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Research Article

Minimum Overlap Difference in Dual Masting of GSM Network Design

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Abstract: Generally, cell radio systems are enormous scale engineering designs and consist of various technical entities. They represent significant financial investments and hence there is the need for scientific design approach in network designs. This calls for finding a minimum overlap difference using a dual masting. In the process of finding the minimum cardinality of the disk in the Global System for Mobile Communication (GSM) cover problem, the octagon-square Archimedean tessellate was deduced by manipulating a four cusps hypocycloid known as Astroid. Since this Archimedean tessellate is formed with two different polygons, it becomes a possible consideration for the dual signal masting. The overlap difference using the octagon-square Archimedean tessellate was found to be 11.73%. This is a 12.46% reduction of the hexagonal tessellation with overlap difference of 13.4%. It was found that for any octagon-square tessellate, the circumcircle radius of the octagon is always approximately 1.847 longer than that of the square and the apothem of the octagon is approximately 2.414 longer than that of the square.

Keywords: network, dual masting, minimum cardinality, radio systems, communication, overlap ratios, node points

MSC: 94A13

1. Introduction

Since the beginning of the human race, effective communication is one of the fundamental surviving skills forged for human existence. Communication comes in different forms, but the underlying purpose is still held in its rightful place. Like a toddler start with sound, later on with words and full sentences as time passes. Our ability to communicate with one another is a gift and children learn this survival skill in a developmental stage. Communication takes alternative forms, a modest smile or a sophisticated speech can equally send the message that one is delighted. Animals also communicate, whales, and dolphins using high and low pitched sound travelling through a medium (water) can communicate over many kilometres [1].

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GSM radio receiving wires and base stations cannot be set wherever on the plane, their potential territory can be affected by a combination of the physical and money related matters. Along these lines, only a couple of urban regions (or parts of urban territories) appear as though a standard plane that can be exploited by as a potential location of a facility. It is eminent that there are two notable well-known tessellations of a plane with typical polygons of a comparative kind: the square and the hexagon [2].

A lot of structures rely upon those two notable regular polygons. The idea of using an Archimedean semi-regular tessellation has to do with its ability to provide smaller overlaps with different signal strengths.

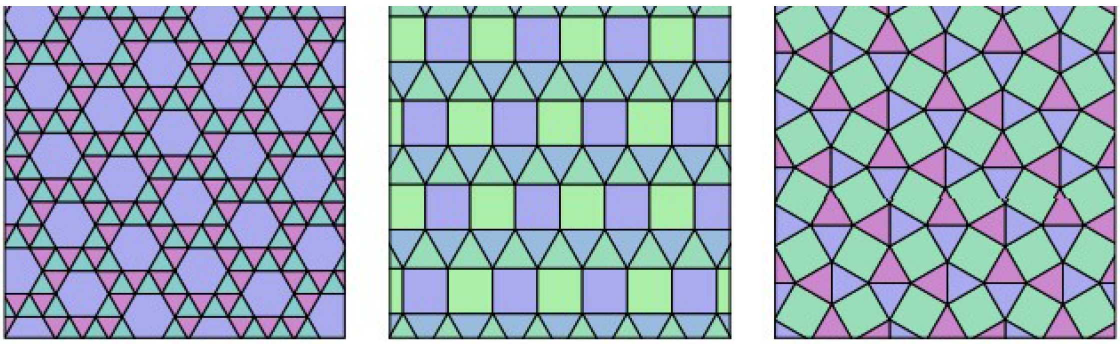
Two modes are possible in the transmission of signals in a telecommunication system, which is either by cable or electromagnetic radiation. The shortest path algorithms are applicable in the cable transmission where the shortest node and edge are optimally selected.

A special case of the cover set problem in a geometric setting is the geometric cover set problem. Given $\mathscr{O}$ called range as a family of subsets of the universe $\mathscr{R}$. If the range is defined as the intersection of $\mathscr{O}$ and other geometric shapes such as disks, then the special case of the cover set problem is to find the minimum size $\mathscr{C} \subseteq \mathscr{O}$ that will cover every point in $\mathscr{P}$, that is $\sum = (\mathscr{R}, \mathscr{O})$. In geometric covering problems, the aim then is to cover a set of given points in $\mathscr{R}$ using a minimum number of shapes with prescribed properties [3].

There is a unique problem in geometry known widely as the minimum geometric disk cover problem. The problem is coined as given an input which consists of a set of points, how then does one find a set of unit disks with the least cardinality whose union covers the points. Disk centres may be placed at arbitrary points and not unnecessary to be selected from a given discrete set like in Discrete Unit Disk Cover [4].

