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Research Article

Approximate Controllability Outcomes of Impulsive Second-Order Stochastic Neutral Differential Evolution Systems

S. Sivasankar[1], R. Udhayakumar[1*], V. Muthukumaran[2], S. Al-Omari[3]

[1]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore-632014, Tamil Nadu, India
[2]Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur 603203, Tamilnadu, India
[3]Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Amman 11134, Jordan
E-mail: udhayaram.v@gmail.com

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Abstract: In this manuscript, we present a collection of suitable requirements for the approximate controllability outcomes of impulsive second-order stochastic neutral differential evolution systems. We use the ideas from the sine and cosine functions and the fixed point strategy to demonstrate the primary findings. The analysis then moves to second-order nonlocal stochastic neutral differential systems. Eventually, in order to make our topic more useful, we will provide an epistemological implementation.

***Keywords*:** second-order stochastic differential system, impulsive differential evolution system, neutral system, approximate controllability, nonlocal conditions

MSC: 34G20, 34K40, 47H04, 93B05

1. Introduction

Numerous partial differential equations, such as those involving flexible bar vibration and extendable strip longitudinal motion, can be described as second-order functional differential systems in infinite-dimensional spaces. Strongly continuous cosine groups of operators are a valuable notion for comprehending second-order abstract differential systems. Differential architectures with impulses are needed for understanding transcendent rotations in annual population growth categories, electromagnetic fields, and organic formations, in addition to other things. In certain unique situations, the state can change whenever, and this is joined by the advancement of improvements that coordinate the component's continued progress. It is reasonable to assume that such disruptions affect impulses since the frequency of perturbations is broad in comparison to the duration of each process; for further information, check the books [1-3].

Impulse perturbations have been widely studied in ecology, population dynamics, and biological systems. Various population dynamical systems have been used in the study of impulsive differential equations [4-8]. Delayed impulse models have also been investigated by many researchers [9-10], for delayed impulse models better simulate the actual situation, that is, the impulse effect usually takes some time to appear. However, the models with delayed impulse

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are usually ordinary differential systems, and reaction-diffusion systems with a delayed impulse are rarely seen in the existing literature. Recently, the researchers Li et al. [11] studied the stability analysis of multi-point boundary conditions for fractional differential equations with non-instantaneous integral impulses. Further, the investigators Xia et al. [12] established the stability analysis for a class of stochastic differential equations with impulses.

It is worth noting that significant progress throughout the development of impulsive functional differential systems was primarily driven by the use of numerous alluringly implemented complexities, including those listed in the following references: [13-17]. It is clear that controllability's potential is a significant segment of equipment and numerical systems theory. Finding a suitable control process to guide the numerical technique under inquiry to a desired point is known as the controllability issue. As a result, several academics have recently investigated the controllability of various nonlinear architectures, and the findings are documented in the articles, for instance, [18-22]. The investigators [23-26], established the approximate controllability results for neutral differential equations with delay.

In several fields of operational arithmetic, neutral systems are in sight, and as a result, consumer interest has grown in recent years [27]. Additionally, it should be recognized that interference or stochastic pain can't be eliminated in natural systems or even amorphous ones. Due to their wide variety of applications in describing a spectrum of complex applied mathematics in the scientific, pharmaceutical, and healthcare sectors, stochastic differential systems have piqued interest. One can verify [28-32]. Many physical processes, including fluid flow, aerodynamics, and others, are represented computationally by differential equations; for further information, see [13, 33-35]. Recently, Arthi et al. [36] studied the approximate controllability of nonlinear fractional stochastic systems involving impulsive effects and state-dependent delay by using Krasnoselskii's fixed point theorem and semigroup theory. Nevertheless, our publication was inspired by the need to further explore the approximate controllability revealed in our analysis.

Inspired by the above-mentioned work, this paper aims to fill this gap. The purpose of this paper is to show the existence of solutions and the impulsive second-order stochastic neutral differential evolution systems have the following form:

$$\frac{d}{d\iota}\left[\xi'(\iota)+\mathfrak{H}(\mathfrak{B},\, \xi_\iota)\right] \in A(\iota)\xi(\iota)+Bu(\iota)+\mathscr{G}(\iota,\, \xi_\iota)\frac{dW(\iota)}{d\iota},$$

$$\iota \in \mathcal{V}=[0,\, c],\ \iota \neq \iota_\imath,\ \imath = 1,\ \cdots,\ n, \tag{1}$$

$$\Delta\xi\,|_{\iota=\iota_\imath} = \chi_\imath(\xi(\iota_\imath^-)),\ \Delta\xi'\,|_{\iota=\iota_\imath} = \overline{\chi_\imath}(\xi(\iota_\imath^-)),\ \imath = 1,\ \cdots,\ n, \tag{2}$$

$$\xi(\iota)=\alpha(\iota)\in L^2(\Omega,\, \mathcal{G}_g),\ \iota \in (-\infty,\, 0],\ \xi'(0)=\xi_1 \in \mathbf{Z}, \tag{3}$$

where values of $\xi(\cdot)$ accepts in a Hilbert space $\mathbf{Z}$. The bounded, nonempty, closed and convex multivalued map $\mathscr{G}$ mapping from $\mathcal{V} \times \mathcal{G}_g$ into $L_2^0(\mathcal{V},\, \mathbf{Z})$. The control $u(\cdot)$ belongs to $L^2(\mathcal{V},\, \mathbb{U})$, where $\mathbb{U}$ is a Hilbert space. The histories ξ_ι : $(-\infty, 0] \rightarrow \mathcal{G}_g$, $\xi_\iota(\alpha) = \xi(\iota + \alpha)$, $\alpha \leq 0$ in relation to phase space $\mathcal{G}_g$. The bounded linear operator $B : \mathbb{U} \rightarrow \mathbf{Z}$. $\chi_\imath, \overline{\chi_\imath} : \mathbf{Z} \rightarrow \mathbf{Z}$, $\Delta\xi|_{\iota=\iota_\imath} = \xi(\iota_\imath^+) - \xi(\iota_\imath^-)$, $\Delta\xi'|_{\iota=\iota_\imath} = \xi'(\iota_\imath^+) - \xi'(\iota_\imath^-)$, $\forall\ \imath = 1, 2, \cdots, n$. $0 = \iota_0 < \iota_1 < \iota_2 < \cdots < \iota_\imath < \iota_{\imath+1} = c$. Here $\xi(\iota_\imath^+)$, $\xi(\iota_\imath^-)$, $\xi'(\iota_\imath^+)$ and $\xi'(\iota_\imath^-)$ stands for the right and left limits of $\xi(\iota)$ at $\iota = \iota_\imath$ and $\xi'(\iota)$ at $\iota = \iota_\imath$ accordingly.

The main contribution and advantage of this article can be summarized as follows:

1. In this manuscript, we investigate the approximate controllability outcomes of impulsive second-order stochastic neutral differential evolution systems.

