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Research Article

Group Classification of Second-Order Linear Neutral Differential Equations

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Abstract: In this paper, we shall extend the method of obtaining symmetries of ordinary differential equations to second-order non-homogeneous functional differential equations with variable coefficients. The existing research for delay differential equations defines a Lie-Bäcklund operator and uses the invariant manifold theorem to obtain the infinitesimal generators of the Lie group. However, we shall use a different approach that requires Taylor's theorem for a function of several variables to obtain a Lie invariance condition and the determining equations for second-order functional differential equations. Certain standard results from the theory of ordinary differential equations have been employed to simplify the equation under study. The symmetry analysis of this equation was found to be non-trivial for arbitrary variable coefficients. In such cases, by selecting certain specific functions, arising in most practical models, we find the symmetries that are seen to be in terms of Bessel's functions, Mathieu functions, etc. We then make a complete group classification of the second-order linear neutral differential equation, for which there is no existing literature.

***Keywords*:** infinitesimals, invariance, Lie group, functional differential equations, symmetries

MSC: 34K40, 34C14, 22E99

1. Introduction

Functional differential equations (FDEs) are differential equations in which the unknown function appears with deviating arguments. One type of FDE of particular interest is called neutral differential equations (NDEs), which are differential equations in which the derivative term appears with delay. These equations are a more powerful and accurate way to model systems involving flip-flop circuits, the controlled motion of a rigid body, the human postural balance model, classical electrodynamics, etc. More information on the applications and methods of solving NDEs can be found in [1-4], and in general, information on FDEs can be found in [5].

As pointed out in [6], symmetries are transformations that leave an object unchanged or invariant and are very useful in the formation and study of laws of nature. In the past, the solvability of algebraic equations of a high degree posed a challenge to mathematicians. It was Galois who then associated a group with such equations, called the Galois group, to determine the solvability of such equations. It turned out that the solvability of the Galois group, an easier task, led to determining if the corresponding algebraic equation was solvable. This motivated Sophus Lie to study the

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solvability of differential equations by associating with them a group called the Lie group.

In [7, 8], symmetries of delay differential equations are obtained by defining a certain operator equivalent to the canonical Lie-Bäcklund operator. In [9], equivalent symmetries of a second-order delay differential equation are obtained. However, in [9], an operator equivalent to the canonical Lie-Bäcklund operator and suitable other operators are defined. In [10], the authors exhaustively describe the Lie symmetries of systems of second-order linear ordinary differential equations with constant coefficients over both real and complex fields. They propose an algebraic approach to obtain bounds for the dimensions of the maximal Lie invariance algebras possessed by such systems. Further, such systems are thoroughly provided with their group classification in [11, 12], with extensions to linear systems of second-order ordinary differential equations with more than two equations. Higher-order symmetries for ordinary differential equations are studied in [13]. In [14], the author suggests a group method to study FDEs based on a search of symmetries of underdetermined differential equations by methods of classical and modern group analysis, using the principle of factorization. The method therein encompasses the use of a basis of invariants consisting of universal and differential invariants. Then, in [15], an admitted Lie group for first-order delay differential equations with constant coefficients is defined, and corresponding generators of the Lie group for this equation are obtained. The approach in [15] consists of using Lie-Bäcklund operators to obtain the determining equations. Lie symmetries of first-order NDEs with a general time delay have been found using a Lie type invariance condition obtained from Taylor's theorem for a function of several variables in [16] and for the case with constant delay in [17]. While a thorough group classification for delay differential equations with applications is seen in [18-20], there has been no work for second-order NDEs with variable coefficients. As these equations would more accurately model several systems, this present paper attempts to bridge the gap and extend the results to second-order NDEs with variable coefficients. Recent studies include a complete classification of FDEs with constant coefficients to solvable Lie algebras that can be found in [21, 22].

In this paper, we do a symmetry analysis of the second-order FDE.

$$x''(t) = f(t,\ x(t),\ x'(t),\ x(t-r),\ x'(t-r),\ x''(t-r)). \tag{1}$$

Here, f is defined on $I \times E^5$ where E^5 means $E \times E \times E \times E \times E$, E is an open interval or the union of open intervals in $\mathbb{R}$, I is any interval in $\mathbb{R}$, $r > 0$ is the delay, and $x'(t-r)$ and $x''(t-r)$ mean $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ evaluated at $t - r$, respectively. Our focus is to obtain the equivalent symmetries and the corresponding generators of the Lie group under which NDEs are invariant. We shall first find the admitted Lie group for equation (1). We then use this group to obtain the desired symmetries. In the absence of the term $x''(t-r)$, equation (1) reduces to a second-order delay differential equation, the group classification of which can be found in [23]. By following a completely different approach from the existing literature for delay differential equations, we, in this paper, extend the results of obtaining symmetries of ordinary differential equations found in [24] to obtain a complete group classification of second-order non-homogeneous linear NDEs with twice differentiable variable coefficients.

$$x''(t) + \alpha(t)x'(t) + \beta(t)x'(t-r) + \gamma(t)x(t) + \rho(t)x(t-r) + \kappa(t)x''(t-r) = h(t). \tag{2}$$

We shall use Taylor's theorem for a function of several variables to obtain a Lie-type invariance condition for NDEs. Using this, we obtain our determining equations. These equations are then split with respect to the independent variables to obtain an overdetermined system of partial differential equations, which are then solved to obtain the most general generator of the Lie group and the corresponding equivalent symmetries. It may be noted that a pro of our approach is that it does not lead to any magnification of the delay terms in the determining equations as compared to the existing literature. However, as a con of the approach, it is seen that in most cases, we do not get an explicit solution due to the arbitrariness of the variable coefficients. As such, we do not get explicit infinitesimal generators. By then choosing particular values of the variable coefficients or restricting our equation by choosing certain values of the obtained constants (which does not alter the symmetries obtained), we illustrate the infinitesimal generators of the admitted group, which are explicitly obtained, for these special cases. We finally make a thorough group classification of this second-order NDE. It is noteworthy to point out here that there is no existing literature on the group classification of NDEs that will aid in studying the properties of solutions of several models involving the use of such equations.

The rest of the paper is organized as follows: (i) The following section gives the necessary preliminaries. (ii) Section 3 sets up the Lie invariance condition using Taylor's theorem, a novel approach, and proves certain results from the theory of ordinary differential equations required in the simplification of our equation under study. (iii) Section 4 makes a thorough group classification of the second-order NDE by taking various cases that are of importance. (iv) In Section 5, the results are suitably illustrated. (v) The concluding section summarizes the results obtained and brings forth some research questions for interested researchers to continue with further work.

2. Preliminaries

We begin this section by stating the definition of Lie groups.

Definition 2.1 ([25]) Let $t = (t_1, t_2, ..., t_n)$ lie in a region $E \subset \mathbb{R}^n$. The set of transformations $\overline{t} = f(t, \delta)$, defined for each t in E, depending on the parameter δ lying in the set $S \subset \mathbb{R}$, with $\phi(\delta, \epsilon)$ defining a law of composition of parameters δ and ϵ in S, forms a one-parameter Lie group of transformations on E if:

1. For each parameter δ in S the transformations are one-one onto E, in particular, $\overline{t}$ lies in E.
2. S with the law of composition ϕ forms a group G.
3. $\overline{t} = t$ when $\delta = e$, that is $f(t, e) = t$.
4. If $\overline{t} = f(t, \delta)$, $\tilde{t} = f(\overline{t}, \epsilon)$, then $\tilde{t} = f(t, \phi(\delta, \epsilon))$.
5. δ is a continuous parameter, that is, S is an interval in $\mathbb{R}$. Without loss of generality, $\delta = 0$ corresponds to the identity element e.
6. f is infinitely differentiable with respect to t in E and an analytic function of δ in S.
7. $\phi(\delta, \epsilon)$ is an analytic function of δ and ϵ, and $\epsilon, \delta \in S$.