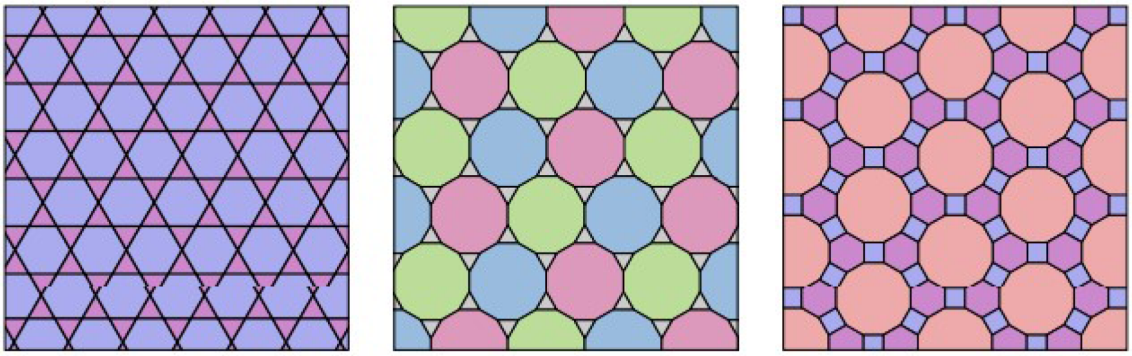
Due to the high capacity loading balance characteristics of wireless networks, its application is enormous and widely used both in military and civil applications. For a successful application, the wireless network must cover a specific or an entire operating area. In view of this, how to position these wireless transceivers to provide connected coverage becomes a critical research issue [5].

Xu and Song [6] studied several and different restricted coverage problems of which the first problem was about K-coverage. This problem was about how one can deploy wireless nodes such that each target is covered by at least K wireless nodes. Yang et al. [7] deployed a minimum number of Relay Stations to satisfy all data rate requests from subscriber stations through cooperative communications. According to [8], there exist eight semi-regular tessellations that is (Two involving triangles and squares; triangles and hexagons; one involving octagons and squares; triangles and dodecagons; squares, hexagons and dodecagons; and triangles, squares and hexagons) and three regular tessellations which is (equilateral triangle, squares and hexagon). According to [9], the study of less constrained polytopes is still largely in its infancy so they determined the minimal regular covers of three of the eight Archimedean tilings. In their second publication on minimal covers of Archimedean tilings, a new technique was developed in order to present the minimal regular covers of certain periodic abstract polytopes [10]. For each Archimedean tiling, a sequence of numbers to represent each of the eight tessellation. This sequence is constructed by considering the faces around each vertex of the covers and writing down the number of edges of each face as shown in Figure 1-3 [11].



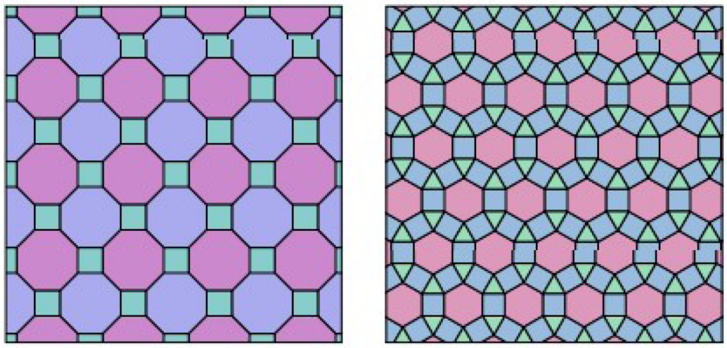
3.3.3.3.6　　3.3.3.4.4　　3.3.4.3.4

Figure 1. Triangles and hexagons; two triangles and squares



3.6.3.6 3.12.12 4.6.12

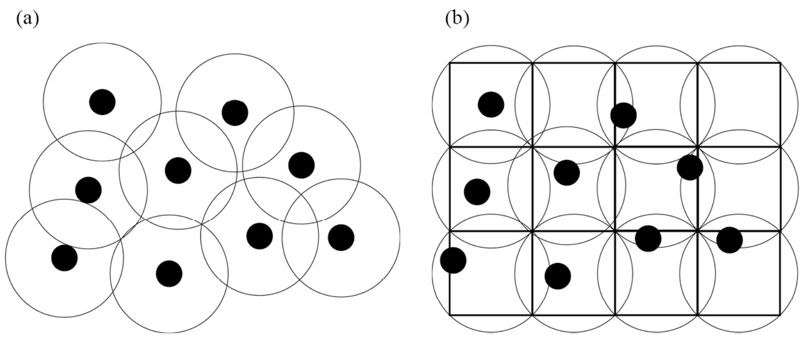
Figure 2. Triangles and hexagons; triangles and dodecagons; squares, hexagons and dodecagons



4.8.8 3.4.6.4

Figure 3. Octagons and squares; triangles, squares and hexagons

The Set Cover problem is inevitably one of the most important and best-known covering problems there is in geometry. Given a set of elements and a collection of sets, the Set Cover problem is to find the smallest sub-collection to cover all the elements. The Geometric Disk Cover problem represent a geometric version of the general cover set problem which is to check whether all points $\mathscr{P}$ is covered by at least one disk $\mathscr{D}$ with the minimum cardinality $\mathscr{D}^* \subseteq \mathscr{D}$ [4].



Disks covering points Square guided covering points

Figure 4. Cover set problem

It can be seen in Figure 4 that all points $\mathscr{P}$ is covered with at least one disk $\mathscr{D}$. This study as stated earlier only considers the electromagnetic mode of transmission. The graph $X = (A, M)$ is a subgraph of $Y = (B, N)$ if $A \subseteq B$ and $M \subseteq N$.