2. A new set of sufficient conditions is established for the approximate controllability outcomes of impulsive second-order stochastic neutral differential evolution systems (1)-(3). This work generalizes many results obtained for multi-valued functions, the evolution system, the cosine and sine functions of operators, and some basic effects.

3. The aim of our technique relies on Dhage's fixed point theorem, which is effectively used to establish the new results.

4. Further, the investigation extended to nonlocal conditions of the system (1)-(3).
5. The proposed results are presented through a theoretical example.

Coming up next is the layout of this article: Section 2 covers conceptual frameworks for multi-valued functions, the evolution system, the cosine and sine functions of operators, and some basic effects. The approximate controllability outcomes for impulsive second-order differential systems with delay are the primary focus of Section 3. Further, we extended our thought of the nonlocal condition of the system (1)-(3) in Section 4. Section 5 concludes our conversation by presenting a related technical argument to support its reliability.

2. Preliminaries

Assume that $(\Omega, \mathscr{F}, P)$ is a complete probability space equipped with a normal filtration $\mathscr{F}_\iota$, $\iota \in \mathcal{V} = [0, c]$. Let W be a $\mathcal{Q}$-Weiner procedure on $(\Omega, \mathscr{F}_c, P)$ with the covariance administrator $\mathcal{Q}$ such that $tr\mathcal{Q} < \infty$. Suppose the full interpolation system exists e_n belongs to E, a bounded sequence of positive real integers λ_n such that $\mathcal{Q}e_n = \lambda_n e_n$, $n = 1, 2, \cdots$ and a sequence β_n of independent Brownian motion such that

$$w(\iota) = \sum_{n=1}^{\infty} \sqrt{\lambda}_n \beta_n(\iota) e_n, \ \iota \in \mathcal{V},$$

and $\mathscr{F}_\iota = \mathscr{F}_\iota^w$, where $\mathscr{F}_\iota^w$ is the σ-algebra referring w. Assume that $L_2^0 = L_2(\mathcal{Q}^{\frac{1}{2}} E; \mathbf{Z})$ be the space of all Hilbert-Schmidt operators along with $\|\psi\|_{\mathcal{Q}}^2 = tr[\psi \mathcal{Q} \psi^*]$. Consider $L^2(\mathscr{F}_c, \mathbf{Z})$ is the Banach space of $\mathscr{F}_c$ measurable square integral random parameters along with values belongs to $\mathbf{Z}$. Assume that Banach space of continuous function $\mathcal{C}([0, c]; L_2(\mathscr{F}_\iota, \mathbf{Z}))$ satisfying the condition $\sup_{\iota \in \mathcal{V}} E\|\xi(\iota)\|^2 < \infty$. Now initiate the spaces

$$\mathbb{H}_2([0, c];\ \mathbf{Z}) = \left\{\xi : \mathcal{V} \to \mathbf{Z},\ \xi|_{(\iota_\iota, \iota_{\iota+1}]} \in \mathcal{C}\big((\iota_\iota, \iota_{\iota+1}], \mathbf{Z}\big), \text{ and } \exists\ \xi(\iota_\iota^+) \text{ for } \iota = 1, \cdots, n\right\}$$

and

$$\mathbb{H}_2'([0, c];\ \mathbf{Z}) = \left\{\xi \in \mathbb{H}_2([0, c];\ \mathbf{Z}),\ \xi|_{(\iota_\iota, \iota_{\iota+1}]} \in \mathcal{C}'\big((\iota_\iota, \iota_{\iota+1}], \mathbf{Z}\big), \text{ and } \exists\ \xi'(\iota_\iota^+) \text{ for } \iota = 1, \cdots, n\right\}.$$

Clearly $\mathbb{H}_2([0, c]; \mathbf{Z})$ endowed with

$$\|\xi\|_{\mathbb{H}_2} = \left(\sup_{\iota \in [0, c]} E\|\xi(\iota)\|_{\mathbf{Z}}^2\right)^{\frac{1}{2}}.$$

Clearly $\mathbb{H}_2'$ is provided with the norm $\|\xi\|_{\mathbb{H}_2'} = \|\xi\|_{\mathbb{H}_2} + \|\xi\|_{\mathbb{H}_2}$.

From [4, 37], we may define $\mathcal{G}_g$ the generalized phase space. Take a look at a continuous map $g : (-\infty, 0] \to (0, +\infty)$ with $j = \int_{-\infty}^{0} g(\iota) d\iota < +\infty$. Now $\exists c > 0$, we define

$$\mathcal{G} = \{\alpha : [-c, 0] \to \mathbf{Z} \text{ such that } \alpha(\iota) \text{ is measurable and bounded}\},$$

and present

$$\|\alpha\|_{[-c,0]} = \sup_{\hbar\in[-c,0]} \|\alpha(\hbar)\|, \ \forall \alpha \in \mathcal{G}.$$

Now, we define

$$\mathcal{G}_g = \{\alpha : (-\infty,\ 0] \to \mathbf{Z} \text{ such that } \forall b > 0,\ (E\|\alpha(\theta)\|^2)^{\frac{1}{2}}$$

is bounded and measurable function on $[-b, 0]$ and

$$\int_{-\infty}^{0} g(\hbar) \sup_{\hbar\le\theta\le 0} (E\|\alpha(\theta)\|^2)^{\frac{1}{2}} d\hbar < +\infty\},$$

and

$$\|\alpha\|_{\mathcal{G}_g} = \int_{-\infty}^{0} g(\hbar) \sup_{\hbar\le\theta\le 0} (E\|\alpha(\theta)\|^2)^{\frac{1}{2}} d\hbar, \forall \alpha \in \mathcal{G}_g,$$

therefore $(\mathcal{G}_g, \|\cdot\|\mathcal{G}_g)$ is a Banach space. Now

$$\mathcal{G}_g' = \Big\{\xi : (-\infty,\ c] \to \mathbf{Z} \ni \xi_\iota \in \mathcal{C}(\mathcal{V}_\iota,\ \mathbf{Z}),$$

$$\exists\, \xi(\iota_\iota^+) \text{ and } \xi(\iota_\iota^-) \text{ with } \xi(\iota_\iota) = \xi(\iota_\iota^-),\ \xi_0 = \alpha \in \mathcal{G}_g\Big\},$$

where ξ_ι is the restriction of ξ to $\mathcal{V}_\iota = (\iota_\iota, \iota_{\iota+1}]$, $\iota = 1, 2, \cdots, n$. Set

$$\|\xi\|_g' = \|\alpha\|_{\mathcal{G}_g} + \sup_{\hbar\in[0,c]} (E\|\xi(\hbar)\|^2)^{\frac{1}{2}},\ \xi \in \mathcal{G}_g'.$$

Lemma 2.1 (See [4]) Suppose $\xi \in \mathcal{G}_g'$, then for $\iota \in \mathcal{V}$, $\xi_\iota \in \mathcal{G}_g$. Also,

$$j(E\|\xi(\iota)\|^2)^{\frac{1}{2}} \le \|\ \xi_\iota\|_{\mathcal{G}_g} \le \|\ \alpha\|_{\mathcal{G}_g} + j \sup_{\hbar\in[0,\iota]} (E\|\xi(\hbar)\|^2)^{\frac{1}{2}},$$

where $j = \int_{-\infty}^{0} g(\iota)d\iota < +\infty$.