Example 2.2 Consider the Lie group of rotation matrices denoted by $SO(2, \mathbb{R})$. These form a subgroup of the group (under multiplication) of 2×2 real invertible matrices denoted by $GL(2, \mathbb{R})$. Let the parameter δ denote the rotation angle. Then we can parametrize this group as follows:

$$SO(2, \mathbb{R}) = \left\{ \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix} : \delta \in \mathbb{R} / 2\pi\mathbb{Z} \right\}.$$

Multiplying any two elements of $SO(2, \mathbb{R})$ yields another element of $SO(2, \mathbb{R})$ with the rotation angle as the addition of the two angles, and on inversion, we see that we get the opposite angle.

Remark 2.3 If the Lie group is given by $\overline{t} = f_1(t, x; \delta)$, $\overline{x} = f_2(t, x; \delta)$ where f_1 and f_2 are smooth functions in t and x having a convergent Taylor series in δ, then $\omega(t, x) = \dfrac{\partial f_1(t, x; 0)}{\partial \delta}$ and $\Upsilon(t, x) = \dfrac{\partial f_2(t, x; 0)}{\partial \delta}$. ω and Υ are called coefficients of the infinitesimal transformations or simply infinitesimals.

Remark 2.4 Throughout this paper, we make the following notations.

$$\Omega = \omega(t - r, \ x(t - r)) \text{ and } \eta = \Upsilon(t - r, \ x(t - r))$$

Definition 2.5 By $F_t, F_x, F_{t-r}, F_{x(t-r)}$, we mean $\dfrac{\partial F}{\partial t}, \dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial (t-r)}, \dfrac{\partial F}{\partial x(t-r)}$, respectively. Similarly, by F_{tt}, F_{xx}, $F_{(t-r)(t-r)}$, $F_{x(t-r)x(t-r)}$, we mean $\dfrac{\partial^2 F}{\partial t^2}, \dfrac{\partial F^2}{\partial x^2}, \dfrac{\partial^2 F}{\partial^2 (t-r)}, \dfrac{\partial^2 F}{\partial^2 x(t-r)}$, respectively.

3. Lie invariance condition for second-order FDEs

In this section, we establish the Lie invariance condition for second-order FDEs. In order to determine this FDE completely, we need to specify the delay term. Otherwise, the problem is not fully determined.

Theorem 3.1 Consider the second-order FDE

$$\frac{d^2x}{dt^2} = F(t,\ x,\ t-r,\ x(t-r),\ x'(t),\ x'(t-r),\ x''(t-r)), \tag{3}$$

where F is defined on a seven-dimensional space $I \times E \times I - r \times E^4$, E is an open interval or the union of open intervals in $\mathbb{R}$, I is any interval in $\mathbb{R}$ and $I - r = \{y - r : y \in I\}$. Then, the Lie invariance condition is given by

$$\omega F_t + \Upsilon F_x + \Omega F_{t-r} + \eta F_{x(t-r)} + \Upsilon_{[t]} F_{x'(t)} + \eta_{[t]} F_{x'(t-r)} + \eta_{[tt]} F_{x''(t-r)}$$

$$= \Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x' x'',$$

where

$$\Upsilon_{[t]} = D_t(\Upsilon) - x' D_t(\omega),$$

$$\Upsilon_{[tt]} = D_t(\Upsilon_{[t]}) - x'' D_t(\omega),$$

$$\eta_{[t]} = (\eta)_{t-r} + ((\eta)_{x(t-r)} - (\Omega)_{t-r})x'(t-r) - (x'(t-r))^2 (\Omega)_{x(t-r)},$$

$$\eta_{[tt]} = (\eta_{(t-r)(t-r)} + (2\eta_{(t-r)x(t-r)} - \Omega_{(t-r)(t-r)})x'(t-r) + (\eta_{x(t-r)x(t-r)} - 2\Omega_{(t-r)x(t-r)})x'(t-r)^2$$

$$-\Omega_{x(t-r)x(t-r)} x'(t-r)^3 + (\eta_{x(t-r)} - 2\Omega_{t-r})x''(t-r) - 3\Omega_{x(t-r)} x'(t-r) x''(t-r)),$$

with $D_t = \frac{\partial}{\partial t} + x' \frac{\partial}{\partial x} + x'' \frac{\partial}{\partial x'} + \cdots$.

Proof. Let the FDE be invariant under the Lie group

$$\overline{t} = t + \delta\omega(t,\ x) + O(\delta^2),\ \ \overline{x} = x + \delta\Upsilon(t,\ x) + O(\delta^2).$$

It may be noted that the coefficients of the infinitesimal transformations $\omega(t, x)$ and $\Upsilon(t, x)$ are as defined earlier in the previous section. We then naturally define

$$\overline{t-r} = t - r + \delta\omega(t-r,\ x(t-r)) + O(\delta^2) \text{ and } \overline{x(t-r)} = x(t-r) + \delta\Upsilon(t-r,\ x(t-r)) + O(\delta^2).$$

It follows that

$$\frac{d\overline{x}}{d\overline{t}} = \left[\frac{dx}{dt} + (\Upsilon_t + \Upsilon_x x')\delta + O(\delta^2)\right]\left[1 - (\omega_t + \omega_x x')\delta + O(\delta^2)\right]$$

$$= \frac{dx}{dt} + \left[\Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2\right]\delta + O(\delta^2).$$

With the notation $D_t = \frac{\partial}{\partial t} + x' \frac{\partial}{\partial x}$, we can write

$$\frac{d\overline{x}}{d\overline{t}} = \frac{dx}{dt} + (D_t(\Upsilon) - x' D_t(\omega))\delta + O(\delta^2) = \frac{dx}{dt} + \Upsilon_{[t]}\delta + O(\delta^2),$$

where $\Upsilon_{[t]} = D_t(\Upsilon) - x' D_t(\omega) = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2$.

Considering the second-order extended infinitesimals, we see that

$$\frac{d^2\overline{x}}{d\overline{t}^2}=\left(\frac{d^2x}{dt^2}+D_t(\Upsilon_{[t]})\delta+O(\delta^2)\right)(1-\delta D_t(\omega)+O(\delta^2))$$

$$=\frac{d^2x}{dt^2}+(D_t(\Upsilon_{[t]})-D_t(\omega)x'')\delta+O(\delta^2).$$

So, $\Upsilon_{[tt]}=D_t(\Upsilon_{[t]})-x''D_t(\omega)$. As $\Upsilon_{[t]}$ contains t, x, and x', we need to extend the definition of D_t. Hence, we have $D_t=\frac{\partial}{\partial t}+x'\frac{\partial}{\partial x}+x''\frac{\partial}{\partial x'}+\cdots$.