The degree noted as $\deg(\rho)$ of a vertex ρ is the number of its neighbours or simply the number of edges incident on the vertex. Given E as the number of edges and V as a vertex,

$$\sum_{\rho \in V} \deg \rho = 2|E| \tag{1}$$

Equation (1) is know as the Handshaking Lemma.

2. Computational experience

This area forms the gap (hypocycloid) when the cell (circles) intersects.

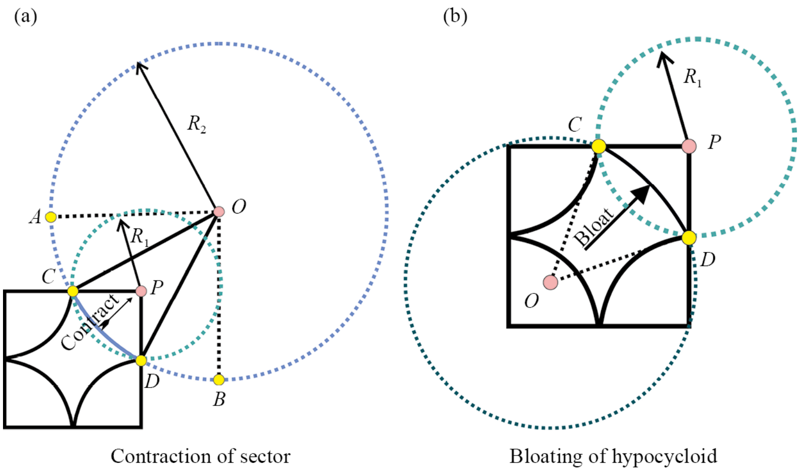


Figure 5. Contraction of sector and bloating of hypocycloid

In Figure 5, we can see two possible manipulations. That is either the sector is contracted inwards (Figure 5a) or the hypocycloid is bloated outwards (Figure 5b). When the sector is contracted the initial sector PCD becomes AOB since a new sector will be formed with properties of another circle. Again, bloating the hypocycloid forms a new sector COP. Among these two situations, the ideal one to create an overlap is bloating the hypocycloid to form a sector because contracting the sector will cause a gap in the signal. Therefore, the overlap will be the lens created by point CD in Figure 5a.

2.1 *Overlap difference in octagon-square inscribe disk*

In cell planning, the presence of overlaps ensures effective and smooth handover, especially in the GSM network. This overlap may have a differential effect which mainly includes fading and attenuation in signals. In the GSM system, this is a major problem and therefore the overlaps must be kept as minimal as possible. The purpose of overlap in the GSM system is to ensure smooth handover of cells (frequency) but also has a disadvantage that is; the presence of these overlaps increases the number of required GSM masts and antenna required in a given area. A typical overlap may arise as a result of both uniform cell radius (disks) and non-uniform cell radius. We shall however in this thesis minimize the overlap difference for both uniform and non-uniform cell range in mobile telephony.

In Figure 6a there exist two overlaps (A) and (B) with different dimensions. One of the overlaps occurs as two identical cells intercept at two points while the other occurs with the intersection of two different radii cells at two points.

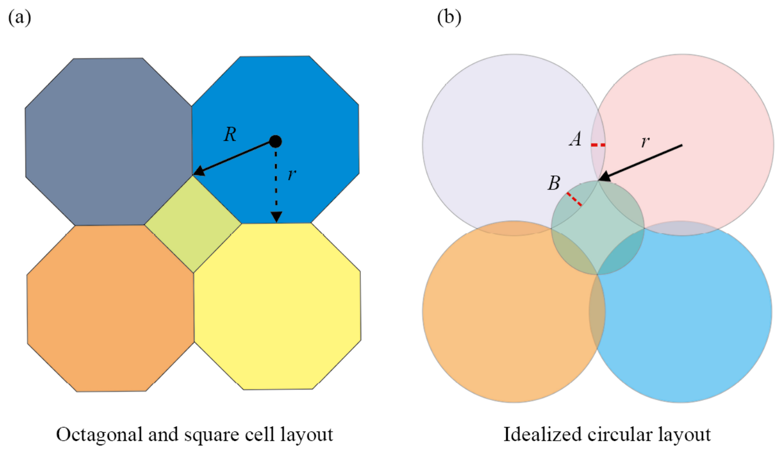


Figure 6. Cell layout models for GSM networks

Let r_n denote the apothem determined by the n sided regular polygon inscribed in a disk of radius R_1.

Theorem 1 The apothem $r_n$defined as above has the expression $r_n = R_1 Cos\left(\frac{\pi}{n}\right)$.

Proof. Consider a regular polygon with sides s inscribed in a circle as shown in Figure 7.

