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Research Article

An Algorithm for Solving Pseudomonotone Variational Inequality Problems in CAT(0) Spaces

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Abstract: In this paper, we presents strong and Δ-convergence results within the framework of CAT(0) space for pseudomonotone mappings. Additionally, we approximate the solution for variational inequality problems in the context of CAT(0) space for such mappings. Lastly, we provide a numerical example to highlight our main result.

***Keywords*:** fixed point, Hadamard space, nonexpansive mapping, Δ-convergence, strong convergence

MSC: 47H09, 47H10, 47J25

1. Introduction and preliminaries

Let σ be a metric and (Ω, σ) a metric space. A geodesic path joining $r \in \Omega$ to $s \in \Omega$ is a map φ from closed interval $[0, l] \subset \mathbb{R}$ to Ω with $l = \sigma(r, s)$ such that $\varphi(0) = r$, and $\varphi(l) = s$, $\sigma(\varphi(\varepsilon), \varphi(\varepsilon')) = |\varepsilon - \varepsilon'|$, for all $\varepsilon, \varepsilon' \in [0, l]$. The image $[r, s]$ of φ is called geodesic segment with endpoints r and s. If every pair of Ω points is connected by a geodesic segment, the pair (Ω, σ) is referred to as geodesic metric space.

A metric space (Ω, σ) is said to be uniquely geodesic if there is exactly one geodesic segment connecting r and s for each $r, s \in \Omega$. Let (Ω, σ) be a geodesic metric space. Geodesic triangle Δ [1] is formed by geodesics joining each pair out of three points on a given surface. In a metric space Ω, geodesic triangle Δ consists of three points $q_1, q_2, q_3 \in \Omega$. Its vertices and choice of three geodesic segments $[q_1, q_2]$, $[q_2, q_3]$, $[q_3, q_1]$ joining them its sides. A comparison triangle for a geodesic triangle $\Delta(q_1, q_2, q_3)$ in (Ω, σ) is the triangle $\overline{\Delta}(q_1, q_2, q_3) = \Delta(\overline{q_1}, \overline{q_2}, \overline{q_3})$ in the Euclidean space $\mathbb{E}^2$ such that $\sigma(q_i, q_j) = \sigma_{\mathbb{E}^2}(\overline{q_i}, \overline{q_j})$ for all $i, j = 1, 2, 3$.

Let (Ω, σ) be a metric space and κ be a real number. Let Δ be a geodesic triangle in Ω and $\overline{\Delta} \subset M_k^2$ be a comparison triangle for Δ. Then, Δ is said to satisfy the CAT(κ) inequality [2] if for all $r, s \in \Delta$ and all comparison points $\overline{r}, \overline{s} \in \overline{\Delta}$, we have $\sigma(r, s) \leq \sigma(\overline{r}, \overline{s})$. A geodesic metric space (Ω, σ) is said to be a CAT(0) space if for each geodesic triangle Δ in Ω and its comparison triangle $\overline{\Delta}$, the CAT(0) inequality (that is, $\sigma(r, s) \leq \sigma_{\mathbb{E}^2}(\overline{r}, \overline{s})$), is satisfied for all $r, s \in \Delta$ and $\overline{r}, \overline{s} \in \overline{\Delta}$.

Let p, q_1, q_2 be three points of CAT(0) space. If q_0 is the midpoint of the segment $[q_1, q_2]$, denoted by $\frac{(q_1 \oplus q_2)}{2}$, then the CAT(0) inequality yields

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$$\sigma^2(p, \frac{q_1 \oplus q_2}{2}) = \sigma^2(p, q_0) \leq \frac{1}{2}\sigma^2(p, q_1) + \frac{1}{2}\sigma^2(p, q_2) - \frac{1}{4}\sigma^2(q_1, q_2). \tag{1}$$

Then (1) referred as (CN) inequality.

A geodesically connected metric space is a CAT(0) space if (CN) inequality is satisfied. Complete CAT(0) spaces are often called Hadamard spaces and a subset Υ of a geodesic metric space (Ω, σ) is said to be convex if for any points v, $w \in \Upsilon$, the geodesic segment $[v, w]$ is contained in Υ.

The concept of quasilinearization for a CAT(0) space Ω was introduced by Berg and Nikolaev. They denote a pair $(v, w) \in \Omega \times \Omega$ by $\overrightarrow{vw}$ and call it a vector. Then quasilinearization map $\langle ., . \rangle : (\Omega \times \Omega) \times (\Omega \times \Omega) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{vw}, \overrightarrow{ez} \rangle = \frac{1}{2}[\sigma^2(v, z) + \sigma^2(w, e) - \sigma^2(v, e) - \sigma^2(w, z)],$$

for all $v, w, e, z \in \Omega$.