To discuss the main results of our manuscript, we first present some basic theories, lemmas, and facts. $B_p(\xi, \mathbf{Z})$ represents for the sealed spherical along with mid ξ and distance $p > 0$ belongs to $\mathbf{Z}$. From [5, 38] we consider

$$\xi'' = A(\iota)\xi(\iota) + E(\iota),\ 0 \le \zeta,\ \iota \le c, \tag{4}$$

$$\xi(0) = \xi_0,\ \xi'(0) = \xi_1. \tag{5}$$

Here $A(\iota) : D(A(\iota)) \subset \mathbf{Z} \to \mathbf{Z}$, $\iota \in \mathcal{V}$ is closed and $E : \mathcal{V} \to \mathbf{Z}$. Existence of solutions for the differential system (4)-(5) is in relation to existence of an evolution operator $\mathcal{S}(\iota, \zeta)$ of the subsequent equations

$$\xi'' = A(\iota)\xi(\iota), \ 0 \le \zeta, \ \iota \le c.$$

Suppose the domain of $A(\iota)$, which is a dense belongs to $\mathbf{Z}$ subspace D that is independent of ι, and $\exists \xi \in D$, a continuous function $\iota \mapsto A(\iota)\xi$.

Definition 2.2 [19] $\mathcal{S}(\iota, \zeta) : \mathcal{V} \times \mathcal{V} \to L(\mathbf{Z})$ a family of linear bounded operators is called an evolution operator given by

(Z1) $\forall \ \xi \in \mathbf{Z}$, the map $(\iota, \zeta) \mapsto \mathcal{S}(\iota, \zeta)\xi \in \mathbf{Z}$ is continuously differentiable and

(1) $\forall \ \iota \in [0, c]$, $\mathcal{S}(\iota, \iota) = 0$,

(2) $\forall \ \iota, \zeta \in [0, c]$, and $\forall \ \xi$ belongs to $\mathbf{Z}$,

$$\frac{\partial}{\partial \iota}\mathcal{S}(\iota, \ \zeta)\xi \,\Big|_{\iota=\zeta} = \xi, \ \frac{\partial}{\partial \zeta}\mathcal{S}(\iota, \ \zeta)\xi \,\Big|_{\iota=\zeta} = -\xi.$$

(Z2) $\forall \ \iota, \zeta \in [0, c]$, provided that $\xi \in D$, $\mathcal{S}(\iota, \zeta)\xi \in D$, the map $(\iota, \zeta) \mapsto \mathcal{S}(\iota, \zeta)\xi \in \mathbf{Z}$ is of class $\mathcal{N}^2$ and

(1) $\dfrac{\partial^2}{\partial \iota^2}\mathcal{S}(\iota, \zeta) = A(\iota)\mathcal{S}(\iota, \zeta)\xi$,

(2) $\dfrac{\partial^2}{\partial \zeta^2}\mathcal{S}(\iota, \zeta)\xi = \mathcal{S}(\iota, \zeta)A(\zeta)\xi$,

(3) $\dfrac{\partial}{\partial \zeta}\dfrac{\partial}{\partial \iota}\mathcal{S}(\iota, \zeta)\xi \,\Big|_{\iota=\zeta} = 0.$

(Z3) $\forall \ \iota, \zeta \in [0, c]$, if $\xi \in D$, $\dfrac{\partial}{\partial \zeta}\mathcal{S}(\iota, \ \zeta)\xi \in D$, $\exists \dfrac{\partial^2}{\partial \iota^2}\dfrac{\partial}{\partial \zeta}\mathcal{S}(\iota, \ \zeta)\xi$, $\dfrac{\partial^2}{\partial \zeta^2}\dfrac{\partial}{\partial \iota}\mathcal{S}(\iota, \ \zeta)\xi$ and

(1) $\dfrac{\partial^2}{\partial \iota^2}\dfrac{\partial}{\partial \zeta}\mathcal{S}(\iota, \ \zeta)\xi = A(\iota)\dfrac{\partial}{\partial \zeta}\mathcal{S}(\iota, \ \zeta)\xi$. Moreover, the map $(\iota, \ \zeta) \mapsto A(\iota)\dfrac{\partial}{\partial \zeta}\mathcal{S}(\iota, \ \zeta)\xi$ is continuous.

(2) $\dfrac{\partial^2}{\partial \zeta^2}\dfrac{\partial}{\partial \iota}\mathcal{S}(\iota, \ \zeta)\xi = \dfrac{\partial}{\partial \iota}\mathcal{S}(\iota, \ \zeta)A(\zeta)\xi$.

We now initiate $\mathcal{N}(\iota, \zeta) = -\dfrac{\partial \mathcal{S}(\iota, \ \zeta)}{\partial \zeta}$. Now we set positive constants P_1 and P_2 such that

$$\sup_{0 \le \iota, \zeta \le c} \| \mathcal{N}(\iota, \ \zeta)\|^2 \le P_1$$

and

$$\sup_{0 \le \iota, \zeta \le c} \| \mathcal{S}(\iota, \ \zeta)\|^2 \le P_2.$$

Further, we determine $P > 0$ such that

$$\|\mathcal{S}(\iota + k, \ \zeta) - \mathcal{S}(\iota, \ \zeta)\|^2 \le P\|k\|^2,$$

$\forall\ \zeta, \iota, \iota + k$ belongs to $\mathcal{V}$. In view of an integrable function $k : \mathcal{V} \to \mathbf{Z}$, $\xi : [0, c] \to \mathbf{Z}$ a mild solution of (4)-(5) is the following form:

$$\xi(\iota) = \mathcal{N}(\iota, \zeta)\xi_0 + \mathcal{S}(\iota, \zeta)\xi_1 + \int_0^\iota \mathcal{S}(\iota, \varsigma)k(\varsigma)d\varsigma.$$

We dismiss the aforementioned idea due to its resemblances. For additional information on the (4)-(5) abstract non-autonomous problem, see for more information on [5, 34, 38-40].

From [41], we initiate $A(\iota) : D(A(\iota)) \subset \mathbf{Z} \to \mathbf{Z}$ and see this articles [33, 42-43] for more information on multivalued function.