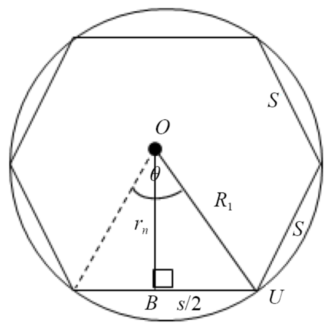


Figure 7. Apothem for regular polygon inscribed in disks

$$\tan\frac{\theta}{2} = \frac{s/2}{r_n}$$

$$\frac{s}{2} = r_n \tan\frac{\theta}{2} \tag{2}$$

$$s = 2r_n \tan\frac{\theta}{2}$$

We remark that the angle subtended between any two (2) line segments (originating from adjacent corners) at the centre of the circle will be equal and since there would be n unique angles totalling 2π, we have:

$$\underbrace{\theta + \theta + ... + \theta} = 2\pi$$

$$n\theta = 2\pi \tag{3}$$

$$\theta = \frac{2\pi}{n}$$

putting equation (3) into equation (2) we have

$$rs = 2r_n \tan\left(\frac{2\pi}{n} \div 2\right)$$

$$= 2r_n \tan\frac{\pi}{n}$$

From $\triangle$ OBU,

$$R_1^2 = r_n^2 + \left(\frac{s}{2}\right)^2$$

$$= r_n^2 + \left(\frac{2r_n \tan\frac{\pi}{n}}{2}\right)^2$$

$$= r_n^2 \left[1 + \tan^2\left(\frac{\pi}{n}\right)\right]$$

$$= r_n^2 \sec^2\left(\frac{\pi}{n}\right) \tag{4}$$

$$R_1 = r_n \sec\left(\frac{\pi}{n}\right)$$

$$r_n = \frac{R_1}{\sec\left(\frac{\pi}{n}\right)}$$

$$r_n = R_1 \cos\left(\frac{\pi}{n}\right)$$

□

Let us calculate the apothem in three particular cases as shown in Figure 8:

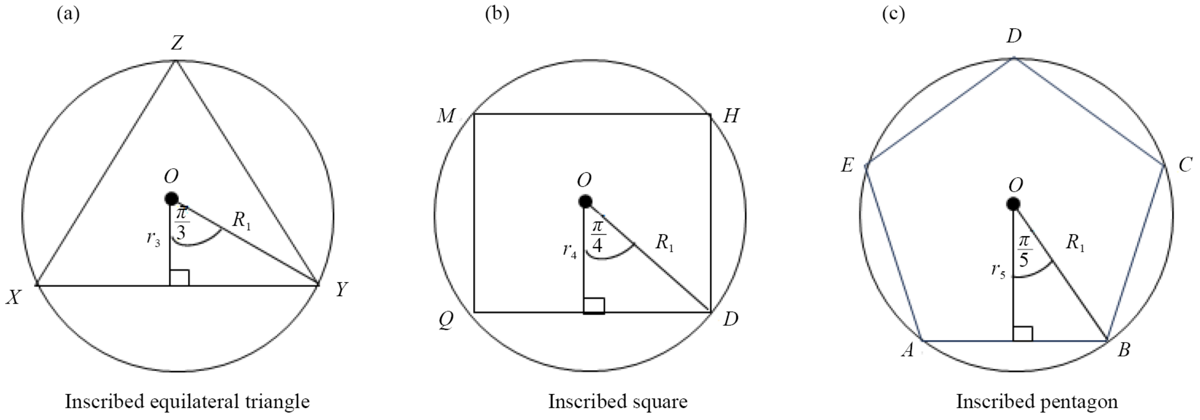


Figure 8. Inscribed regular polygons

Case I: Equilateral Triangle XYZ.
Consider the right triangle in Figure 8a.

$$\cos\left(\frac{\pi}{3}\right) = \frac{r_3}{R_1}$$

$$r_3 = R_1 \cos\left(\frac{\pi}{3}\right)$$

$$= R_1 \times \frac{1}{2}$$

and this gives

$$r_3 = \frac{1}{2} R_1$$

Case II: Square $QDHM$ $(n = 4)$.
Consider the right triangle in Figure 8b.

$$\cos\left(\frac{\pi}{4}\right) = \frac{r_4}{R_1}$$

and this also gives;

$$r_4 = R_1 \cos \frac{\pi}{4}$$

Case III: Pentagon $ABCDE$.

Consider the right triangle in Figure 8c.

$$\cos\left(\frac{\pi}{5}\right) = \frac{r_5}{R_1}$$

and this gives.

$$r_5 = R_1 \cos\frac{\pi}{5}$$

Remarks 1 Geometrical construction strongly confirms that for all $n \geq 3$, $r_n = R_1 \cos\left(\frac{\pi}{n}\right)$.

The apothem formulae in equation (4) lessen computation and helps us propose the total overlap difference formula in Theorem 2.

Theorem 2 The one-dimensional total overlap difference created by two different $n(n_1,\ n_2)$ sided tessellated regular polygons (octagon and square) inscribed in a disk for covering with radius $R(R_1,\ R_2)$ is

$$n_1 R_1\left[1-\cos\left(\frac{\pi}{n_1}\right)\right] + \frac{n_1}{2} R_1\left[1-\cos\left(\frac{\pi}{n_1}\right)\right] + \frac{n_1}{2} R_2\left[1-\cos\left(\frac{\pi}{n_2}\right)\right]$$

Each overlap difference is $2R_1\left[1-\cos\left(\frac{\pi}{n_1}\right)\right]$ and $R_1\left[1-\cos\left(\frac{\pi}{n_1}\right)\right] + R_2\left[1-\cos\left(\frac{\pi}{n_2}\right)\right]$.