It is easily seen that

$$\langle \overrightarrow{vw}, \overrightarrow{ez} \rangle = \langle \overrightarrow{ez}, \overrightarrow{vw} \rangle,$$

$$\langle \overrightarrow{vw}, \overrightarrow{ez} \rangle = -\langle \overrightarrow{wv}, \overrightarrow{ez} \rangle,$$

$$\langle \overrightarrow{vw}, \overrightarrow{vw} \rangle = \sigma^2(v, w),$$

and

$$\langle \overrightarrow{vw}, \overrightarrow{ez} \rangle = \langle \overrightarrow{vx}, \overrightarrow{ez} \rangle + \langle \overrightarrow{xw}, \overrightarrow{ez} \rangle,$$

for all $v, w, e, z, x \in \Omega$. It is said that Ω satisfies the Cauchy-Schwarz inequality if and only if

$$\langle \overrightarrow{vw}, \overrightarrow{ez} \rangle \leq \sigma(v, w)\sigma(e, z), \ \forall \ v, w, e, z \in \Omega.$$

A geodesic space is a CAT(0) space. It goes without saying that the quasilinearization map $\langle \cdot, \cdot \rangle$ expands the idea of the inner product from Hilbert space to CAT(0) space Ω.

Let Ω be a complete CAT(0) space, and assume that (v_n) is a bounded sequence in a complete CAT(0) space Ω and $\pi(., (v_n)) : \Omega \to [0, \infty)$ is a continuous functional for $v \in \Omega$ defined as

$$\pi(v, (v_n)) = \limsup_{n\to\infty} \sigma(v, v_n).$$

The asymptotic radius $r((v_n))$ of (v_n) is given by

$$r((v_n)) = \inf\{\pi(v, (v_n)) : v \in \Omega\}.$$

The asymptotic center of the bounded sequence (v_n) in Ω is the set $A((v_n))$ defined as

$$A((v_n)) = \inf\{v \in \Omega : \pi(v, (v_n))\}$$

and let $\omega_w(v_n) = \cup A((v_n))$, where union is taken over all subsequences of (v_n). Remember that a bounded sequence (v_n) in Ω is considered to be regular [1] if $r((v_n)) = r((v_{n_k}))$ for each subsequence (v_{n_k}) of (v_n).

Assume a sequence (v_n) which is bounded in complete CAT(0) space (Ω, σ).

Then (v_n) is called Δ-converges to $v \in \Omega$, if

$$A((v_{n_k})) = \{v\}$$

for each subsequence (v_{n_k}) of (v_n). When expressing the Δ-limit of (v_n), we start by writing $\Delta\text{-}\lim_n v_n = v$ and refer to v as the limit. In a complete CAT(0) space, it is well recognized that $A((v_n))$ consists of exactly one point, i.e., $\omega_w((v_n)) = \cup A((x_n)) = \{x\}$.

The classical variational inequality problem (V.I.P.) is defined in a real Hilbert space setting as: find $x \in D$ such that $\langle Tx, y - x\rangle \geq 0$ for all $y \in D$ where T is a nonlinear operator defined on D and D is a nonempty subset of the Hilbert space. The theory of V.I.P. combines concepts of nonlinear operators and convex analysis in such a way that it generalizes both and is used to model nonlinear problems of physical phenomena in economics, sciences and engineering [3-6].

In 1967, Stampacchia [7] first established the V.I.P. in finite dimensional spaces and proved the uniqueness and he presence of solutions to the problem of variational inequality. In fact, the concept of variational inequalities is a very general form of the theory of boundary value problems as well as actually allows us to consider brand-new issues that arise in a wide range of fields of applied mathematics, which would include mechanics, physics, engineering, the concept of convex program development, and control theory. The V.I.P. incorporates the idea of nonlinear operators with convex analysis to represent nonlinear challenges of observable systems in economic concepts, fields of science, and architecture.

Recently many authors [8] have proposed to solve V.I.P. by using many iterative schemes. In this connection several iterative algorithm have been developed for figuring out the V.I.P.'s. Projections techniques offer an efficient way for solving the V.I.P.'s and are easier than numerical calculations. Khatibzadeh and Ranjbar [1] converted the concept of V.I.P. in Hilbert space to the study of complete CAT(0) spaces. Assume that $\Upsilon \neq \varnothing$, which is closed and convex subset of CAT(0) space (Ω, Υ) and define a mapping $\psi : \Omega \to \Upsilon$. Find $\zeta \in \Upsilon$ such that

$$\langle \overrightarrow{\psi(\zeta)\zeta}, \overrightarrow{\zeta\kappa}\rangle \geq 0, \ \forall\ \kappa \in \Upsilon. \tag{2}$$

The above inequality referred as variational inequality and denoted by $VI(\Upsilon, \psi)$.

A point $x \in \Omega$ is said to be fixed point of mapping T if $T(x) = x$. If Υ is the subset of a complete CAT(0) space Ω, then $P_\Upsilon : \Omega \to \Upsilon$ is the metric projection.

Throughout the paper, ψ is nonexpansive mapping, F represents the set of fixed points and the solution set of V.I.P. (2) is denoted by $Sol(\Upsilon, \psi)$.

We assume that the following conditions hold:

Important conditions for main results:

C1: The solution set $Sol(\Upsilon, \psi)$ is nonempty.