Definition 2.3 [33, 42-43] The multivalued map $\mathscr{G} : \mathcal{V} \times \mathcal{G}_g \to BCC(\mathbf{Z})$ ($BCC(\mathbf{Z})$ represents the family of all bounded, closed, nonempty and convex subset of $\mathbf{Z}$) is called L^2-Caratheodory provided that

(i) $\iota \mapsto \mathscr{G}(\iota, \xi)$ is measurable $\forall\ \xi \in \mathcal{G}_g$;

(ii) $\xi \mapsto \mathscr{G}(\iota, \xi)$ is u.s.c. $\forall\ \iota \in \mathcal{V}$;

(iii) $\forall\ p > 0$, $\exists \varrho_p \in L^1(\mathcal{V}, \mathbb{R})$ such that

$$\|\mathscr{G}(\iota,\ \xi)\|^2 = \sup_{h \in \mathscr{G}(\iota, \xi)} E\|g\|^2 \le \varrho_p(\iota), \forall\ \|\xi\|^2_{\mathcal{G}_g} \le p \text{ and for a.e. } \iota \in \mathcal{V}.$$

Now, we will go over the key operators and basic results in the following order:

$$\aleph_0^c = \int_0^c \mathcal{S}(c - \varpi)BB^*\mathcal{S}^*(c - \varpi)d\varpi : \mathbf{Z} \to \mathbf{Z},$$

$$R(\delta, \aleph_0^c) = (\delta I + \aleph_0^c)^{-1} : \mathbf{Z} \to \mathbf{Z}.$$

Here B^* and $\mathcal{S}^*(c)$ represents the adjoints of B and $\mathcal{S}(c)$ individually. As a result, $\aleph_0^c$ is bounded.

The following statement is produced to confirm the approximation controllability for impulsive second-order stochastic neutral differential systems (1)-(3):

$\mathbf{H_0}$ $\delta(\delta I + \aleph_0^c)^{-1} \to 0$ when $\delta \to 0^+$ belongs to the strong operator topology.

From [44], $\mathbf{H_0}$ satisfied if and only if second-order linear differential system

$$\frac{d^2}{d\iota^2}(\xi(\iota)) = A(\iota)\xi(\iota) + (Bu)(\iota),\ \iota \in \mathcal{V}, \tag{6}$$

$$\xi(0) = \xi_0,\ \xi'(0) = \xi_1, \tag{7}$$

is approximately controllable on $[0, c]$.

Lemma 2.4 [43]. Assume $\mathbf{Z}$ be a Hilbert space. $\phi_1 : \mathbf{Z} \to BCC(\mathbf{Z})$ and $\phi_2 : \mathbf{Z} \to BCC(\mathbf{Z})$ be two multivalued operators satisfies:

(a) ϕ_1 is a contraction, and

(b) ϕ_2 is completely continuous.

Then either

(i) the operator inclusion $\lambda\varkappa \in \phi_1\varkappa + \phi_2\varkappa$ has a solution when $\lambda = 1$, or

(ii) $\mathscr{G} = \{\varkappa \in \mathbf{Z} : \lambda\varkappa \in \phi_1\varkappa + \phi_2\varkappa, \lambda > 1\}$ is unbounded for $\lambda \in (0, 1)$.

3. Approximate controllability

The results of approximate controllability for the impulsive second-order stochastic neutral differential systems (1)-(3) are explored here. Take into account the mild solution of (1)-(3) as follows:

Definition 3.1 An $\mathscr{F}_\iota$-adapted stochastic process $\xi : (-\infty, c] \rightarrow \mathbf{Z}$ is said to be a mild solution of (1)-(3) given by $\xi_0 = \alpha \in L^2(\Omega, \mathcal{G}_g)$, $\xi'(0) = \xi_1 \in \mathbf{Z}$ on $(-\infty, 0]$, and the impulsive conditions $\Delta\xi|_{\iota=\iota_\iota} = \chi_\iota(\xi(\iota_\iota^-))$, $\Delta\xi'|_{\iota=\iota_\iota} = \overline{\chi}_\iota(\xi(\iota_\iota^-))$, $\iota = 1, \cdots, n$; $\xi(\cdot)$ to χ_ι is continuous and

(i) $\xi(\iota)$ is measurable and adapted to $\mathscr{F}_\iota$, $\iota \geq 0$.

(ii) $\xi(\iota) \in \mathbf{Z}$ has càdlàg paths on $\iota \in \mathcal{V}$ and, $\forall\ \iota \in \mathcal{V}$, $\xi(\iota)$ satisfies the integral equation

$$\xi(\iota) = \mathcal{N}(\iota, 0)\alpha(0) + \mathcal{S}(\iota, 0)[\xi_1 + \mathfrak{H}(\iota, \alpha(0))] - \int_0^\iota \mathcal{N}(\iota, \varpi)\mathfrak{H}(\varpi, \xi_\varpi)d\varpi$$

$$+\int_0^\iota \mathcal{S}(\iota, \varpi)g(\varpi)dW(\varpi) + \int_0^\iota \mathcal{S}(\iota, \varpi)Bu(\varpi)d\varpi$$

$$+\sum_{0<\iota_\iota<\iota} \mathcal{N}(\iota, \iota_\iota)\chi_\iota(\xi_{\iota_\iota}) + \sum_{0<\iota_\iota<\iota} \mathcal{S}(\iota, \iota_\iota)\overline{\chi_\iota}(\xi_{\iota_\iota}),\ \iota \in \mathcal{V}.$$

We establish the primary assumptions for demonstrating the main theorem as follows:

$\mathbf{H_1}$ $\mathcal{N}(\iota, 0)$, $\iota > 0$ is compact.

$\mathbf{H_2}$ $\mathfrak{H} : \mathcal{V} \times \mathcal{G}_g$ into $\mathbf{Z}$ is continuous and $\exists\, \widetilde{L}_h > 0$ for $\iota \in \mathcal{V}$ and $v, w \in \mathcal{G}_g$ such that

$$E\|\mathfrak{H}(\iota, v)\|^2 \leq \widetilde{L_h}\left[1 + \|v\|^2_{\mathcal{G}_g}\right],\ v \in \mathcal{G}_g,$$

$$E\|\mathfrak{H}(\iota, v) - \mathfrak{H}(\iota, w)\|^2 \leq \widetilde{L_h}\|v - w\|^2_{\mathcal{G}_g},\ v, w \in \mathcal{G}_g,$$

and $\widetilde{L}_h = \sup_{\iota\in\mathcal{V}} \|\mathfrak{H}(\iota, 0)\|$.

$\mathbf{H_3}$ $\mathscr{G}$ mapping from $\mathcal{V} \times \mathcal{G}_g$ into $BCC(L(\mathbb{U}, \mathbf{Z}))$ is L^2-Caratheodory and that fulfilled the subsequent requirements: $\forall\ \iota \in \mathcal{V}$, $\mathscr{G}(\iota, \cdot)$ is upper semicontinuous; $\forall\ \xi \in \mathcal{G}_g$, $\mathscr{G}(\cdot, \xi)$ stand for measurable and ξ belongs to $\mathcal{G}_g$,

$$T_{\mathscr{G},\,\xi} = \left\{g \in L^2(\mathcal{C}, L(\mathbb{U}, \mathbf{Z})) : g(\iota) \in \mathscr{G}(\iota, \xi_\iota),\ \forall \iota \in \mathcal{V}\right\},$$

is nonempty.