Consider the two geometrical shapes in Figure 9a and Figure 9b showing the overlap difference of octagon-octagon tessellation and octagon-square tessellation respectively.

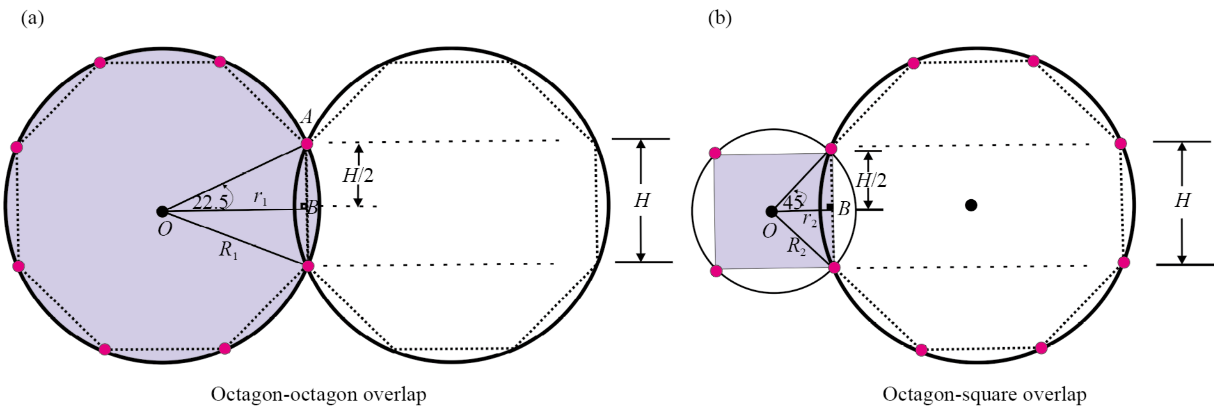


Figure 9. Overlap difference for uniform and non-uniform disks

Proof. Let d denote the one-dimensional overlap difference and n_1 and n_2 be the sides of a regular octagon and square respectively. If R_1 and R_2 are respectively the radii of the Octagon and Square, then;

$$d = d_a + d_b$$

Case I: Overlaps between two octagons

$$
\begin{aligned}
d_a &= 2(R_1 - r_1) \\
&= 2\left[R_1 - R_1 \cos\left(\frac{\pi}{n_1}\right)\right] \\
&= 2R_1\left[1 - \cos\left(\frac{\pi}{n_1}\right)\right]
\end{aligned} \tag{5}
$$

Case II: Overlaps between octagon-square

$$
\begin{aligned}
d_b &= (R_1 - r_1) + (R_2 - r_2) \\
&= \left[R_1 - R_1 \cos\left(\frac{\pi}{n_1}\right)\right] + \left[R_2 - R_2 \cos\left(\frac{\pi}{n_2}\right)\right] \\
&= R_1\left[1 - \cos\left(\frac{\pi}{n_1}\right)\right] + R_2\left[1 - \cos\left(\frac{\pi}{n_2}\right)\right]
\end{aligned} \tag{6}
$$

But a tessellable two regular polygons with n different sides have n overlaps. Where n is the side for the larger sided polygon, $n = n_1$.

$$
\begin{aligned}
d &= \frac{n_1}{2}(d_a + d_b) \\
&= \frac{n_1}{2}\left[2R_1\left(1 - \cos\left(\frac{\pi}{n_1}\right)\right) + R_1\left(1 - \cos(\frac{\pi}{n_1})\right) + R_2\left(1 - \cos\left(\frac{\pi}{n_2}\right)\right)\right] \\
&= n_1 R_1\left(1 - \cos\left(\frac{\pi}{n_1}\right)\right) + \frac{n_1}{2} R_1\left(1 - \cos\left(\frac{\pi}{n_1}\right)\right) + \frac{n_1}{2} R_2\left(1 - \cos\left(\frac{\pi}{n_2}\right)\right)
\end{aligned}
$$

□

Theorem 3 The total overlap area created by $n(n_1,\ n_2)$ sided tessellable regular polygons (octagon and square) inscribed in a disks for covering of uniform radius $R(R_1,\ R_2)$ is $4A_1 + 4A_2$, where

$$
A_1 = \frac{1}{4}\left[\pi - 4\sin\left(\frac{\pi}{4}\right)\right]R_1^2 \text{ and } A_2 = \frac{1}{4}\left[\left[\pi - 2\sin\left(\frac{\pi}{2}\right)\right]R_2^2 + \frac{1}{2}\left[\pi - 4\sin\left(\frac{\pi}{4}\right)\right]R_1^2\right]
$$

Proof. We let n_1 be the sides of the largest sided figure, in this case the octagon and let n_2 be the side of the square. Respectively, R_1 and R_2 are the uniform radius inscribing the octagon and square.