Expanding $\Upsilon_{[tt]}$, gives

$$\Upsilon_{[tt]}=\Upsilon_{tt}+(2\Upsilon_{tx}-\omega_{tt})x'+(\Upsilon_{xx}-2\omega_{tx})x'^2-\omega_{xx}x'^3+(\Upsilon_x-2\omega_t)x''-3\omega_x x'x''.$$

With the notations $\Omega=\omega(t-r,\ x(t-r))$ and $\eta=\Upsilon(t-r,\ x(t-r))$, it follows that

$$\overline{x'(t-r)}=x'(t-r)+[(\eta)_{t-r}+((\eta)_{x(t-r)}-(\Omega)_{t-r})x'(t-r)-(x'(t-r))^2(\Omega)_{x(t-r)}]\delta+O(\delta^2),$$

and

$$\overline{x''(t-r)}=x''(t-r)+\Big[\eta_{(t-r)(t-r)}+(2\eta_{(t-r)x(t-r)}-\Omega_{(t-r)(t-r)})x'(t-r)$$

$$+(\eta_{x(t-r)x(t-r)}-2\Omega_{(t-r)x(t-r)})x'(t-r)^2-\Omega_{x(t-r)x(t-r)}x'(t-r)^3$$

$$+(\eta_{x(t-r)}-2\Omega_{t-r})x''(t-r)-3\Omega_{x(t-r)}x'(t-r)x''(t-r)\Big]\delta+O(\delta^2).$$

Let $\eta_{[t]}=(\eta)_{t-r}+((\eta)_{x(t-r)}-(\Omega)_{t-r})x'(t-r)-(x'(t-r))^2(\Omega)_{x(t-r)}$ and $\eta_{[tt]}=(\eta_{(t-r)(t-r)}+2\eta_{(t-r)x(t-r)})-\Omega_{(t-r)(t-r)})x'(t-r)+(\eta_{x(t-r)x(t-r)}-2\Omega_{(t-r)x(t-r)})x'(t-r)^2-\Omega_{x(t-r)x(t-r)}x'(t-r)^3+(\eta_{x(t-r)}-2\Omega_{(t-r)})x''(t-r)-3\Omega_{x(t-r)})x'(t-r)\,x''(t-r)$. For invariance,

$$\frac{d^2\overline{x}}{d\overline{t}^2}=F(\overline{t},\ \overline{x},\ \overline{t-r},\ \overline{x(t-r)},\ \frac{d\overline{x}}{d\overline{t}},\ \overline{x}'(\overline{t-r}),\ \overline{x}''(\overline{t-r})).$$

This gives

$$\frac{d^2x}{dt^2}+\Upsilon_{[tt]}\delta+O(\delta^2)=F(t+\delta\omega+O(\delta^2),\ x+\delta\Upsilon+O(\delta^2),\ t-r+\delta\Omega+O(\delta^2),$$

$$x(t-r)+\delta\eta+O(\delta^2),\ \frac{dx}{dt}+\delta\Upsilon_{[t]}+O(\delta^2),\ \frac{dx}{dt}(t-r)$$

$$+\eta_{[t]}\delta+O(\delta^2),\ \frac{d^2x}{dt^2}(t-r)+\eta_{[tt]}\delta+O(\delta^2))$$

$$= F(t,\ x,\ t-r,\ x(t-r),\ x'(t),\ x'(t-r),\ x''(t-r))$$

$$+(\omega F_t + \Upsilon F_x + \Omega F_{t-r} + \eta F_{x(t-r)} + \Upsilon_{[t]} F_{x'(t)}$$

$$+\eta_{[t]} F_{x'(t-r)} + \eta_{[tt]} F_{x''(t-r)})\delta + O(\delta^2).$$

Comparing the coefficient of δ, we get

$$\begin{aligned} &\omega F_t + \Upsilon F_x + \Omega F_{t-r} + \eta F_{x(t-r)} + \Upsilon_{[t]} F_{x'(t)} + \eta_{[t]} F_{x'(t-r)} + \eta_{[tt]} F_{x''(t-r)} \\ &= \Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x' x''. \end{aligned} \tag{4}$$

Equation (4) is a Lie invariance condition.

We then, naturally define the extended operator, for equation (2) as

$$\zeta^{(1)} = \omega\frac{\partial}{\partial t} + \omega^r \frac{\partial}{\partial(t-r)} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r \frac{\partial}{\partial x(t-r)} + \Upsilon_{[t]}\frac{\partial}{\partial x'} + \Upsilon^r_{[t]}\frac{\partial}{\partial x'(t-r)} + \Upsilon_{[tt]}\frac{\partial}{\partial x''} + \Upsilon^r_{[tt]}\frac{\partial}{\partial x''(t-r)}. \tag{5}$$

The operator given by equation (5) will be acted upon the NDE under study to get the determining equations, from which the symmetries of the equation can be found.

Defining $\Delta = \dfrac{d^2x}{dt^2} - F(t,\ x(t),\ t-r,\ x(t-r),\ x'(t),\ x'(t-r),\ x''(t-r)) = 0$, we get

$$\zeta^{(1)}\Delta = \Upsilon_{[tt]} - \omega F_t - \Upsilon F_x - \Omega F_{t-r} - \eta F_{x(t-r)} - \Upsilon_{[t]} F_{x'(t)} - \eta_{[t]} F_{x'(t-r)} - \eta_{[tt]} F_{x''(t-r)}. \tag{6}$$

Comparing equation (6) with equation (4), we get

$$\Upsilon_{[tt]} = \Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x' x''.$$

On substituting $x'' = F$ into $\zeta^{(1)}\Delta = 0$, we get an invariance condition $\zeta^{(1)}\Delta\big|_{\Delta=0} = 0$, for equation (2), from which we shall obtain the determining equations.

The following lemmas shall be used for simplifying our analysis.

Lemma 3.2 If $x_1(t)$ is an arbitrary solution of

$$x''(t) + a(t)x'(t) + b(t)x'(t-r) + c(t)x(t) + d(t)x(t-r) + k(t)x''(t-r) = h(t),$$

then by employing the change of variables $\overline{t} = t,\ \overline{x} = x_1(t) - x$, the non-homogeneous NDE gets transformed into a homogeneous NDE

$$x''(t) + a(t)x'(t) + b(t)x'(t-r) + c(t)x(t) + d(t)x(t-r) + k(t)x''(t-r) = 0. \tag{7}$$

Proof. For invariance, we must have

$$\overline{x}''(\overline{t}) + \alpha(\overline{t})\overline{x}'(\overline{t}) + b(\overline{t})\overline{x}'(\overline{t}-r) + c(\overline{t})\overline{x}(\overline{t}) + d(\overline{t})\overline{x}(\overline{t}-r) + k(\overline{t})\overline{x}''(\overline{t}-r) = h(\overline{t}).$$

Using the given change of variables, we get

$$x_1''((t)+a(t)x_1'(t)+b(t)x_1'(t-r)+c(t)x_1(t)+d(t)x_1(t-r)+k(t)x_1''((t-r)$$

$$-(x''(t)+a(t)x'(t)+b(t)x'(t-r)+c(t)x(t)+d(t)x(t-r)+k(t)x''(t-r))=h(t),$$

which by the hypothesis on $x_1(t)$ reduces the above equation to equation (7).

Lemma 3.3 By employing a suitable transformation, the NDE with twice differentiable variable coefficients

$$x''(t)+a_1(t)x'(t)+b_1(t)x'(t-r)+c_1(t)x(t)+d_1(t)x(t-r)+k_1(t)x''(t-r)=0, \tag{8}$$

can be reduced to a one in which the first-order ordinary derivative term is missing.