C2: The mapping $\psi : \Omega \to \Omega$ is pseudomonotone on Ω, that is

$$\langle \overrightarrow{\psi(\zeta)\zeta}, \overrightarrow{\zeta\kappa}\rangle \geq 0 \Rightarrow \langle \overrightarrow{\psi(\kappa)\zeta}, \overrightarrow{\zeta\kappa}\rangle \geq 0, \ \forall \zeta, \kappa \in \Upsilon.$$

C3: The mapping $\psi : \Omega \to \Omega$ is nonexpansive mapping, that is

$$\sigma(\psi(\zeta), \psi(\kappa)) \leq \sigma(\zeta, \kappa), \ \forall \zeta, \kappa \in \Upsilon.$$

Now these are some facts that we need in the presented paper.

Lemma 1 ([1], Lemma 2.4)

Let (Ω, σ) be a CAT(0) Space. Then for all $\zeta_1, \zeta_2, \zeta_3 \in \Omega$ and $\beta \in [0, 1]$.

(i) $\sigma(\beta\zeta_1 \oplus (1 - \beta)\zeta_2, \zeta_3) \leq \beta\sigma(\zeta_1, \zeta_3) + (1 - \beta)\sigma(\zeta_2, \zeta_3)$,

(ii) $\sigma^2(\beta\zeta_1 \oplus (1 - \beta)\zeta_2, \zeta_3) \leq \beta\sigma^2(\zeta_1, \zeta_3) + (1 - \beta)\sigma^2(\zeta_2, \zeta_3) - \beta(1 - \beta)\sigma^2(\zeta_1, \zeta_2)$.

Lemma 2 [9]

Consider (Ω, σ) a Hadamard space, Υ be it's nonempty subset which is convex, $\zeta \in \Omega$ and $\upsilon \in \Upsilon$. Then

$$\upsilon = P_\Upsilon\zeta \Leftrightarrow \langle \overrightarrow{\zeta\upsilon}, \overrightarrow{\upsilon\kappa}\rangle \geq 0, \ \forall\kappa \in \Upsilon.$$

Lemma 3 [10]

Let $\zeta_1, \zeta_2, \zeta_3 \in \Omega$ and $\mu, \gamma \in [0, 1]$. Then

(i) $\sigma(\mu\zeta_1 \oplus (1-\mu)\zeta_2, \gamma\eta_1 \oplus (1-\gamma)\zeta_2) = |\mu - \gamma|\sigma(\zeta_1, \zeta_2)$,

(ii) $\sigma(\mu\zeta_1 \oplus (1-\mu)\zeta_2, \mu\zeta_1 \oplus (1-\mu)\zeta_3) \leq \sigma(\zeta_2, \zeta_3)$.

Lemma 4 ([1], Lemma 2.9)

If (Ω, σ) is a CAT(0) space and (ρ_n) is a sequence in Ω, then F is a nonempty subset of Ω that can be used to confirm the following requirements:

(i) for each $\eta \in F$, lim$n$ $\sigma(\rho_n, \eta)$ exists,

(ii) if a subsequence (ρ_{n_p}) of (ρ_n) is Δ-convergent to $\rho \in \Omega$, then $\rho \in F$, then (ρ_n) Δ-convergent to an element of F.

Lemma 5 ([10], Lemma 2.7)

A bounded sequence (ρ_n) in CAT(0) space (Ω, σ) has a Δ-convergent subsequence.

Lemma 6 ([11], Lemma 3.7)

Consider a nonempty closed, convex subset Υ of CAT(0) space Ω and let $\psi : \Upsilon \to \Omega$ be a nonexpansive mapping. Then the conditions (ρ_n) Δ-converges to ρ and $\sigma(\rho_n, \psi(\rho_n)) \to 0$, where $\rho \in \Upsilon$ and $\psi(\rho) = \rho$.

Lemma 7 [1]

Let (Ω, σ) be a CAT(0) and $\zeta, \kappa, \iota \in \Omega$. Then for each $\beta \in [0, 1]$.

$$\sigma^2(\beta\zeta \oplus (1-\beta)\kappa,\ c) \leq \beta^2\sigma^2(\zeta,\ \iota) + (1-\beta)^2\sigma^2(\kappa,\ \iota) + 2\beta(1-\beta)\langle\overrightarrow{\zeta\iota},\ \overrightarrow{\kappa\iota}\rangle.$$

Lemma 8 ([12], Lemma 2.4)

If (ρ_n) is a bounded sequence in a complete CAT(0) space with

$$A((\rho_n)) = \{\rho\}$$

(υ_n) is a subsequence of (ρ_n) with $A((\upsilon_n)) = \{\upsilon\}$ and the sequence $(\sigma\{\rho_n, \upsilon\})$ converges, then $\rho = \upsilon$.

Now define the following algorithm for pseudomonotone V.I.P. by the sequence (z_n).

Algorithm 1

Initialization: Given $t_n, s_n \in [0, 1]$, $\beta_n \in (0, 1)$. Let $z_1 \in \Upsilon$ be randomly chosen.