$$\frac{\text{Area of circle-Area of polygon}}{n} \times 2 \tag{7}$$

Now since the overlaps are formed with two different polygons, the overlapping area will likewise be different. An octagon having eight sides will have four of is sides overlapping with an identical figure while the other four overlap with the square. From Equation (7), the area of each octagon-octagon overlap is,

$$A_1 = \frac{\text{Area of circle-Area of octagon}}{8} \times 2$$

$$= \frac{\text{Area of circle-Area of octagon}}{4}$$

Again the area of each octagon-square overlap is

$$A_2 = \frac{\text{Area of circle-Area of square}}{4} + \frac{A_1}{2}$$

Case I: Octagon-Octagon

$$\begin{aligned} A_1 &= \frac{\text{Area of circle-Area of octagon}}{4} \\ &= \frac{1}{4}\left[\pi R_1^2 - \frac{nR_1^2}{2}\sin\left(\frac{2\pi}{n}\right)\right] \\ &= \frac{1}{4}\left[\pi - \frac{8}{2}\sin\left(\frac{2\pi}{8}\right)\right]R_1^2 \\ &= \frac{1}{4}\left[\pi - 4\sin\left(\frac{\pi}{4}\right)\right]R_1^2 \end{aligned} \tag{8}$$

Case II: Octagon-Square

$$A_2 = \frac{\text{Area of circle-Area of square}}{4} + \frac{A_1}{2}$$

$$= \frac{1}{4}\left[\pi R_2^2 - \frac{nR_2^2}{2}\sin\left(\frac{2\pi}{n}\right)\right] + \frac{A_1}{2}$$

$$= \frac{1}{4}\left[\pi R_2^2 - 2R_2^2\sin\left(\frac{\pi}{2}\right)\right] + \frac{A_1}{2}$$

$$= \frac{1}{4}\left[\pi - 2\sin\left(\frac{\pi}{2}\right)\right]R_2^2 + \frac{A_1}{2}$$

$$= \frac{1}{4}\left[\pi - 2\sin\left(\frac{\pi}{2}\right)\right]R_2^2 + \frac{\frac{1}{4}\left[\pi - 4\sin\left(\frac{\pi}{4}\right)\right]R_1^2}{2}$$

$$= \frac{1}{4}\left[\left[\pi - 2\sin\left(\frac{\pi}{2}\right)\right]R_2^2 + \frac{1}{2}\left[\pi - 4\sin\left(\frac{\pi}{4}\right)\right]R_1^2\right]$$

Case III: Total overlap

$$\begin{aligned} &= 4A_1 + 4\left(\frac{A_1}{2}\right) + 4\left(A_2 - \frac{A_1}{2}\right) \\ &= 4A_1 + 4A_2 \end{aligned} \tag{9}$$

□

Generally, for an n_1 sided octagon and n_2 sided squares in an Archimedean tessellate (as shown in 4.8.8 in Figure 3), n overlays will have area as

$$A = n \times [4A_1 + 4A_2]$$

$$= 4n[A_1 + A_2]$$

$$= 4n\left\{\frac{2}{n_1}\left[\pi - \frac{n_1}{2}\sin\left(\frac{2\pi}{n_1}\right)\right]R_1^2 + \frac{1}{n_2}\left[\pi - \frac{n_2}{2}\sin\left(\frac{2\pi}{n_2}\right)\right]R_2^2 + \frac{2}{n_1}\left[\pi - \frac{n_1}{2}\sin\left(\frac{2\pi}{n_1}\right)\right]R_1^2 \div 2\right\}$$

$$= 4n\left\{\left[\pi - \frac{n_1}{2}\sin\left(\frac{2\pi}{n_1}\right)\right]R_1^2\left(\frac{2}{n_1} + \frac{1}{n_1}\right) + \frac{1}{n_2}\left[\pi - \frac{n_2}{2}\sin\left(\frac{2\pi}{n_2}\right)\right]R_2^2\right\}$$

$$= 4n\left\{\frac{3}{n_1}\left[\pi - \frac{n_1}{2}\sin\left(\frac{2\pi}{n_1}\right)\right]R_1^2 + \frac{1}{n_2}\left[\pi - \frac{n_2}{2}\sin\left(\frac{2\pi}{n_2}\right)\right]R_2^2\right\}$$

Table 1. Total overlap area for some regular polygon

	Square (S)	Hexagon (H)	Octagon (O)
$2\left[\pi - \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)\right]R^2$	$2[\pi - 2]R^2$	$\left[2\pi - 3\sqrt{3}\right]R^2$	$2\left[\pi - 2\sqrt{2}\right]R^2$
Area of disks (D)	πR^2	πR^2	πR^2
Total occupying area ratio	72.7%	34.6%	19.93%

Table 1 gives the summary of the overlap ratio. Using a circumcircle radius of R, the square have the highest overlap ratio, followed by the hexagon and then an octagon. Using equal radius, we can intuitively say that combining the octagon and square will give a higher overlap than the hexagon. Be that as it may, in our tessellation the radius of the square is always smaller as compared to the octagon. This is proved in Theorem 4.

Theorem 4 For any octagon-square tessellate, the circumcircle radius of the octagon is always $\approxeq 1.847$ longer than the circumcircle radius of the square and the apothem of the octagon is $\approxeq 2.414$ longer than the apothem of the square.

Proof. Figure 10 shows the Octagon-square Archimedean tessellate geometric figure.