$\mathbf{H_4}$ Provided that $p > 0$, $\exists\ \varrho_p : \mathcal{V} \rightarrow \mathbb{R}^+$ such that

$$E\|\mathscr{G}(\iota, \xi_\iota)\|^2 = \sup\left\{E\|g\|^2 : g(\iota) \in \mathscr{G}(\iota, \xi_\iota)\right\} \leq \varrho_p(\iota),$$

for almost everywhere $\iota \in \mathcal{V}$.

$\mathbf{H_5}$ $\zeta \rightarrow \varrho_p(\zeta) \in L^1(\mathcal{V}, \mathbb{R}^+)$ and $\exists\ \gamma > 0$ such that

$$\lim_{p\to\infty}\frac{\int_0^{\iota}\varrho_p(\zeta)d\zeta}{p}=\gamma<\infty.$$

$\mathbf{H_6}$ $\chi_\iota \in \mathcal{C}(\mathbf{Z}, \mathbf{Z})$ and $\exists\, P_m : [0, +\infty) \to (0, +\infty)$ which are continuous non-decreasing functions such that

$$E\left\|\chi_\iota(\xi)\right\|^2 \le P_m(E\|\xi\|^2),\ \iota = 1, \cdots, n,\ \xi \text{ belongs to } \mathbf{Z}$$

and

$$\lim_{p\to\infty}\inf\frac{P_m(p)}{p}=\tau_\iota<\infty,\ \iota=1, \cdots, n.$$

$\mathbf{H_7}$ $\overline{\chi_\iota} \in \mathcal{C}(\mathbf{Z}, \mathbf{Z})$ and $\exists\,\overline{P_m} : [0, +\infty) \to (0, +\infty)$ which are continuous non-decreasing functions such that

$$E\left\|\overline{\chi_\iota}(\xi)\right\|^2 \le \overline{P_m}(E\|\xi\|^2),\ \iota = 1, \cdots, n,\ \xi \text{ belongs to } \mathbf{Z}$$

and

$$\lim_{p\to\infty}\inf\frac{\overline{P_m}(p)}{p}=\overline{\tau_\iota}<\infty,\ \iota=1, \cdots, n.$$

Lemma 3.2 For any $\xi_c \in L^2(\mathscr{F}_c, \mathbf{Z})$ there exists $\Upsilon \in L^2_{\mathscr{F}}\left(\Omega;\ L^2([0,\ c];\ L_2^0)\right)$ such that $\xi_c = Ez_c + \int_0^c \Upsilon(\varpi)dW(\varpi)$. Now any $\delta > 0$ and $\xi_c \in L^2(\mathscr{F}_c, \mathbf{Z})$ the control function is defined as follows:

$$\begin{aligned}\hat{u}^{\delta}_{\xi}(\iota) = B^*\mathcal{S}^*(c,\ \iota)\Big[&(\delta I + \aleph_0^c)^{-1}\Big(E\xi_c - \mathcal{N}(c,\ 0)\alpha(0) - \mathcal{S}(c,\ 0)[\xi_1 + \mathfrak{H}(c,\ \alpha(0))]\Big)\\ &+\int_0^c(\delta I + \aleph_0^c)^{-1}\Upsilon(\varpi)dW(\varpi) + \int_0^c(\delta I + \aleph_\varpi^c)^{-1}\mathcal{N}(c,\ \varpi)\mathfrak{H}(\varpi,\ \xi_\varpi^\delta)d\varpi\\ &-\int_0^c(\delta I + \aleph_\varpi^c)^{-1}\mathcal{S}(c,\ \varpi)g(\varpi)dW(\varpi) - \sum_{0<\iota_\iota<c}(\delta I + \aleph_\varpi^c)^{-1}\mathcal{N}(c,\ \iota_\iota)\chi_\iota(\xi_{\iota_\iota})\\ &-\sum_{0<\iota_\iota<c}(\delta I + \aleph_\varpi^c)^{-1}\mathcal{S}(c,\ \iota_\iota)\overline{\chi_\iota}(\xi_{\iota_\iota})\Big],\end{aligned} \tag{8}$$

where $g \in T_{\mathscr{G},\xi} = \{g \in L^2(\mathcal{C}, L(\mathbb{U}, \mathbf{Z}) : \mathscr{G}(\iota) \in \mathscr{G}(\iota, \xi) \text{ a.e. } \iota \in \mathcal{V}\}$.

Theorem 3.3 If $\mathbf{H_1}$-$\mathbf{H_7}$ are fulfilled. Then, the impulsive second-order stochastic neutral differential inclusions (1)-(3) has a mild solution on $\mathcal{V}$ if

$$\left(7+\frac{49}{\delta}P_2^2P_B^2c\right)\left[4P_1\widetilde{L_h}c+4P_2Tr(\mathcal{Q})\gamma j^2+4P_1n^2\sum_{\iota=1}^{n}\tau_\iota+4P_2n^2\sum_{\iota=1}^{n}\overline{\tau_\iota}\right]<1, \tag{9}$$

where $P_B = ||B||^2$.

Proof. If $\varrho > 0$, let us take $\Psi^\varrho : \mathcal{G}_g' \to 2^{\mathcal{G}_g'}$ defined as $\Psi^\varrho\xi$ the set of $\xi \in \mathcal{G}_g'$ such that

$$\xi(\iota)=\begin{cases}\alpha(\iota),\ \iota\in(-\infty,0],\\ \mathcal{N}(\iota,0)\alpha(0)+\mathcal{S}(\iota,0)[\xi_1+\mathfrak{H}(\iota,\alpha(0))]-\int_0^\iota\mathcal{N}(\iota,\varpi)\mathfrak{H}(\varpi,\xi_\varpi)d\varpi\\ +\int_0^\iota\mathcal{S}(\iota,\varpi)g(\varpi)dW(\varpi)+\int_0^\iota\mathcal{S}(\iota,\varpi)Bu_\delta(\varpi,\xi)d\varpi\\ +\sum_{0<\iota_\imath<\iota}\mathcal{N}(\iota,\iota_\imath)\chi_\imath(\xi_{\iota_\imath})+\sum_{0<\iota_\imath<\iota}\mathcal{S}(\iota,\iota_\imath)\overline{\chi_\imath}(\xi_{\iota_\imath}),\ \iota\in\mathcal{V},\end{cases}$$

where $g \in T_{\mathscr{G},\xi}$. It is necessary to check Ψ^ϱ has a fixed point and which is the solution of the system (1)-(3). It is obvious, $\xi_c = \xi(c) \in (\Psi^\varrho\xi)(c)$, it signifies that $u_\delta(\xi, \iota)$ drives the system (1)-(3) from $\xi_0 \to \xi_c$ belongs to limited time c.