Iterative Steps: For the given iterate z_n, determine $z_n + 1$ as follows:

Step 1. Calculate

$$y_n = P_\Upsilon(\beta_n z_n \oplus (1-\beta_n)\psi(z_n))$$

Step 2. Calculate

$$l_n = s_n\psi y_n \oplus (1-s_n)y_n,$$

Step 3. Calculate

$$z_{n+1} = t_n\psi(l_n) \oplus (1-t_n)y_n.$$

If $z_n = z_{n+1}$ with suitable tolerance, then stop and z_n consider to be the solution of V.I.P. (2). Otherwise

Set $n := n + 1$ and go to Step 1.

In this paper, our main goal is to study the existence and approximation of the solution for V.I.P. (2) by using Algorithm 1 and to get the strong and Δ-convergence results for Algorithm 1 in CAT(0) spaces.

2. Main results

Now we are going to prove, in this section, some important facts for Algorithm 1 in CAT(0) spaces.

First of all we prove the lemma which is very useful for the next results.

Lemma 9 Let (Ω, σ) be Hadamard space, Υ be a nonempty, closed and convex subset of Ω and consider a mapping $\psi : \Upsilon \to \Upsilon$. Assume that C1-C3 hold and a sequence $(z_n) \subset \Upsilon$ be defined by Algorithm 1. Then

(i) limn $\sigma(z_n, x^*)$ exists.

(ii) limn $\sigma(z_n, \psi(z_n)) = 0$.

Proof. Since $Sol(\Upsilon, \psi)$ is nonempty. Let x^* be any point in $Sol(\Upsilon, \psi)$

$$\sigma^2(z_{n+1}, x^*) = \sigma^2(t_n\psi(l_n) \oplus (1 - t_n)y_n), x^*,$$

by using Lemma 1, we have

$$\sigma^2(z_{n+1}, x^*) \le t_n\sigma^2(\psi(l_n), x^*) + (1-t_n)\sigma^2(y_n, x^*) - t_n(1-t_n)d^2(\psi(l_n), y_n)$$

$$\le t_n\sigma^2(\psi(l_n), x^*) + (1-t_n)\sigma^2(y_n, x^*),$$

since ψ is nonexpansive, so we have

$$\sigma^2(z_{n+1}, x^*) \le t_n\sigma^2((l_n), x^*) + (1-t_n)\sigma^2(y_n, x^*). \tag{3}$$

Now, since $l_n = s_n\psi y_n \oplus (1 - s_n)y_n$, we have

$$\sigma^2(l_n, x^*) = \sigma^2(s_n\psi y_n \oplus (1-s_n)y_n, x^*),$$

by using Lemma 7, we get

$$\sigma^2(l_n, x^*) \le (s_n)^2\sigma^2(\psi y_n, x^*) + (1-s_n)^2\sigma^2(y_n, x^*) + 2s_n(1-s_n)\langle\overrightarrow{\psi(y_n)x^*}, \overrightarrow{y_nx^*}\rangle,$$

$$\sigma^2(l_n, x^*) \le (s_n)^2\sigma^2(\psi y_n, x^*) + (1-s_n)^2\sigma^2(y_n, x^*) - 2s_n(1-s_n)\langle\overrightarrow{\psi(y_n)x^*}, \overrightarrow{x^*y_n}\rangle.$$

Since by quasi-linearization

$$\langle\overrightarrow{a_1a_2}, \overrightarrow{a_3a_4}\rangle = -\langle\overrightarrow{a_2a_1}, \overrightarrow{a_3a_4}\rangle.$$

By C3, we have

$$\langle\overrightarrow{\psi^*(x)x^*}, \overrightarrow{x^*y_n}\rangle \ge 0 \Rightarrow \langle\overrightarrow{\psi(y_n)x^*}, \overrightarrow{x^*y_n}\rangle \ge 0.$$

We get

$$\sigma^2(l_n, x^*) \le (s_n)^2\sigma^2(\psi y_n, x^*) + (1-s_n)^2\sigma^2(y_n, x^*).$$

Since ψ is nonexpansive,

$$\sigma^2(l_n, x^*) \le (s_n)^2\sigma^2(y_n, x^*) + (1-s_n^2)\sigma^2(y_n, x^*)$$

$$\sigma^2(l_n, x^*) \le (s_n)^2\sigma^2(y_n, x^*) + (1-2s_n+(s_n)^2)\sigma^2(y_n, x^*)$$

$$\sigma^2(l_n, x^*) \le \sigma^2(y_n, x^*) + 2s_n(s_n-1)\sigma^2(y_n, x^*). \tag{4}$$

put in (3), we get

$$\sigma^2(z_{n+1}, x^*) \leq \ t_n\sigma^2((l_n), x^*) + (1 - t_n)\sigma^2(y_n, x^*)$$

$$\sigma^2(z_{n+1}, x^*) \leq t_n(\sigma^2(y_n, x^*) + 2s_n(s_n - 1)\sigma^2(y_n, x^*)) + (1 - t_n)\sigma^2(y_n, x^*)$$

$$\sigma^2(z_{n+1}, x^*) \leq t_n\sigma^2(y_n), x^*) + 2t_ns_n(s_n - 1)\sigma^2(y_n, x^*) + (1 - t_n)\sigma^2(y_n, x^*)$$

$$\sigma^2(z_{n+1}, x^*) \leq \sigma^2(y_n, x^*) + 2t_ns_n(s_n - 1)\sigma^2(y_n, x^*). \tag{5}$$