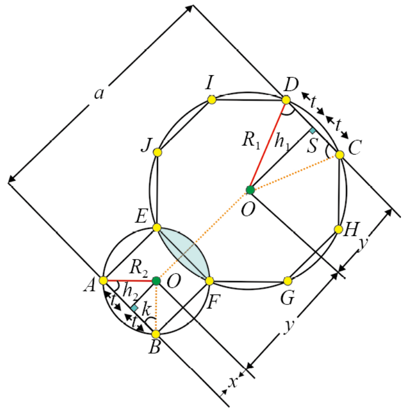


Figure 10. Octagon-square geometric figure

For an octagon and a square be use in covering a plane, the length of the side of the square must be the same for the length of all the sides of the octagon. Therefore the sides AB, AE, EF, and FB must be of the same length as EF, FG, GH, HC, CD, DI, IJ, and JE of the octagon. The longest length of the figure is the sum of apothem of the square OK, the distance between the centres of the two polygons and the apothem of the octagon OS ie. $(a = x + y + g)$. Since the length of each sides of both figure are the same, this is always the case.

Consider the octagon $CDIJEFGH$, we can deduce from the isosceles $\triangle DOC$ that $\angle DOC$ is 45°, $\angle ODS$ is 67.5°, $\angle OCS$ is 67.5° therefore $\angle DOS$ is 22.5°. $\sin(67.5) = \frac{h_1}{R_1}$, $\cos(67.5) = \frac{t}{R_1}$, $\tan(67.5) = \frac{h_1}{t}$.

Again, consider the square $ABFE$, we can deduce from the isosceles $\triangle AOB$ that $\angle AOB$ is 90°, $\angle OAB$ is 45°, $\angle OBA$ is 45° therefore $\angle KOB$ is 45°. $\sin(45) = \frac{h_2}{R_2}$, $\cos(45) = \frac{t}{R_2}$, $\tan(45) = \frac{h_2}{t}$.

Case I: Radius

$$\cos(67.5) = \frac{t}{R_1} \implies t = R_1 \cos(67.5) \tag{10}$$

$$\cos(45) = \frac{t}{R_2} \implies t = R_2 \cos(45) \tag{11}$$

Equating Equations (10) and (11),

$$R_1 \cos(67.5) = R_2 \cos(45)$$

$$\frac{R_1}{R_2} = \frac{\cos(45)}{\cos(67.5)} \approxeq 1.847$$

$$\therefore R_1 = 1.847 R_2$$

Case II: Apothem

$$\tan(67.5) = \frac{h_1}{t} \implies t = \frac{h_1}{\tan(67.5)} \tag{12}$$

$$\tan(45) = \frac{h_2}{t} \implies t = \frac{h_2}{\tan(45)} \tag{13}$$

Equating Equations (12) and (13),

$$\frac{h_1}{\tan(67.5)} = \frac{h_2}{\tan(45)}$$

$$\frac{h_1}{h_2} = \frac{\tan(67.5)}{\tan(45)} \approxeq 2.414$$

$$\therefore h_1 = 2.414 h_2$$

□

It has been established that to cover a given plane or point sets with disks of radius R_1 and Octagonal apothem r_1, we require an overlap difference of $2(R_1 - r_1)$. We shall, however, deduce formulas for calculating the width of any octagonal disks covering in terms of the apothem (r_1) or the radius of the disks (R_1) or the height of the overlap area (H). Consider two intersecting uniform disks shown in Figure 11. Consider triangle AOB in Figure 11.

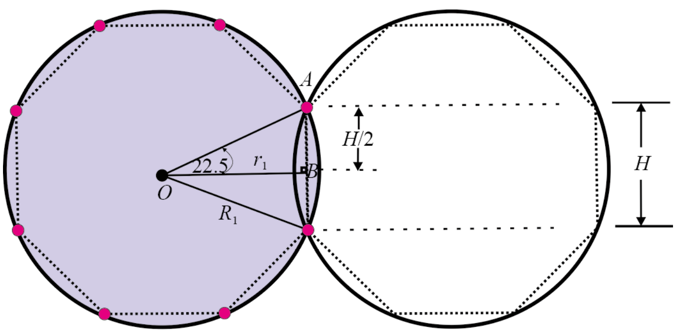


Figure 11. Overlap width for uniform disks (cell radius)

$$\cos\left(\frac{\pi}{8}\right) = \frac{OB}{OA}$$

$$\cos\left(\frac{\pi}{8}\right) = \frac{r_1}{R_1}$$

$$r_1 = \cos\left(\frac{\pi}{8}\right) R1$$

$$r_1 = 0.9239 R_1 \tag{14}$$

$$\text{Width} = 2(R_1 - r_1)$$

$$\therefore \text{Width} = 2(R_1 - 0.9239 R_1)$$

$$= 2(1 - 0.9239) R_1$$

$$= 2(1 - 0.9239) 1.3065 H$$

Generally for n full overlaps the difference is obtain to be

$$d = \sum_{k=1}^{n} 2(1 - 0.9239) R_k$$

$$= 2(1 - 0.9239) \sum_{k=1}^{n} R_k$$

But $R_k = R_q$ for all $k,\ q \in \mathbb{N}$, hence

$$d = 2(1 - 0.9239) \frac{n}{2} R_k \tag{15}$$

Equation (14) establish the formula for calculating the width of a disks covering via octagonal tilling. Again, to cover a given point sets with disks of radii (R_1, R_2) such the $R_1 > R_2$ and Octagonal and Square apothems (r_1, r_2), it has been established that we require an overlap difference of $(R_2 - r_2) + (R_1 - r_1)$. Consider two intersecting uniform disks shown in Figure 12.