As $\beta \in \mathcal{G}_g$, we determine $\hat{\beta}$ by

$$\hat{\beta}(\iota)=\begin{cases}\beta(\iota),\ \iota\in(-\infty,0],\\ \mathcal{N}(\iota,0)\alpha(0)+\mathcal{S}(\iota,0)[\xi_1+\mathfrak{H}(\iota,\alpha(0))],\ \iota\in\mathcal{V},\end{cases}$$

then $\hat{\beta} \in \mathcal{G}_g'$. Assume $\xi(\iota) = \varkappa(\iota) + \hat{\beta}(\iota)$, $-\infty < \iota \le c$. Now, we conclude y satisfies $x_0 = 0$ and

$$\varkappa(\iota)=-\int_0^\iota\mathcal{N}(\iota,\varpi)\mathfrak{H}(\varpi,\varkappa_\varpi+\hat{\beta}_\varpi)d\varpi+\int_0^\iota\mathcal{S}(\iota,\varpi)g(\varpi)dW(\varpi)$$

$$+\int_0^\iota\mathcal{S}(\iota,\varpi)BB^*\mathcal{S}^*(c,\varpi)(\delta I+\aleph_0^c)^{-1}\Big[E\xi_c+\int_0^c\Upsilon(\varsigma)dW(\varsigma)-\mathcal{N}(c,0)\alpha(0)$$

$$-\mathcal{S}(c,0)[\xi_1+\mathfrak{H}(c,\alpha(0))]+\int_0^c\mathcal{N}(c,\varsigma)\mathfrak{H}(\varsigma,\varkappa_\varsigma+\hat{\beta}_\varsigma)d\varsigma-\int_0^c\mathcal{S}(c,\varsigma)g(\varsigma)dW(\varsigma)$$

$$-\sum_{0<\iota_\imath<c}\mathcal{N}(c,\iota_\imath)\chi_\imath\big(\varkappa(\iota_\imath^-)+\hat{\beta}(\iota_\imath^-)\big)-\sum_{0<\iota_\imath<c}\mathcal{S}(c,\iota_\imath)\overline{\chi_\imath}\big(\varkappa(\iota_\imath^-)+\hat{\beta}(\iota_\imath^-)\big)\Big](\varpi)d\varpi$$

$$+\sum_{0<\iota_\imath<\iota}\mathcal{N}(\iota,\iota_\imath)\chi_\imath\big(\varkappa(\iota_\imath^-)+\hat{\beta}(\iota_\imath^-)\big)+\sum_{0<\iota_\imath<\iota}\mathcal{S}(\iota,\iota_\imath)\overline{\chi_\imath}\big(\varkappa(\iota_\imath^-)+\hat{\beta}(\iota_\imath^-)\big),\ \iota\in\mathcal{V},$$

$\Leftrightarrow$ $\varkappa$ satisfies the following

$$\xi(\iota)=\mathcal{N}(\iota,0)\alpha(0)+\mathcal{S}(\iota,0)[\xi_1+\mathfrak{H}(\iota,\alpha(0))]-\int_0^\iota\mathcal{N}(\iota,\varpi)\mathfrak{H}(\varpi,\varkappa_\varpi+\hat{\beta}_\varpi)d\varpi$$

$$+\int_0^\iota\mathcal{S}(\iota,\varpi)g(\varpi)dW(\varpi)+\int_0^\iota\mathcal{S}(\iota,\varpi)BB^*\mathcal{S}^*(c,\varpi)(\delta I+\aleph_0^c)^{-1}\Big[E\xi_c+\int_0^c\Upsilon(\varsigma)dW(\varsigma)$$

$$-\mathcal{N}(c, 0)\alpha(0) - \mathcal{S}(c, 0)[\xi_1 + \mathfrak{H}(c, \alpha(0))] + \int_0^c \mathcal{N}(c, \varsigma)\mathfrak{H}(\varsigma, \varkappa_\varsigma + \hat{\beta}_\varsigma)d\varsigma$$

$$-\int_0^c \mathcal{S}(c, \varsigma)g(\varsigma)dW(\varsigma) - \sum_{0<\iota_l<c} \mathcal{N}(c, \iota_l)\chi_l\left(\varkappa(\iota_l^-) + \hat{\beta}(\iota_l^-)\right)$$

$$-\sum_{0<\iota_l<c} \mathcal{S}(c, \iota_l)\overline{\chi_l}\left(\varkappa(\iota_l^-) + \hat{\beta}(\iota_l^-)\right)\Big](\varpi)d\varpi + \sum_{0<\iota_l<\iota} \mathcal{N}(\iota, \iota_l)\chi_l\left(\varkappa(\iota_l^-) + \hat{\beta}(\iota_l^-)\right)$$

$$+\sum_{0<\iota_l<\iota} \mathcal{S}(\iota, \iota_l)\overline{\chi_l}\left(\varkappa(\iota_l^-) + \hat{\beta}(\iota_l^-)\right), \ \iota \in \mathcal{V},$$

and $\xi(\iota) = \beta(\iota)$, $\iota \in (-\infty, 0]$.

Assume $\mathcal{G}_g'' = \{\varkappa \in \mathcal{G}_g' : \varkappa_0 = 0 \in \mathcal{G}_g\}$, $\forall\, \varkappa \in \mathcal{G}_g''$,

$$\|\varkappa\|_c = \|\varkappa_0\|_{\mathcal{G}_g} + \sup_{0\le\hbar\le c}\left(E\|\varkappa(\hbar)\|^2\right)^{\frac{1}{2}} = \sup_{0\le\hbar\le c}\left(E\|\varkappa(\hbar)\|^2\right)^{\frac{1}{2}},$$

$\Rightarrow (\mathcal{G}_g'', \|\cdot\|_c)$ is a Banach space. Take $B_p = \{\varkappa \in \mathcal{G}_g'' : \|\varkappa\|_c^2 \le p\}$ $\forall\, p > 0$, $\Rightarrow B_p \subseteq \mathcal{G}_g''$ is uniformly bounded, $\forall\, \varkappa \in B_p$, according to Lemma 2.1, we have

$$\begin{aligned} E\left\|\varkappa_\iota + \hat{\beta}_\iota\right\|_{\mathcal{G}_g}^2 &\le 2E\left(\left\|\varkappa_\iota\right\|_{\mathcal{G}_g}^2 + \left\|\hat{\beta}_\iota\right\|_{\mathcal{G}_g}^2\right) \\ &\le 4\left(j^2 \sup_{\hbar\in[0,\iota]} E\left\|x(\hbar)\right\|^2 + \left\|\varkappa_0\right\|_{\mathcal{G}_g}^2 + j^2 \sup_{\hbar\in[0,\iota]} E\|\hat{\beta}(\hbar)\|^2 + \|\hat{\beta}_0\|_{\mathcal{G}_g}^2\right) \\ &\le 4j^2\left(p + 2(P_1E\|\hat{\beta}(0)\|^2 + P_2E\|\xi_1\|^2)\right) + 4\left\|\hat{\beta}_0\right\|_{\mathcal{G}_g}^2 = p'. \end{aligned} \tag{10}$$