Now we find $\sigma^2(y_n, x^*)$. Since $P_\Upsilon(\beta_nz_n \oplus (1 - \beta_n)\psi(z_n) = (\beta_nz_n \oplus (1 - \beta_n)\psi(z_n)$, we get

$$\sigma^2(y_n, x^*) = \sigma^2(P_\Upsilon(\beta_nz_n \oplus (1 - \beta_n)\psi(z_n)), x^*)$$

$$\sigma^2(y_n, x^*) = \sigma^2(\beta_nz_n \oplus (1 - \beta_n)\psi(z_n)), x^*)$$

again by Lemma 1

$$\sigma^2(y_n, x^*) \leq \beta_nd^2(z_n, x^*) + (1 - \beta_n)\sigma^2(\psi(z_n)), x^*) - \beta_n(1 - \beta_n)\sigma^2(z_n,\ \psi(z_n))$$

since ψ is nonexpansive, so we have

$$\sigma^2(y_n, x^*) \leq \beta_n\sigma^2(z_n, x^*) + (1 - \beta_n)\sigma^2(z_n), x^*) - \beta_n(1 - \beta_n)\sigma^2(z_n,\ \psi(z_n))$$

$$\sigma^2(y_n, x^*) \leq \sigma^2(z_n, x^*) - \beta_n(1 - \beta_n)\sigma^2(z_n,\ \psi(z_n)).$$

Put above inequality in (5) implies that

$$\sigma^2(z_{n+1}, x^*) \leq \sigma^2(z_n, x^*) - \beta_n(1 - \beta_n)\sigma^2(z_n,\ \psi(z_n)) + 2t_ns_n(s_n - 1)\sigma^2(y_n, x^*)$$

$$\sigma^2(z_{n+1}, x^*) \leq \sigma^2(u_n, x^*) - \beta_n(1 - \beta_n)\sigma^2(u_n,\ \psi(u_n)) - 2t_ns_n(1 - s_n)\sigma^2(y_n, x^*)$$

$$\sigma^2(z_{n+1}, x^*) \leq \sigma^2(z_n, x^*) - \beta_n(1 - \beta_n)\sigma^2(z_n,\ \psi(z_n)). \tag{6}$$

Thus we have

$$\sigma^2(z_{n+1}, x^*) \leq \sigma^2(z_n, x^*)$$

$$\sigma(z_{n+1}, x^*) \leq \sigma(z_n, x^*).$$

Hence $\sigma(z_n, x^*)$ is a decreasing sequence and $\lim_n \sigma(z_n, x^*)$ exists.
Without lose of generality, we consider that

$$\lim_n \sigma(z_n, x^*) = c. \tag{7}$$

In the presence of (7), we deduce that the sequences (z_n), (y_n), $(\psi(z_n))$ and $(\psi(u_n))$ are bounded.
Now again by (6)

$$\sigma^2(z_{n+1}, x^*) \leq \sigma^2(z_n, x^*) - \beta_n(1 - \beta_n)\sigma^2(z_n,\ \psi(z_n))$$

$$\beta_n(1 - \beta_n)\sigma^2(z_n,\ \psi(z_n)) \leq \sigma^2(z_n, x^*) - \sigma^2(z_{n+1}, x^*),$$

taking limit on both sides,

$$\beta_n(1-\beta_n)\sigma^2(z_n,\ \psi(z_n)) \le \sigma^2(z_n,\ x^*) - \sigma^2(z_{n+1},\ x^*)$$

$$\lim_n \beta_n(1-\beta_n)\sigma^2(z_n,\ \psi(z_n)) \le \lim_n(\sigma^2(z_n,\ x^*) - \sigma^2(z_{n+1},\ x^*))$$

$$\le \lim_n(\sigma^2(z_n,\ x^*)) - \lim_n(\sigma^2(z_{n+1},\ x^*))$$

$$\le 0$$

$$\lim_n \beta_n(1-\beta_n)\sigma^2(z_n,\ \psi(z_n)) = 0,$$

Since $\beta_n \in (0, 1)$ and $\lim_n \beta_n(1-\beta_n) > 0$. We get

$$\lim_n \sigma^2(z_n,\ \psi(z_n)) = 0,$$

$$\lim_n \sigma(z_n,\ \psi(z_n)) = 0. \qquad \square$$

Theorem 1 Consider a Hadamard space (Ω, σ), Υ and ψ be as in Lemma 9. Then the sequence (z_n) Δ-converges to $x^* \in Sol(\Upsilon, \psi)$.

Proof. By Lemma 9, we get $\lim_n \sigma(z_n, x^*)$ exists for all $x^* \in F(\psi)$ and $\lim_n \sigma(z_n, \psi(z_n)) = 0$. The sequence $\{z_n\}$ is bounded. Now we have to prove

$$\omega_w(z_n) = \bigcup_{\{\vartheta_n\}\subset\{z_n\}} \{A(\{\vartheta_n\})\} \subset Sol(\Upsilon,\ \psi).$$

Let $\vartheta \in \omega_w(z_n)$. Then there is a subsequence (ϑ_n) of (z_n) that has the characteristic that

$$A((\vartheta_n)) = (\vartheta).$$

Since by Lemma 5, (ϑ_n) has Δ-convergent subsequence. Let (v_n) be subsequence of (ϑ_n) which is Δ-converges to $v \in \Upsilon$.