Proof. By employing the transformation, $x=u(t)s(t)$, where $u(t)\neq 0$ is some twice differentiable function in t, $s(t)=\exp\left(-\int^{t}\frac{a_1(\xi)d\xi}{2}\right)+c^*$, c^* is an arbitrary constant, equation (8), can be reduced to $u''(t)+b_2(t)u'(t-r)+c_2(t)u(t)+d_2(t)u(t-r)+k_2(t)u''(t-r)=0$, where

$$b_2(t)=\frac{b_1(t)s(t-r)+2k(t)s'(t-r)}{s(t)},\ c_2(t)=\frac{s''(t)+a_1(t)s'(t)+c_1(t)s(t)}{s(t)},$$

$$d_2(t)=\frac{b_1(t)s'(t-r)+d_1(t)s(t-r)+k_1(t)s''(t-r)}{s(t)}$$

and $k_2(t)=\dfrac{k_1(t)}{u(t)}$.

4. Symmetries of the non-homogeneous linear second-order NDE

In this section, we shall obtain symmetries of the second-order non-homogeneous NDE with continuously differentiable variable coefficients given by

$$x''(t)+a(t)x'(t)+b(t)x'(t-r)+c(t)x(t)+d(t)x(t-r)+k(t)x''(t-r)=h(t). \tag{9}$$

Because of Lemma 3.3, we shall consider equivalent symmetries of

$$x''(t)+b(t)x'(t-r)+c(t)x(t)+d(t)x(t-r)+k(t)x''(t-r)=0. \tag{10}$$

4.1 *Symmetry analysis of equation (10)*

Let us specify the delay point

$$g(t)=t-r. \tag{11}$$

Applying operator $\zeta^{(1)}$ defined by equation (5) to equation (11), we get

$$\Omega=\omega. \tag{12}$$

Applying operator $\zeta^{(1)}$ defined by equation (5) to equation (10), we get

$$\Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)(-b(t)x'(t-r) - c(t)x - d(t)x(t-r)$$
$$-k(t)x''(t-r)) - 3\omega_x x'(-b(t)x'(t-r) - c(t)x - d(t)x(t-r) - k(t)x''(t-r))$$
$$= -\big[\omega(b'(t)x'(t-r) + c'(t)x(t) + d'(t)x(t-r) + k'(t)x''(t-r)) + c(t)\Upsilon$$
$$+d(t)\eta + b(t)(\eta_{t-r} + (\eta_{x(t-r)} - \Omega_{t-r})x'(t-r) - \Omega_{x(t-r)}x'^2(t-r)) + k(t)(\eta_{(t-r)(t-r)} \tag{13}$$
$$+(2\eta_{(t-r)x(t-r)} - \Omega_{(t-r)(t-r)})x'(t-r) + (\eta_{x(t-r)x(t-r)} - 2\Omega_{(t-r)x(t-r)})x'^2(t-r)$$
$$-\Omega_{x(t-r)x(t-r)}x'^3(t-r) + (\eta_{x(t-r)} - 2\Omega_{t-r})x''(t-r) - 3\Omega_{x(t-r)}x'(t-r)x''(t-r))\big].$$

Splitting equation (13) with respect to $x'^3(t-r)$, we get $k(t)\Omega_{x(t-r)x(t-r)} = 0$, which we can easily solve to get

$$\omega(t, x) = \alpha(t)x + \beta(t). \tag{14}$$

Differentiating equation (13) with respect to $x''(t-r)$, we get

$$k(t)(2\omega_t - \Upsilon_x) + 3k(t)\omega_x x' = 3k(t)\Omega_{x(t-r)}x'(t-r) - (\omega k'(t) + k(t)(\eta_{x(t-r)} - 2\Omega_{t-r})).$$

Splitting this equation with respect to $x'(t-r)$ and using the fact that $k(t) \neq 0$, we get $\omega_x = 0$. This with equation (14) gives

$$\omega(t, x) = \beta(t). \tag{15}$$

Splitting equation (13) with x'^2, we get $\Upsilon_{xx} = 0$, which solves to give

$$\Upsilon(t, x) = \gamma(t)x + \rho(t). \tag{16}$$

Substituting equations (15) and (16) into the determining equation (13), we get

$$\gamma''(t)x + \rho''(t) + (2\gamma'(t) - \beta''(t))x' + (\gamma(t) - 2\beta'(t))(-b(t)x'(t-r) - c(t)x - d(t)x(t-r) - k(t)x''(t-r))$$
$$= -\big[\beta(t)(b'(t)x'(t-r) + c'(t)x + d'(t)x(t-r) + k'(t)x''(t-r)) + c(t)(\gamma(t)x + \rho(t)) + d(t)(\gamma(t-r)x(t-r)$$
$$+\rho(t-r)) + b(t)(\gamma'(t-r)x(t-r) + \rho'(t-r) + (\gamma(t-r) - \beta'(t-r))x'(t-r)) + k(t)(\gamma''(t-r)x(t-r) \tag{17}$$
$$+\rho''(t-r) + (2\gamma'(t-r) - \beta''(t-r))x'(t-r) + (\gamma(t-r) - 2\beta'(t-r))x''(t-r))\big].$$

From equation (12), we have

$$\beta(t) = \beta(t-r). \tag{18}$$

Splitting equation (17) with respect to $x(t)$, we get

$$\gamma''(t) + 2\beta'(t)c(t) + \beta(t)c'(t) = 0. \tag{19}$$

Splitting equation (17) with respect to $x'(t)$, we get

$$\gamma(t) = \frac{1}{2}[\beta'(t) + c_1]. \tag{20}$$

Using equation (18), we get

$$\gamma(t) = \gamma(t - r). \tag{21}$$

Splitting equation (17) with respect to the constant terms, we get

$$\rho''(t) + b(t)\rho'(t - r) + c(t)\rho(t) + d(t)\rho(t - r) + k(t)\rho''(t - r) = 0. \tag{22}$$

That is, $\rho(t)$ satisfies the homogeneous NDE of second-order given by equation (7). Splitting equation (17) with respect to $x''(t - r)$, and using equations (18) and (21), we get

$$\beta(t)k'(t) = 0. \tag{23}$$

4.2 *Group classification of equation (10)*

Theorem 4.1 The NDE (10) for which $k(t)$ is not a constant, admits a two-dimensional group generated by $\zeta_1^* = x\frac{\partial}{\partial x}$, $\zeta_2^* = \rho(t)\frac{\partial}{\partial x}$.

Proof. Equation (23) having to be true for an arbitrary $\beta(t)$ and $k(t)$ implies that for a nonconstant $k(t)$, we must have, $\beta(t) = 0$ and consequently, $\omega(t, x) = 0$ and $\Upsilon(t, x) = \frac{c_1}{2}x + \rho(t)$.

The infinitesimal generator of the Lie group is given by

$$\zeta^*(t, x) = \frac{c_1}{2}x\frac{\partial}{\partial x} + \rho(t)\frac{\partial}{\partial x}, \tag{24}$$

where c_1 is an arbitrary constant and $\rho(t)$ satisfies equation (7).

Remark 4.2 Theorem 4.1 is useful in studying the symmetry properties of solutions of NDEs arising in the modeling of networks containing lossless transmission lines.

Having obtained the infinitesimal generator for the case when $k(t)$ is non-constant, we now perform symmetry analysis and a complete group classification of the second-order NDE (9), for which

$$k(t) = c_2, \tag{25}$$

where c_2 is an arbitrary constant.

Splitting equation (17) with respect to $x(t - r)$ and using equation (21), we get

$$k(t)\beta'''(t) + 2\beta(t)d'(t) + 4\beta'(t)d(t) + 2b(t)\gamma'(t) = 0. \tag{26}$$

Splitting equation (17) with respect to $x'(t - r)$ and using equations (18) and (21), we get

$$b(t)\beta'(t) + \beta(t)b'(t) = 0. \tag{27}$$

Equation (27) can be easily integrated to give

$$b(t)\beta(t) = c_3, \tag{28}$$

where c_3 is an arbitrary constant.