From Lemma 2.1, $\forall\, \iota \in \mathcal{V}$,

$$E\left\|\varkappa(\iota) + \hat{\beta}(\iota)\right\|^2 \le j^{-2}E\left\|\varkappa_\iota + \hat{\beta}_\iota\right\|_{\mathcal{G}_g}^2.$$

For each $\iota \in \mathcal{V}$, $y \in B_p$, from the equation (10), $\mathbf{H_6}$ and $\mathbf{H_7}$, we obtain

$$\sup_{\iota\in\mathcal{V}} E\left\|\varkappa(\iota) + \hat{\beta}(\iota)\right\|^2 \le j^{-2}E\left\|y_\iota + \hat{\beta}_\iota\right\|_{\mathcal{G}_g}^2 \le j^{-2}p'.$$

Therefore

$$E\left\|\chi_\imath\left(\varkappa(\imath_\imath^-)+\hat{\beta}(\imath_\imath^-)\right)\right\|^2 \le P_m\left(E\left\|\varkappa(\imath_\imath^-)+\hat{\beta}(\imath_\imath^-)\right\|^2\right)$$

$$\le P_m\left(\sup_{\imath\in\mathcal{V}} E\left\|\varkappa(\imath)+\hat{\beta}(\imath)\right\|^2\right)$$

$$\le P_m\left(j^{-2}p\right),\ \imath=1,\ 2,\ \cdots,\ n,$$

and

$$E\left\|\overline{\chi_\imath}\left(\varkappa(\imath_\imath^-)+\hat{\beta}(\imath_\imath^-)\right)\right\|^2 \le \overline{P_m}\left(E\left\|\varkappa(\imath_\imath^-)+\hat{\beta}(\imath_\imath^-)\right\|^2\right)$$

$$\le \overline{P_m}\left(\sup_{\imath\in\mathcal{V}} E\left\|\varkappa(\imath)+\hat{\beta}(\imath)\right\|^2\right)$$

$$\le \overline{P_m}(j^{-2}p'),\ \imath=1,\ 2,\ \cdots,\ n.$$

Let $\Upsilon : \mathcal{G}_g'' \to \mathcal{G}_g''$ given by Υy the collection of $\bar{\xi} \in \mathcal{G}_g''$ such that

$$\bar{\xi}(\imath)=\begin{cases} 0,\ \imath\in(-\infty,\ 0], \\ -\int_0^\imath \mathcal{N}(\imath,\ \varpi)\mathfrak{H}(\varpi,\ \varkappa_\varpi+\hat{\beta}_\varpi)d\varpi+\int_0^\imath \mathcal{S}(\imath,\ \varpi)g(\varpi)dW(\varpi) \\ +\int_0^\imath \mathcal{S}(\imath,\ \varpi)BB^*\mathcal{S}^*(c,\ \varpi)(\delta I+\aleph_0^c)^{-1}\Big[E\xi_c+\int_0^c \Upsilon(\varsigma)dW(\varsigma)-\mathcal{N}(c,\ 0)\alpha(0) \\ -\mathcal{S}(c,\ 0)[\xi_1+\mathfrak{H}(c,\ \alpha(0))]+\int_0^c \mathcal{N}(c,\ \varsigma)\mathfrak{H}(\varsigma,\ \varkappa_\varsigma+\hat{\beta}_\varsigma)d\varsigma-\int_0^c \mathcal{S}(c,\ \varsigma)g(\varsigma)dW(\varsigma) \\ -\sum_{0<\imath_\imath<c} \mathcal{N}(c,\ \imath_\imath)\chi_\imath\left(\varkappa(\imath_\imath^-)+\hat{\alpha}(\imath_\imath^-)\right) \\ -\sum_{0<\imath_\imath<c} \mathcal{S}(c,\ \imath_\imath)\overline{\chi_\imath}\left(\varkappa(\imath_\imath^-)+\hat{\beta}(\imath_\imath^-)\right)\Big](\varpi)d\varpi \\ +\sum_{0<\imath_\imath<\imath} \mathcal{N}(\imath,\ \imath_\imath)\chi_\imath\left(\varkappa(\imath_\imath^-)+\hat{\beta}(\imath_\imath^-)\right) \\ +\sum_{0<\imath_\imath<\imath} \mathcal{S}(\imath,\ \imath_\imath)\overline{\chi_\imath}\left(\varkappa(\imath_\imath^-)+\hat{\beta}(\imath_\imath^-)\right),\ \imath\in\mathcal{V}. \end{cases}$$

Now, we assume the operators Υ_1 and Υ_2 defined by

$$\Upsilon_1\varkappa(\imath)=\int_0^\imath \mathcal{N}(\imath,\ \varpi)\mathfrak{H}(\varpi,\ \varkappa_\varpi+\hat{\beta}_\varpi)d\varpi,\ \imath\in\mathcal{V}$$

$$\Upsilon_2\varkappa(\iota) = \int_0^\iota \mathcal{S}(\iota, \varpi)g(\varpi)dW(\varpi) + \int_0^\iota \mathcal{S}(\iota, \varpi)BB^*\mathcal{S}^*(c, \varpi)(\delta I + \aleph_0^c)^{-1}\Big[E\xi_c + \int_0^c \Upsilon(\varsigma)dW(\varsigma)$$

$$-\mathcal{N}(c,0)\alpha(0) - \mathcal{S}(c, 0)[\xi_1 + \mathfrak{H}(c, \alpha(0))] + \int_0^c \mathcal{N}(c, \varsigma)\mathfrak{H}(\varsigma, \varkappa_\varsigma + \hat{\beta}_\varsigma)d\varsigma$$

$$-\int_0^c \mathcal{S}(c, \varsigma)g(\varsigma)dW(\varsigma) - \sum_{0<\iota_\imath<c} \mathcal{N}(c, \iota_\imath)\chi_\imath\big(\varkappa(\iota_\imath^-) + \hat{\beta}(\iota_\imath^-)\big)$$

$$-\sum_{0<\iota_\imath<c} \mathcal{S}(c, \iota_\imath)\overline{\chi_\imath}\big(\varkappa(\iota_\imath^-) + \hat{\alpha}(\iota_\imath^-)\big)\Big](\varpi)d\varpi$$

$$+\sum_{0<\iota_\imath<\iota} \mathcal{N}(\iota, \iota_\imath)\chi_\imath\big(\varkappa(\iota_\imath^-) + \hat{\beta}(\iota_\imath^-)\big)$$

$$+\sum_{0<\iota_\imath<\iota} \mathcal{S}(\iota, \iota_\imath)\overline{\chi_\imath}\big(\varkappa(\iota_\imath^-) + \hat{\beta}(\iota_\imath^-)\big), \ \iota \in \mathcal{V}.$$

$\Rightarrow$ the fixed point of $\Upsilon = \Upsilon_1 + \Upsilon_2$.