Now by (8) we have

$$\lim_n \sigma(z_n,\ \psi(z_n)) = 0$$

so, $d(v_n, \psi(v_n)) \to 0$

$$\lim_n \sigma(v_n,\ \psi(v_n)) = 0$$

by using Lemma 6 we get

$$\psi(v) = v. \tag{8}$$

$v \in F(\psi)$. Now by (7), since $\lim_n \sigma(z_n, v)$ exists, hence by Lemma 8, $\vartheta = v$. We get $\varpi\Delta \subset F(\psi)$.

Finally we show that sequence (z_n), Δ-convergence to a point in $F(\psi)$. Now showing that $\omega_w(z_n)$ consists of precisely one point is adequate to attain this goal.

Let (ϑ_n) be subsequence of (z_n) with

$$A((\vartheta_n)) = \{\vartheta\}.$$

Let

$$A((z_n)) = \{x^*\}.$$

Since $\vartheta \in \omega_w(z_n)$

$$\omega_w(z_n) \subset F(\psi)$$

and $(\sigma(z_n, \vartheta))$ converges by (8), we have again by Lemma 8 $x^* = \vartheta$. Hence

$$\omega_w(z_n) = \{x^*\}$$

implies that $\{z_n\}$ is Δ-converges to $x^* \in Sol(\Upsilon, \psi)$, which complete's the proof. □

Theorem 2 Let (Ω, σ) be Hadamard space and Υ, ψ be as in Lemma 9. Then the sequence (z_n) strongly converges to $x^* \in F(\psi)$ if and only if $\liminf_n \sigma(z_n, F(\psi)) = 0$, where $\sigma(z, F(\psi)) = \inf\{\sigma(z, x^*) : x^* \in F(\psi)\}$.

Proof. Necessity is obvious, we have to show converse part only.

Let $\liminf_n \sigma(z_n, \psi(z_n)) = 0$. Already proved in Lemma 9, we have

$$\sigma(z_{n+1}, x^*) \leq \sigma(z_n, x^*),$$

for all $x^* \in F(\psi)$. So

$$\sigma(z_{n+1}, F(\psi)) \leq \sigma(z_n, F(\psi)),$$

this implies that $\lim_n \sigma(z_n, F(\psi))$ exists. By given condition so we have $\lim_n \sigma(z_n, F(\psi)) = 0$.

Now, we will show that the sequence (z_n) in Υ is a Cauchy sequence. Choose an arbitrary $\epsilon > 0$. Since $\lim_n \sigma(z_n, F(\psi)) = 0$, then there exists n_1 a positive integer, such that

$$\sigma(z_n, F(\psi)) < \frac{\epsilon}{4}, \ \forall\ n \geq n_1.$$

To be specific, $\inf\{\sigma(z_{n_1}, x^*) : x^* \in F(\psi)\} < \frac{\epsilon}{4}$. Thus there exist $\breve{\varsigma} \in F(\psi)$ such that

$$\sigma(z_{n_1}, \breve{\varsigma}) < \frac{\epsilon}{2}.$$

Now, for all $m, n \geq n_1$, we have

$$\begin{aligned}\sigma(z_{m+n}, z_n) &\leq \sigma(z_{m+n}, \breve{\varsigma}) + \sigma(\breve{\varsigma}, z_n)\\ &\leq 2\sigma(z_{n_1}, \breve{\varsigma})\\ &< 2\left(\frac{\epsilon}{2}\right)\\ &= \epsilon.\end{aligned}$$

This implies that the sequence (z_n) is a Cauchy sequence in Υ. By the closedness of Υ which is a subset of a complete CAT(0) space, the (z_n) sequence must converges to a point $\breve{b} \in \Upsilon$. □

Now, by the limn $\sigma(z_n, F(\psi)) = 0$, we get $\sigma(\breve{b}, F(\psi)) = 0$ and by the closedness of $F(\psi)$ gives that $\breve{b} \in F(\psi)$.

3. Application and numerical example

Now we are going to define this section a numerical example for the convergence of a sequence which is defined by Algorithm 1.

Example 1 Let $\Omega = \mathbb{R}^4$ be a CAT(0) space with Euclidean metric and $\Upsilon = \{(\zeta_1, 0, \zeta_3, 0) : \zeta_1, \zeta_3 \in [-1, 1]\}$ be a close, convex subset of Ω. A mapping $\psi : \Upsilon \to \Upsilon$ be a nonexpansive and pseudomonotone as well. And $\psi : = (\Upsilon_{ij})_{1 \le i, j \le 4}$ is the 4×4 matrix the entries of which are given by:

$$\Upsilon_{ij} = \begin{cases} -0.25, \text{ if } i = j \\ 0, \text{ otherwise.} \end{cases} \tag{9}$$

Solution 1 Given

$$\psi_{4\times4} = \begin{bmatrix} -0.25 & 0 & 0 & 0 \\ 0 & -0.25 & 0 & 0 \\ 0 & 0 & -0.25 & 0 \\ 0 & 0 & 0 & -0.25 \end{bmatrix}$$

$\psi(\zeta_1, 0, \zeta_3, 0) = (-0.25\zeta_1, 0, -0.25\zeta_3, 0)$.

