.
Define

$$\mathscr{G} = \left\{ I_0 \subset I : \forall\, I' \subset I \text{ finite},\ \emptyset \neq \bigcap_{i\in I_0\cup I'} A_i \in \mathscr{E}_q(X,\ d) \right\}.$$

Then we get by Proposition 19 that $\mathscr{G} \neq \emptyset$. Indeed, one sees that $\emptyset \in \mathscr{G}$. Consider now a totally ordered set $G_k \in \mathscr{G}$ and some finite set $I' \subset I$, then by setting

$$A_{G_k} = \bigcap_{i\in G_k\cup I'} A_i,$$

we get a decreasing sequence of nonempty externally q-hyperconvex sets. Let

$$G = \bigcup_k G_k.$$

Then we have by Theorem 21 that

$$\emptyset \neq A = \bigcap_{i\in I_0\cup I'} A_i = \bigcap_k A_{G_k} \in \mathscr{E}_q(X,\ d).$$

We therefore have that $G \in \mathscr{G}$ is an upper bound of G_k. By applying Zorn's lemma, we conclude that there exists some maximal element, $J_0 \in \mathscr{G}$ say.

For $i \in I$, we have that $J_0 \cup \{i\} \in \mathscr{G}$ and since J_0 is maximal, we conclude that $I = J_0 \in \mathscr{G}$. □

As an application of Proposition 23, we have the following corollary.

Corollary 24 For a q-hyperconvex T_0-quasi-metric space $(X,\ d)$ and a family $\{A_i\}_{i\in I}$ of pairwise intersecting externally q-hyperconvex subsets with the property that at least one of them is bounded, we have that

$$\bigcap_{i\in I} A_i \neq \emptyset.$$

We end this note with the following proposition.

Proposition 25 Let Y be a subset of a T_0-quasi-metric space $(X,\ d)$ with the property that $Y \in \mathscr{E}_q(X,\ d)$. If A is externally q-hyperconvex (relative to Y), then $A \in \mathscr{E}_q(X,\ d)$.

Proof. Let $[(C_d(x_i,\ \alpha_i))_{i\in I};\ (C_{d^{-1}}(x_i,\ \beta_i))_{i\in I}]$ be a family of double balls, where $x_i \in X$ and $\alpha_i,\ \beta_i \geq 0$ whenever $i \in I$, $d(x_i,\ x_j) \leq \alpha_i + \beta_j$, $d(x_i,\ A) \leq \alpha_i$ and $d(A,\ x_i) \leq \beta_i$. We have that

$$A_i := C_{x_i}(\alpha_i,\ \beta_i) \cap Y \in \mathscr{E}_q(X,\ d)$$

and therefore the sets A_i are also externally q-hyperconvex (relative to Y). Notice that $A_i \cap A \neq \emptyset$. As a result of Y being externally q-hyperconvex, we find that

$$A_i \cap A_j = C_{x_i}(\alpha_i,\ \beta_i) \cap C_{x_j}(\alpha_j,\ \beta_j) \cap Y \neq \emptyset.$$

This then gives us a collection of pairwise intersecting externally q-hyperconvex subsets of Y and by applying Corollary 24 we get

$$A \cap \bigcap_{i\in I} C_{x_i}(\alpha_i,\ \beta_i) = A \cap \bigcap_{i\in I} A_i \neq \emptyset.$$

□

4. Summary

We continued recent works about external q-hyperconvexity in quasi-pseudometric spaces. We were able to show that for any family of pairwise intersecting externally q-hyperconvex subsets of a quasi-pseudometric space with the property that at least one of them is bounded, we find that the whole intersection is nonempty. We also showed that if Y is an externally q-hyperconvex subset of a quasi-pseudometric space X and A is a subset of X which is externally q-hyperconvex (relative to Y), then A is externally q-hyperconvex (relative to X).

The results of this study will be very vital in our future study of the characterization of all q-hyperconvex subsets of the space of all bounded real-valued functions defined on a set X, equipped with the usual quasi-pseudometric.

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Conflict of interest

We declare that we have no personal relationships or financial interests that could have influenced the work presented in this manuscript.

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