Using equation (15), we can rewrite equations (16), (19), (20), (26), and (28), respectively as

$$\Upsilon(t,\ x)=\left[\frac{1}{2}(\omega_t+c_1)\right]x+\rho(t), \tag{29}$$

$$\omega_{ttt}+4c(t)\omega_t+2c'(t)\omega=0, \tag{30}$$

$$\gamma(t)=\frac{1}{2}(\omega_t+c_1), \tag{31}$$

$$c_2\omega_{ttt}+2d'(t)\omega(t)+4d(t)\omega_t+b(t)\omega_{tt}=0, \tag{32}$$

and

$$\omega(t,\ x)=\frac{c_3}{b(t)}, \tag{33}$$

where c_1, c_2, and c_3 are arbitrary constants.

Next, we shall obtain a complete classification of equation (10).

Theorem 4.3 The NDE (10) with $b(t)\neq 0,\ d(t)\neq 0,\ k(t)=c_2$ admits a three-dimensional group generated by $\zeta_1^*=x\frac{\partial}{\partial x},\ \zeta_2^*=\frac{1}{b(t)}\frac{\partial}{\partial t}+\frac{x}{2}\left(\frac{1}{b(t)}\right)'\frac{\partial}{\partial x},\ \zeta_3^*=\rho(t)\frac{\partial}{\partial x}.$

Proof. If $c_3\neq 0$, from equation (33), we get

$$b(t)=\frac{c_3}{\omega(t,\ x)}. \tag{34}$$

From equation (29), we can write

$$\Upsilon(t,\ x)=\frac{1}{2}\left[c_2\left(\frac{1}{b(t)}\right)'+c_1\right]x+\rho(t). \tag{35}$$

Using equation (34) in equation (32), we get

$$c_2\omega\omega_{ttt}+2\omega^2 d'(t)+4\omega\omega_t d(t)+c_3\omega_{tt}=0. \tag{36}$$

Equation (36) can be easily integrated to give

$$c_2\omega\omega_{tt}-\frac{c_0}{2}\omega_t^2+2\omega^2 d(t)+c_3\omega_t=c_4, \tag{37}$$

where c_4 is an arbitrary constant. Using equation (33), we can solve equation (37) for $d(t)$ to get $d(t)=\frac{1}{2}\left[c_5 b^2(t)+b'(t)+c_2\left(\frac{b''(t)}{b(t)}-2\left(\frac{b'(t)}{b(t)}\right)^2+\frac{b'(t)}{b^2(t)}\right)\right]$, where $c_5=c_2c_3^2$.

Since $\omega=\Omega$, we get $b(t)=b(t-r)$. Using equation (33) in equation (30), we get

$$c'(t)-2\frac{b'(t)}{b(t)}c(t)=-\frac{1}{2}b(t)\left[6\frac{b'(t)b''(t)}{b^3(t)}-\frac{b'''(t)}{b^2(t)}-6\frac{b'^3(t)}{b^4(t)}\right]. \tag{38}$$

Equation (38) is a first-order linear ordinary differential equation, which can be solved to give

$$c(t)=\frac{1}{2}\left[\frac{b''(t)}{b(t)}-\frac{3}{2}\left(\frac{b'(t)}{b(t)}\right)^2+\frac{c_6}{2}b^2(t)\right], \tag{39}$$

where c_6 is an arbitrary constant.

In this case, we have obtained the coefficients of the infinitesimal transformation as

$$\omega(t)=\frac{c_3}{b(t)},\ \Upsilon(t,\ x)=\frac{x}{2}\left[c_3\left(\frac{1}{b(t)}\right)'+c_1\right]+\rho(t). \tag{40}$$

The infinitesimal generator in this case is given by

$$\zeta^*(t,\ x)=\frac{c_1}{2}x\frac{\partial}{\partial x}+c_3\left(\frac{1}{b(t)}\frac{\partial}{\partial t}+\frac{x}{2}\left(\frac{1}{b(t)}\right)'\frac{\partial}{\partial x}\right)+\rho(t)\frac{\partial}{\partial x}, \tag{41}$$

where $\rho(t)$ is an arbitrary solution of equation (10). If $c_3 = 0$, then

$$\omega(t,\ x)=0,\ \Upsilon(t,\ x)=\frac{c_1}{2}x+\rho(t). \tag{42}$$

The infinitesimal generator is given by

$$\zeta^*(t,\ x)=\left(\frac{c_1}{2}x+\rho(t)\right)\frac{\partial}{\partial x}. \tag{43}$$

Remark 4.4 Theorem 4.3 finds applications in the symmetry analysis of models involving the study of vibrating masses attached to an elastic bar.

Theorem 4.5 The NDE (10) with $b(t)\neq 0,\ d(t)=0,\ k(t)=c_2$ admits the infinitesimal generator given by $\zeta^* =\Phi_1(t)\frac{\partial}{\partial t}+\Psi_1(t,\ x)\frac{\partial}{\partial x}$, where $\Phi_1(t)$, solves $\int^{\omega(t)}\frac{c_2}{E\tan A}\mathrm{d}\theta-t-c_9=0$ for $\omega(t)$ and A is a root (or zero) of $\left[B\ln\left(\frac{B^2(1+\tan^2 y)}{c_2\theta}\right)+D+2c_3y\right]$ for y with $B=\sqrt{2c_7c_2^2-c_3^2}$, $D=c_8B$, $E=c_3+B$ and $\Psi_1(t,x)=\frac{1}{2}\left[(\Phi_1(t))_t+c_1\right]x+\rho(t)$.

Proof. If $c_3\neq 0$, then substituting equation (34) into equation (32), we get

$$c_2\omega\omega_{ttt}+c_3\omega_{tt}=0. \tag{44}$$

This is a non-linear third-order differential equation, the solution $\omega(t)$ of which is given by

$$\int^{\omega(t)}\frac{c_2}{E\tan A}d\theta-t-c_9=0, \tag{45}$$

where A is a root (or zero) of $\left[B\ln\left(\dfrac{B^2(1+\tan^2 y)}{c_2\theta}\right)+D+2c_3 y\right]$ for y with $B=\sqrt{2c_7c_2^2-c_3^2}$, $D=c_8B$ and $E=c_3+B$.

It is to be noted that the expression in equation (45) may be complex valued, and we are finding the zeroes for y. In this solution, c_7, c_8, and c_9 are arbitrary constants. To obtain the corresponding infinitesimal generator, we have to solve equation (45) for $\omega(t)$. The infinitesimal generator in this case is given by

$$\zeta^*(t,\ x)=\Phi_1(t)\frac{\partial}{\partial t}+\Psi_1(t,\ x)\frac{\partial}{\partial x}, \tag{46}$$

where $\Phi_1(t)$ solves equation (45) for $\omega(t)$ and $\Psi_1(t,\ x)=\frac{1}{2}\left[(\Phi_1(t))_t+c_1\right]x+\rho(t)$.

Remark 4.6 In the above, we see that ζ^* is not easy to solve in general. So, choosing $B=0$, that is $k(t)=\dfrac{c_3}{\sqrt{2c_7}}$, we see that $\omega(t,\ x)=\dfrac{c_3}{c_2}(t+c_{10})$ solves equation (45). But the condition $\omega=\Omega$ gives $c_3=0$.