To prove ψ is nonexpansive:

Let $\zeta = (\zeta_1, 0, \zeta_3, 0)$, $\kappa = (\kappa_1, 0, \kappa_3, 0) \in \Omega$.

$$\begin{aligned}
\sigma(\psi(\zeta), \psi(\kappa)) &= \sigma(\psi(\zeta_1, 0, \zeta_3, 0), \psi(\kappa_1, 0, \kappa_3, 0)) \\
&= \sigma((-0.25\zeta_1, 0, -0.25\zeta_3, 0), (-0.25\kappa_1, 0, -0.25\kappa_3, 0)) \\
&= \sqrt{(-0.25\zeta_1 - (-0.25\kappa_1))^2 + 0 + (-0.25\zeta_3 - (-0.25\kappa_3))^2 + 0} \\
&= 0.25\sqrt{(\zeta_1 - (\kappa_1))^2 + 0 + (\zeta_3 - (\kappa_3))^2 + 0} \\
&= 0.25\sigma(\zeta, \kappa) \\
&\le \sigma(\zeta, \kappa).
\end{aligned}$$

To prove ψ is pseudomonotone:

$$\langle \overrightarrow{\psi(\zeta)\zeta}, \overrightarrow{\zeta\kappa} \rangle \ge 0 \Rightarrow \langle \overrightarrow{\psi(\kappa)\zeta}, \overrightarrow{\zeta\kappa} \rangle \ge 0 \ \forall \ \zeta, \kappa \in \Upsilon.$$

$$\langle\overrightarrow{\psi(\kappa)\zeta},\ \overrightarrow{\zeta\kappa}\rangle=\frac{1}{2}[\sigma^2((-0.25\kappa_1,\ 0,\ -0.25\kappa_3,\ 0),(\kappa_1,\ 0,\ \kappa_3,\ 0))+\sigma^2((\zeta_1,\ 0,\ \zeta_3,\ 0),\ (\zeta_1,\ 0,\ \zeta_3,\ 0))$$

$$-\sigma^2((-0.25\kappa_1,\ 0,\ -0.25\kappa_3,\ 0),\ (\zeta_1,\ 0,\ \zeta_3,\ 0))-\sigma^2((\zeta_1,\ 0,\ \zeta_3,\ 0),\ (\kappa_1,\ 0,\ \kappa_3,\ 0))],$$

$$=\frac{1}{2}[(-0.25\kappa_1)^2+(\kappa_1)^2+(-0.25\kappa_3)^2+(\kappa_3)^2+(0.5)(\kappa_1)^2+(0.5)(\kappa_3)^2-(0.25)^2(\kappa_1)^2-(0.25)^2(\kappa_3)^2$$

$$-(\zeta_1)^2-(\zeta_3)^2-(\zeta_1)^2-(\zeta_3)^2-(\kappa_1)^2-(\kappa_3)^2-0.5(\zeta_1)(\kappa_1)-0.5(\zeta_3)(\kappa_3)-2(\zeta_1)(\kappa_1)-2(\zeta_3)(\kappa_3)],$$

$$=\frac{1}{2}[-2(\zeta_1)^2-2(\zeta_3)^2+0.5(\kappa_1)^2+0.5(\kappa_3)^2-0.5(\zeta_1)(\kappa_1)-0.5(\zeta_3)(\kappa_3)+2(\zeta_1)(\kappa_1)+2(\zeta_3)(\kappa_3)],$$

$$\geq 0.$$

since $\langle\overrightarrow{\psi(\zeta)\zeta},\ \overrightarrow{\zeta\kappa}\rangle\geq 0$, where $\zeta_1=\zeta_3=0$.

Now

$$\langle\overrightarrow{\psi(\zeta)\zeta},\ \overrightarrow{\zeta\kappa}\rangle=\langle\overrightarrow{(-0.25\zeta_1,\ 0,\ -0.25\zeta_3,\ 0)(\zeta_1,\ 0,\ \zeta_3,\ 0)},\ \overrightarrow{(\zeta_1,\ 0,\ \zeta_3,\ 0)(\kappa_1,\ 0,\ \kappa_3,\ 0)}\rangle,$$

$$=\frac{1}{2}[\sigma^2((-0.25\zeta_1,\ 0,\ -0.25\zeta_3,\ 0),\ (\kappa_1,\ 0,\ \kappa_3,\ 0))+\sigma^2((\zeta_1,\ 0,\ \zeta_3,\ 0),\ (\zeta_1,\ 0,\ \zeta_3,\ 0))$$