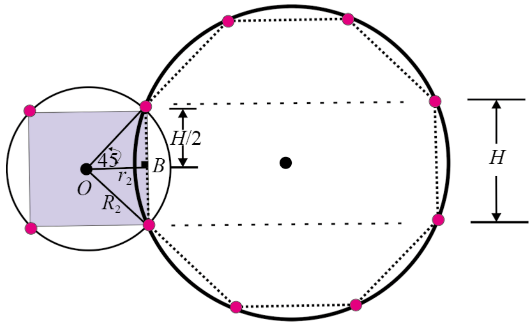


Figure 12. Overlap width for non-uniform disks (cell radius)

Consider triangle AOB in Figure 12.

$$\cos\left(\frac{\pi}{4}\right) = \frac{OB}{OA}$$

$$\cos\left(\frac{\pi}{4}\right) = \frac{r_2}{R_2}$$

$$r_2 = \left(\frac{\sqrt{2}}{2}\right)R_2$$

$$\text{Width} = (R_1 - r_1) + (R_2 - r_2)$$

From Equation (14) $r_1 = \cos\left(\frac{\pi}{8}\right)R_1$,

$$\begin{aligned} \therefore \text{Width} &= \left(R_1 - 0.9239R_1\right) + \left(R_2 - \frac{\sqrt{2}}{2}R_2\right) \\ &= \left(1 - 0.9239\right)R_1 + \left(1 - \frac{\sqrt{2}}{2}\right)R_2 \end{aligned} \tag{16}$$

Generally for n full overlaps, R_t and R_p are the radius from the circumcircle of the octagon and square respectively then the difference is formulated as

$$d = \sum_{p=1}^{n} (1-0.9239)R_p + \sum_{t=1}^{n} \left(1-\frac{\sqrt{2}}{2}\right) R_t$$

$$= (1-0.9239) \sum_{p=1}^{n} R_p + \left(1-\frac{\sqrt{2}}{2}\right) \sum_{t=1}^{n} R_t \tag{17}$$

$$d = (1-0.9239)\frac{n}{2}R_p + \left(1-\frac{\sqrt{2}}{2}\right)\frac{n}{2}R_t$$

Equation (17) establish the formula for calculating the width of a disks covering via square-octagonal tilling.

2.2 *Overlap ratios*

For uniform octagon overlaps, $d = 2(1-0.9239)R$ from Equation (14).

$$0.\text{Ratio} = \frac{2(1-0.9239)R_1}{2R_1} \times 100\%$$
$$= 7.61\% \tag{18}$$

For non-uniform octagon-square overlaps, $\big(1-0.9239\big)R_1 + \left(1-\frac{\sqrt{2}}{2}\right)R_2$ and from Equation (17) and from Theorem 4 where $R_1 = 1.847R_2$.

$$\text{Ratio} = \frac{\big(1-0.9239\big)R_1 + \left(1-\frac{\sqrt{2}}{2}\right)R_2}{2R_1} \times 100\%$$

$$= \frac{\big(1-0.9239\big)R_1 + \left(1-\frac{\sqrt{2}}{2}\right)0.5414R_1}{2R_1} \times 100\% \tag{19}$$

$$= 11.73\%$$

Table 2. Summary of occupying overlap difference and ratio

Regular polygon	Octagon	Oct-square	Hexagon	Square
Ratio	7.61%	11.73%	13.397%	29.289%

Octagon-square tessellate have an overlap ratio lesser than 13.39% for each possible side in Table 2.

3. Discussion

It has been established that using octagon-square tessellate provides a dual mast with the least overlaps as compared to the hexagons from [12]. In comparing overlap difference of some regular polygons, the total occupying area ratio of a square tessellate was estimated to 72.7%, hexagon tessellate of 34.6% and octagon was 19.93%. The overlap ration of octagon-octagon side overlap was found to be 7.61% and an octagon-square ratio was 11.73%. It is established that for any octagon-square tessellate, the circumcircle radius of the octagon is always $\approxeq$ 1.847 longer than the circumcircle radius of the square where the apothem of the octagon is $\approxeq$ 2.414 longer than the apothem of the square. This work also established that, the apothem r_n created by n sided regular polygon inscribed in a disk of radius R_1 is $r_n = R_1 \cos\left(\frac{\pi}{n}\right)$.

4. Conclusion

The findings in this study provide an optimal dual mast (disks) tessellation model using octagon-square tilling with least occupying overlap difference ratio of 11.73%. This is a 12.46% reduction of the existing hexagonal tesselation model proposed by [12]. We used geometry of Archimedean tilling approach to geometric disks covering of GSM masts to reach optimality and it is the first study that combines two tessellable non-uniform regular polygons to arrive at least covering with minimum overlap difference and ratio.

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Conflict of interest

Authors declare that there are no conflict of interest regarding the publication of this work.

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