Consequently, $\omega(t,\ x)=0$, $\Upsilon(t,\ x)=\frac{1}{2}c_1x+\rho(t)$ and the infinitesimal generator is given by

$$\zeta^*(t,\ x)=\frac{1}{2}x\frac{\partial}{\partial x}+\rho(t)\frac{\partial}{\partial x}. \tag{47}$$

By considering a very special case in which $c_2=1=c_3$, we obtain from equation (32)

$$\omega\omega_{ttt}+\omega_{tt}=0. \tag{48}$$

Equation (48) yields a solution for which some infinitesimal generators can be explicitly found. This solution, in which c_{11}, c_{12}, and c_{13} are arbitrary constants, is given by

$$\int^{\omega(t)}\frac{d\theta}{1+c_{11}\tan G}-t-c_{13}=0, \tag{49}$$

where G is a root (or zero) of $\left[\ln\left(\dfrac{c_{11}^2}{\cos^2 y}\right)c_{11}-c_{11}\ln\theta+c_{11}c_{12}+2y\right]$ for y.

The infinitesimal generator in this case is

$$\zeta^*(t,\ x)=\Phi_2(t)\frac{\partial}{\partial t}+\Psi_2(t,\ x)\frac{\partial}{\partial x}, \tag{50}$$

where $\Phi_2(t)$ solves equation (49) for $\omega(t)$ and $\Psi_2(t,\ x)=\frac{1}{2}\left[(\Phi_2(t))_t+c_1\right]x+\rho(t)$.

Corollary 4.7 The NDE (10) with $b(t)\neq 0$, $d(t)=0$, $k(t)=1=c_3$, $c_{11}=0$ admits the three-dimensional group given by

$$\zeta_1^*=\frac{\partial}{\partial t},\ \zeta_2^*=\frac{x}{2}\frac{\partial}{\partial x},\ \zeta_3^*=\rho(t)\frac{\partial}{\partial x}.$$

Proof. It can be easily seen that the generators corresponding to $c_{11}=0$ can be explicitly obtained. In this case, $\omega(t,\ x)=c_{14}t+c_{15}$ is a solution to equation (48), where c_{14} and c_{15} are arbitrary constants. The condition $\omega=\Omega$ implies $c_{14}=0$. Hence, $\omega(t,\ x)=c_{15}$ and $\Upsilon(t,\ x)=\frac{c_1}{2}x+\rho(t)$.

If $c_{15} \neq 0$, then infinitesimal generator is given by

$$\zeta^*(t, x) = c_{15}\frac{\partial}{\partial t} + \frac{1}{2}c_1 x\frac{\partial}{\partial x} + \rho(t)\frac{\partial}{\partial x}. \tag{51}$$

If $c_{15} = 0$, then the infinitesimal generator is given by equation (47). Finally, if $c_3 = 0$, then the infinitesimal generator is given by equation (43). It may be noted that, using equation (39), $c(t) = \frac{1}{4}\frac{c_6 c_3^2}{c_{15}^2}$.

Theorem 4.8 The NDE given by equation (10) for which $b(t) = 0$, $d(t) \neq 0$, $k(t) = c_2$ admits the infinitesimal generator given by

$$\zeta^* = \Phi_3(t)\frac{\partial}{\partial t} + \Psi_3(t, x)\frac{\partial}{\partial x},$$

where $\Phi_3(t)$ solves $c_2\omega\omega_{tt} - c_2\frac{\omega_t^2}{2} + 2\omega^2(t)d(t) = c_{16}$, for $\omega(t)$ and $\Psi_3(t, x) = \frac{1}{2}\left[(\Phi_3(t))_t + c_1\right]x + \rho(t)$.

Proof. From equation (32), we get

$$c_2\omega_{ttt} + 2d'(t)\omega(t) + 4d(t)\omega_t = 0. \tag{52}$$

Equation (52) can be integrated once to obtain

$$c_2\omega\omega_{tt} - c_2\frac{\omega_t^2}{2} + 2\omega^2(t)d(t) = c_{16}, \tag{53}$$

where c_{16} is an arbitrary constant.

Equation (53) is extremely difficult to solve for an arbitrary *d*(*t*). If $\omega(t) = \Phi_3(t)$ solves equation (53), then the infinitesimal generator in this case is given by

$$\zeta^*(t, x) = \Phi_3(t)\frac{\partial}{\partial t} + \Psi_3(t, x)\frac{\partial}{\partial x}, \tag{54}$$

where $\Phi_3(t)$ solves equation (53) for $\omega(t)$ and $\Psi_3(t, x) = \frac{1}{2}\left[(\Phi_2(t))_t + c_1\right]x + \rho(t)$.

Remark 4.9 As the infinitesimal generator cannot always be explicitly solved due to the arbitrariness of *d*(*t*), we shall choose a few explicit values of *d*(*t*).

Corollary 4.10 The NDE (10) with $b(t) = 0, d(t) = e^t, k(t) = c_2$ admits the five-dimensional Lie group generated by

$$\zeta_1^* = \left(J_0(2\lambda)\right)^2\frac{\partial}{\partial t} + \frac{xe^t}{\sqrt{k(t)e^t}}J_0(2\lambda)J_1(2\lambda)\frac{\partial}{\partial x},$$

$$\zeta_2^* = \left(Y_0(2\lambda)\right)^2\frac{\partial}{\partial t} - \frac{xe^t}{\sqrt{k(t)e^t}}Y_0(2\lambda)Y_1(2\lambda)\frac{\partial}{\partial x},$$

$$\zeta_3^* = J_0(2\lambda)Y_0(2\lambda)\frac{\partial}{\partial t} - \frac{xe^t}{\sqrt{k(t)e^t}}\left(J_1(2\lambda)Y_0(2\lambda) + J_0(2\lambda)Y_1(2\lambda)\right)\frac{\partial}{\partial x},$$

$$\zeta_4^* = \frac{x}{2}\frac{\partial}{\partial x} \text{ and } \zeta_5^* = \rho(t)\frac{\partial}{\partial x}, \text{ where } \lambda = \frac{\sqrt{k(t)e^t}}{k(t)}.$$

Proof. Taking $d(t) = e^t$, equation (53) becomes $c_2\omega\omega_{tt} - c_2\frac{\omega_t^2}{2} + 2e^t\omega^2(t) = c_{16}$, which can be solved to give

$$\omega(t) = \frac{1}{4}\frac{c_{23}{}^2(1+2c_{16})}{c_{22}}\left(J_0\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)\right)^2 + c_{22}\left(Y_0\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)\right)^2 + c_{23}J_0\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)Y_0\left(2\frac{\sqrt{c_2e^t}}{c_2}\right), \tag{55}$$

where c_{22} and c_{23} are arbitrary constants. From equation (29), we get

$$\Upsilon(t, x) = \frac{1}{2}\left[\frac{-1}{2}\frac{e^tc_{23}{}^2(1+2c_{16})}{c_{22}\sqrt{c_2e^t}}J_0\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)J_1\left(2\frac{\sqrt{c_2e^t}}{c_2}\right) - 2\frac{c_{22}e^t}{\sqrt{c_2e^t}}Y_0\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)Y_1\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)\right.$$

$$\left. - \frac{c_{23}e^t}{\sqrt{c_2e^t}}J_1\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)Y_0\left(2\frac{\sqrt{c_2e^t}}{c_2}\right) - \frac{c_{23}e^t}{\sqrt{c_2e^t}}J_0\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)Y_1\left(2\frac{\sqrt{c_2e^t}}{c_2}\right) + c_1\right]x + \rho(t).$$