$$-\sigma^2((-0.25\zeta_1,\ 0,\ -0.25\zeta_3,\ 0),\ (\zeta_1,\ 0,\ \zeta_3,\ 0))-\sigma^2((\zeta_1,\ 0,\ \zeta_3,\ 0),\ (\kappa_1,\ 0,\ \kappa_3,\ 0))],$$

$$=\frac{1}{2}[(-0.25\zeta_1)^2+(\kappa_1)^2+(-0.25\zeta_3)^2+(\kappa_3)^2-(-0.25\zeta_1)^2-(-0.25\zeta_3)^2-(\kappa_1)^2-(\kappa_3)^2-(\zeta_1)^2$$

$$-(\zeta_3)^2-(\zeta_1)^2-(\zeta_3)^2+0.5(\zeta_1)(\kappa_1)+0.5(\zeta_3)(\kappa_3)-0.5(\zeta_1)^2-0.5(\zeta_3)^2+2(\zeta_1)(\kappa_1)+2(\zeta_3)(\kappa_3)],$$

$$=\frac{1}{2}[-2(\zeta_1)^2-2(\zeta_3)^2-0.5(\zeta_1)^2-0.5(\zeta_3)^2+0.5(\zeta_1)(\kappa_1)+0.5(\zeta_3)(\kappa_3)+2(\zeta_1)(y_1)+2(\zeta_3)(\kappa_3)],$$

$$=(\frac{1}{2})[(-2(\zeta_1)^2-0.5(\zeta_1)^2)-2(\zeta_3)^2-0.5(\zeta_3)^2+0.5(\zeta_1)(\kappa_1)+0.5(\zeta_3)(\kappa_3)+2(\zeta_1)(\kappa_1)+2(\zeta_3)(\kappa_3)].$$

If $(-2(\zeta_1)^2-0.5(\zeta_1)^2)=(-2(\zeta_3)^2-0.5(\zeta_3)^2)=0$, then

$$\langle\overrightarrow{\psi(\zeta)\zeta},\ \overrightarrow{\zeta\kappa}\rangle\geq 0.$$

This implies that the solution of this example is only the zero vector $z=(0,0,0,0)$. To compute the algorithm, we take $\beta_n=\frac{2}{3n}$, $t_n=\frac{1}{2n}$, $s_n=\frac{1}{n}$, and let $z_1=(0.1,0,0.2,0)$ be initial points with tolerance 10^{-8}.

Table 1 and Figure 1 shows the convergence behavior.

Table 1. Convergence behavior of x-component, y-component, z-component, l-componentof xyzl-vec towards fixed point and elapse-time

Steps	x-componenet	y-component	z-component	l-component	cpu-time
0	0.100000000000000	0.000000000000000	0.200000000000000	0.000000000000000	1.023608
1	0.309895833333333	0.000000000000000	0.061979166666667	0.000000000000000	1.012378
2	0.003954399956597	0.000000000000000	0.007908799913194	0.000000000000000	1.011876
3	0.000092681248983	0.000000000000000	0.000185362497965	0.000000000000000	0.996328
4	−0.000003409173547	0.000000000000000	−0.000006818347093	0.000000000000000	0.999069
5	0.000000257463627	0.000000000000000	0.000000514927254	0.000000000000000	0.996935
6	−0.000000026372142	0.000000000000000	−0.000000052744285	0.000000000000000	1.000142
7	0.000000003222234	0.00000000000000	0.000000006444468	0.000000000000000	1.015133

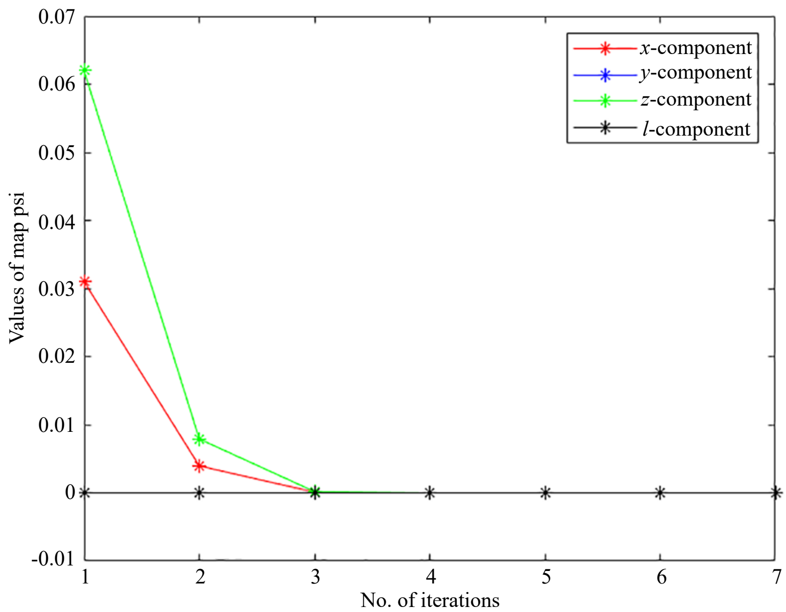

Figure 1. Convergence behaviour of mapping psi

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This paper is in final form and no version of it will be submitted for publication elsewhere.

Conflict of interest

The authors declare that there is no conflict of interest.

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