The infinitesimal generator is given by

$$\begin{aligned}\zeta^*(t, x) = {} & \frac{c_{23}^2(1+2c_{16})}{4c_{22}}\left[\left(J_0\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)\right)^2\frac{\partial}{\partial t} - \frac{xe^t}{\sqrt{c_2e^t}}J_0\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)J_1\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)\frac{\partial}{\partial x}\right] \\ & + c_{22}\left[\left(Y_0\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)\right)^2\frac{\partial}{\partial t} - \frac{xe^t}{\sqrt{c_2e^t}}Y_0\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)Y_1\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)\frac{\partial}{\partial x}\right] \\ & + c_{23}\left[J_0\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)Y_0\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)\frac{\partial}{\partial t} - \frac{xe^t}{\sqrt{c_2e^t}}\left(J_1\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)Y_0\left(2\frac{\sqrt{c_2e^t}}{c_2}\right) + J_0\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)Y_1\left(2\frac{\sqrt{c_2e^t}}{c_2}\right)\right)\frac{\partial}{\partial x}\right] \\ & + c_1\frac{x}{2}\frac{\partial}{\partial x} + \rho(t)\frac{\partial}{\partial x}.\end{aligned} \tag{56}$$

Remark 4.11 Corollary 4.10 finds applications in the theory of automatic control, in which the attached coefficients are variables.

Notation: Let $\tau = \left(0, -\frac{2}{k(t)}, \frac{-\pi}{4} + \frac{t}{2}\right)$, $M_C = \text{MathieuC}(\tau)$, $M_{C'} = \text{MathieuCPrime}(\tau)$,

$M_S = \text{MathieuS}(\tau)$ and $M_{S'} = \text{MathieuSPrime}(\tau)$.

Corollary 4.12 The NDE (10) with $b(t) = 0, d(t) = \sin t,\ k(t) = c_2$ admits the five-dimensional Lie group generated by

$$\zeta_1^* = (M_C)^2 \frac{\partial}{\partial t} + \frac{x}{2} M_C M_{C'} \frac{\partial}{\partial x},\ \zeta_2^* = (M_S)^2 \frac{\partial}{\partial t} + \frac{x}{2} M_S M_{S'} \frac{\partial}{\partial x},$$

$$\zeta_3^* = M_C M_S \frac{\partial}{\partial t} + \frac{x}{4} M_{C'} M_S + M_C M_{S'} \frac{\partial}{\partial x},\ \zeta_4^* = \frac{x}{2}\frac{\partial}{\partial x},\ \zeta_5^* = \rho(t)\frac{\partial}{\partial x}.$$

Proof. Taking $d(t) = \sin t$, equation (53) becomes $c_2 \omega\omega_{tt} - c_2 \frac{\omega_t^2}{2} + 2\omega^2(t)\sin t = c_{16}$, which can be solved to give

$$\omega(t) = \frac{1}{4}\frac{c_{26}{}^2}{c_{25}}(1+8c_{16})\left(M_C\right)^2 + c_{25}\left(M_S\right)^2 + c_{26} M_C M_S, \tag{57}$$

where c_{25} and c_{26} are arbitrary constants. Using equation (57), equation (29) gives

$$\Upsilon(t,\ x) = \frac{1}{2}\left[\frac{1}{4}\frac{c_{26}{}^2}{c_{25}}(1+8c_{16}) M_C M_{C'} + c_{25} M_S M_{S'} + \frac{1}{2} c_{26} M_{C'} M_S + \frac{1}{2} c_{26} M_C M_{S'} + c_1\right] x + \rho(t).$$

The infinitesimal generator in this case is explicitly given by

$$\begin{aligned} \zeta^*(t,\ x) = {} & \frac{1}{4}\frac{c_{26}^2}{c_{25}}(1+8c_{16})\left[(M_C)^2\frac{\partial}{\partial t} + M_C M_{C'}\frac{\partial}{\partial x}\right] + c_{25}\left[(M_S)^2\frac{\partial}{\partial t} + \frac{x}{2} M_S M_{S'}\right] \\ & + c_{26}\left[M_C M_S \frac{\partial}{\partial t} + \frac{x}{4}(M_{C'} M_S + M_C M_{S'})\frac{\partial}{\partial x}\right] + c_1 \frac{x}{2}\frac{\partial}{\partial x} + \rho(t)\frac{\partial}{\partial x}. \end{aligned} \tag{58}$$

Remark 4.13 Corollary 4.12 is extremely important in studying the properties of solutions of models arising in the postural balance models for humans described in [26].

Corollary 4.14 The NDE given by equation (10) for which $b(t) = 0$, $d(t) = t^m$ where m is any constant, $k(t) = c_2$ admits the five-dimensional Lie group generated by

$$\zeta_1^* = t(J_\nu(\mu))^2 \frac{\partial}{\partial t} + x\left(\frac{1}{2}(J_\nu(\mu))^2 + \frac{2}{m+2} J_\nu(\mu)\left(-J_{\nu+1}(\mu) + \frac{J_\nu(\mu)}{2\tau}\right)\tau(m/2+1)\right)\frac{\partial}{\partial x},$$

$$\zeta_2^* = t(Y_\nu(\mu))^2 \frac{\partial}{\partial t} + x\left(\frac{1}{2}(Y_\nu(\mu))^2 + \frac{2}{m+2} Y_\nu(\mu)\left(-Y_{\nu+1}(\mu) + \frac{Y_\nu(\mu)}{2\tau}\right)\tau(m/2+1)\right)\frac{\partial}{\partial x},$$

$$\zeta_3^* = tJ_\nu(\mu)Y_\nu(\mu)\frac{\partial}{\partial t} + x\left(\frac{1}{2}J_\nu(\mu)Y_\nu(\mu) + \frac{1}{m+2}\left(-J_{\nu+1}(\mu) + \frac{J_\nu(\mu)}{2\tau}Y_\nu(\mu) + J_\nu(\mu)\left(-Y_{\nu+1}(\mu) + \frac{Y_\nu(\mu)}{2\tau}\right)\tau(m/2+1)\right)\right)\frac{\partial}{\partial x},$$

$$\zeta_4^* = \frac{x}{2}\frac{\partial}{\partial x} \text{ and } \zeta_5^* = \rho(t)\frac{\partial}{\partial x},$$

where $\nu=(m+2)^{-1}$, $\tau=\sqrt{(k(t))^{-1}}t^{m/2+1}$, $\mu=2\tau\nu$.

Proof. Taking $d(t)=t^m$, where m is any constant, equation (53) becomes $c_2\omega\omega_{tt}-c_2\dfrac{\omega_t^2}{2}+2\omega^2(t)t^m=c_{16}$, which can be solved to give

$$\omega(t,\ x)=\frac{1}{4}\frac{c_{29}{}^2t}{c_{28}}(1+2c_{16})\left(J_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\right)^2+c_{28}t\left(Y_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\right)^2$$
$$+c_{29}tJ_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)Y_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right),\tag{59}$$

where c_{28} and c_{29} are arbitrary constants. From equation (29), we get

$$\Upsilon(t,\ x)=\frac{1}{2}\left[\frac{1}{4}\frac{c_{29}{}^2}{c_{28}}(1+2c_{16})\left(J_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\right)^2+\frac{1+2c_{16}}{c_{28}(m+2)}\left(c_{29}^2J_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\left(-J_{(m+2)^{-1}+1}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\right.\right.\right.$$
$$\left.\left.+\frac{J_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)}{2\sqrt{c_2^{-1}}t^{m/2+1}}\right)\sqrt{c_2^{-1}}t^{m/2+1}\left(m/2+1\right)\right)+c_{28}\left(Y_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\right)^2$$
$$+\frac{1}{m+2}\left(4c_{28}Y_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\left(-Y_{(m+2)^{-1}+1}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\right.\right.$$
$$\left.\left.+\frac{Y_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)}{2\sqrt{c_2^{-1}}t^{m/2+1}}\right)\sqrt{c_2^{-1}}t^{m/2+1}\left(m/2+1\right)\right)+c_{29}J_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)Y_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)$$
$$+\frac{1}{m+2}\left(2c_{29}\left(-J_{(m+2)^{-1}+1}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)+\frac{J_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)}{2\sqrt{c_2^{-1}}t^{m/2+1}}\right)\sqrt{c_2^{-1}}t^{m/2+1}\left(m/2+1\right)Y_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\right.$$

$$+\frac{1}{m+2}\left(2c_{29}J_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\left(-Y_{(m+2)^{-1}+1}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)+\frac{Y_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)}{2\sqrt{c_2^{-1}}t^{m/2+1}}\right)\sqrt{c_2^{-1}}t^{m/2+1}\left(m/2+1\right)\right)+c_1\Bigg]$$

$$x+\rho(t).$$

The infinitesimal generator is given by

$$\zeta^*(t,\ x)=\frac{1}{4}\frac{c_{29}^2}{c_{28}}(1+2c_{16})\left[t\left(J_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\right)^2\frac{\partial}{\partial t}+\left(\frac{x}{2}\left(J_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\right)^2+\frac{2x}{m+2}J_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\left(-J_{(m+2)^{-1}+1}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)+\frac{J_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)}{2\sqrt{c_2^{-1}}t^{m/2+1}}\right)\sqrt{c_2^{-1}}t^{m/2+1}\left(m/2+1\right)\right)\frac{\partial}{\partial x}\right]$$

$$+c_{28}\left[t\left(Y_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\right)^2\frac{\partial}{\partial t}+\left(\frac{x}{2}\left(Y_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\right)^2+\frac{2x}{m+2}Y_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\left(-Y_{(m+2)^{-1}+1}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)+\frac{Y_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)}{2\sqrt{c_2^{-1}}t^{m/2+1}}\right)\sqrt{c_2^{-1}}t^{m/2+1}\left(m/2+1\right)\right)\frac{\partial}{\partial x}\right]$$

$$
+c_{29}\left[\begin{array}{l}
tJ_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)Y_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\frac{\partial}{\partial t}+\left(\frac{x}{2}J_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)Y_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\right. \\
+\frac{x}{m+2}\left(-J_{(m+2)^{-1}+1}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)+\frac{J_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)}{2\sqrt{c_2^{-1}}t^{m/2+1}}\right)\sqrt{c_2^{-1}}t^{m/2+1}\left(m/2+1\right)Y_{(m+2)^{-1}} \\
\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)+\frac{x}{m+2}J_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)\left(-Y_{(m+2)^{-1}+1}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)+\frac{Y_{(m+2)^{-1}}\left(2\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\right)}{2\sqrt{c_2^{-1}}t^{m/2+1}}\right) \\
\left.\sqrt{c_2^{-1}}t^{m/2+1}\left(m/2+1\right)\right)\frac{\partial}{\partial x}
\end{array}\right] \tag{60}
$$

$$
+c_1\frac{x}{2}\frac{\partial}{\partial x}+\rho(t)\frac{\partial}{\partial x}.
$$

Remark 4.15 Corollary 4.14 is useful in the analysis of the Cauchy-Euler differential equation with neutral delay appearing in modeling time-harmonic vibrations of a thin elastic rod, problems on annual and solid discs, wave mechanics, etc.

5. Illustrative examples

Example 5.1 Consider the second order NDE given by $x''(t)+x''(t-\pi)=0$. A solution of this differential equation is $x(t)=\sin t$. Following the procedure given in the previous section, we can show that $\omega(t, x) = c_{38}$, a constant and $\Upsilon(t,\, x)=\frac{c_1}{2}x+\sin t$.

Solving the system $\frac{d\overline{t}}{d\delta}=\omega(\overline{t},\,\overline{x})=c_{38}$ and $\frac{d\overline{x}}{d\delta}=\Upsilon(\overline{t},\,\overline{x})=\frac{c_1}{2}x+\sin t$, subject to the conditions $\overline{t}=t$ and $\overline{x}=x$ when $\delta=0$, we get the above NDE invariant under the Lie group

$$
\overline{t}=t+c_{38}\delta,\ \overline{x}=\frac{2}{c_1}\left[e^{c_1\delta/2}\left(\frac{c_1}{2}x+\sin t\right)-\sin(t+c_{38}\delta)\right].
$$

The generators of the Lie group (or vector fields of the symmetry algebra) corresponding to this NDE are given by

$$
\zeta_1^*(t,\, x)=\frac{\partial}{\partial t},\ \zeta_2^*=x\frac{\partial}{\partial x},\ \text{and}\ \zeta_3^*=\sin t\frac{\partial}{\partial x}.
$$

Example 5.2 Consider the Cauchy problem $x'(t)=\int_{-r}^{0}x(s)ds$.

This is equivalent to the second order delay differential equation (special case of the NDE with $k(t)=0$) given by $x''(t)-x(t)+x(t-r)=0$.

Following the procedure in the previous section, we get $\omega(t, x) = c_{39}$, where $c_{39} = \sqrt{c_{37}}$ is a constant and $\Upsilon(t, x) = \frac{c_1}{2}x + \tilde{x}(t)$.

Solving the system $\frac{d\bar{t}}{d\delta} = \omega(\bar{t}, \bar{x}) = c_{39}$ and $\frac{d\bar{x}}{d\delta} = \Upsilon(\bar{t}, \bar{x}) = \frac{c_1}{2}x + \tilde{x}(t)$, subject to the conditions $\bar{t} = t$ and $\bar{x} = x$ when $\delta = 0$, we get the above NDE invariant under the Lie group

$$\bar{t} = t + c_{39}\delta,\ \bar{x} = \frac{2}{c_1}\left[e^{c_1\delta/2}\left(\frac{c_1}{2}x + \tilde{x}(t)\right) - \tilde{x}(t + c_{39}\delta)\right].$$

The generators of the Lie group (or vector fields of the symmetry algebra) corresponding to this delay differential equation are given by $\zeta_1^*(t, x) = \frac{\partial}{\partial t}$, $\zeta_2^* = x\frac{\partial}{\partial x}$, and $\zeta_3^* = \tilde{x}(t)\frac{\partial}{\partial x}$.

6. Conclusion

The approach presented in this paper has the potential to expand the current methods for studying second-order non-homogeneous FDEs with variable coefficients. By using Taylor's theorem for a function of several variables, we have derived a Lie invariance condition and the determining equations for these equations, which can lead to new insights and techniques for solving them. Additionally, our complete group classification of the second-order linear NDE can provide a foundation for further research in this area. Our study has presented a novel approach for studying second-order non-homogeneous FDEs with variable coefficients. Overall, our findings can contribute to the advancement of the field of differential equations. The research findings can be extended to include more choices of variable coefficients. This classification can be used in studying the properties of solutions arising in several models of practical importance, like those found in [27].

Conflict of interest

There is no conflict interest in this study.

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