To determine mild solutions of (1)-(3) is shortened to determine the solutions of $\varkappa \in \Upsilon_1(\varkappa) + \Upsilon_2(\varkappa)$.

Hereafter, we verify Υ_1 and Υ_2 fulfills the requirements of Lemma 2.4.

Step 1: Υ_1 is a contraction on $\mathcal{G}_g''$. Assume that $u, v \in \mathcal{G}_g''$. From $\mathbf{H_2}$, we get

$$E\left\|\Upsilon_1 u(\iota) - \Upsilon_2 v(\iota)\right\|^2 \le E\left\|\int_0^\iota \mathcal{N}(\iota, \varpi)\mathfrak{H}(\varpi, u_\varpi + \hat{\alpha}_\varpi)d\varpi - \int_0^\iota \mathcal{N}(\iota, \varpi)\mathfrak{H}(\varpi, v_\varpi + \hat{\beta}_\varpi)d\varpi\right\|^2$$

$$\le P_1\int_0^\iota \widetilde{L_h}\left\|u_\varpi - v_\varpi\right\|_{\mathcal{G}_g}^2 d\varpi$$

$$\le 2P_1\widetilde{L_h}\int_0^\iota \Big(j^2 \sup_{\varpi\in[0,c]} E\left\|u(\varpi) - v(\varpi)\right\|_{\mathcal{G}_g}^2 + \left\|u_0\right\|_{\mathcal{G}_g}^2\Big)d\varpi.$$

Since $\|u_0\|^2 = 0$. Taking supremum over ϖ,

$$E\left\|\Upsilon_1 u(\iota) - \Upsilon_1 v(\iota)\right\|^2 \le C_0 E\left\|u - v\right\|^2.$$

In the above $C_0 = 2P_1\widetilde{L_h}j^2c < 1$. As a result, Υ_1 is a contraction on $\mathcal{G}_g''$.

Step 2: Υ_2 is completely continuous and has compact, convex values.

CLAIM 1: Υ_2 is convex $\forall\ \varkappa \in \mathcal{G}_g''$.

Assume that $\bar{\xi}_1, \bar{\xi}_2 \in \Upsilon_2(\varkappa)$, and $g_1, g_2 \in T_{\mathscr{G},\xi}$ such that $\forall\ \iota \in \mathcal{V}$,

$$\overline{\xi_\imath}(\iota) = \int_0^\iota \mathcal{S}(\iota, \varpi) g_\imath(\varpi) dW(\varpi) + \int_0^\iota \mathcal{S}(\iota, \varpi) BB^* \mathcal{S}^*(c, \varpi)(\delta I + \aleph_0^c)^{-1} \Big[E\xi_c + \int_0^c \Upsilon(\varsigma) dW(\varsigma)$$

$$-\mathcal{N}(c, 0)\alpha(0) - \mathcal{S}(c, 0)[\xi_1 + \mathfrak{H}(c, \alpha(0))] + \int_0^c \mathcal{N}(c, \varsigma)\mathfrak{H}(\varsigma, \varkappa_\varsigma + \hat{\beta}_\varsigma) d\varsigma$$

$$-\int_0^c \mathcal{S}(c, \varsigma) g_\imath(\varsigma) dW(\varsigma) - \sum_{0<\iota_l<c} \mathcal{N}(c, \iota_l)\chi_l\big(\varkappa(\iota_l^-) + \hat{\beta}(\iota_l^-)\big)$$

$$-\sum_{0<\iota_l<c} \mathcal{S}(c, \iota_l)\overline{\chi_l}\big(\varkappa(\iota_l^-) + \hat{\beta}(\iota_l^-)\big)\Big](\varpi) d\varpi$$

$$+\sum_{0<\iota_l<\iota} \mathcal{N}(\iota, \iota_l)\chi_l\big(\varkappa(\iota_l^-) + \hat{\beta}(\iota_l^-)\big)$$

$$+\sum_{0<\iota_l<\iota} \mathcal{S}(\iota, \iota_l)\overline{\chi_l}\big(\varkappa(\iota_l^-) + \hat{\beta}(\iota_l^-)\big), \imath = 1, 2.$$

Let $\beta \in [0, 1]$. Then $\forall\ \iota \in \mathcal{V}$, we get

$$(\beta\overline{\xi_1} + (1-\beta)\overline{\xi_2})(\iota) = \int_0^\iota \mathcal{S}(\iota, \varpi)[\beta g_1(\varpi) + (1-\beta) g_2(\varpi)] dW(\varpi)$$

$$+\int_0^\iota \mathcal{S}(\iota, \varpi) BB^* \mathcal{S}^*(c, \varpi)(\delta I + \aleph_0^c)^{-1} \Big[E\xi_c + \int_0^c \Upsilon(\varsigma) dW(\varsigma) - \mathcal{N}(c, 0)\alpha(0)$$

$$-\mathcal{S}(c, 0)[\xi_1 + \mathfrak{H}(c, \alpha(0))] + \int_0^c \mathcal{N}(c, \varsigma)\mathfrak{H}(\varsigma, \varkappa_\varsigma + \hat{\beta}_\varsigma) d\varsigma$$

$$-\int_0^c \mathcal{S}(c, \varsigma)[\beta g_1(\varsigma) + (1-\beta) g_2(\varsigma)] dW(\varsigma)$$

$$-\sum_{0<\iota_l<c} \mathcal{N}(c, \iota_l)\chi_l\big(\varkappa(\iota_l^-) + \hat{\beta}(\iota_l^-)\big)$$

$$-\sum_{0<\iota_l<c} \mathcal{S}(c, \iota_l)\overline{\chi_l}\big(\varkappa(\iota_l^-) + \hat{\beta}(\iota_l^-)\big)\Big](\varpi) d\varpi$$

$$+\sum_{0<\iota_l<\iota} \mathcal{N}(\iota, \iota_l)\chi_l\big(\varkappa(\iota_l^-) + \hat{\beta}(\iota_l^-)\big)$$

$$+\sum_{0<\iota_l<\iota} \mathcal{S}(\iota, \iota_i)\overline{\chi_l}\big(\varkappa(\iota_l^-) + \hat{\beta}(\iota_l^-)\big).$$

Clearly, $T_{\mathscr{G}, \xi}$ is convex, sine $\mathscr{G}$ has convex values.

$\Rightarrow \beta g_1 + (1-\beta)g_2 \in T_{\mathscr{G},\xi}$.

Consequently,

$$\beta\overline{\xi_1} + (1-\beta)\overline{\xi_2} \in \Upsilon_2(\varkappa).$$

CLAIM 2: In $\mathcal{G}_g''$, Υ_2 maps bounded sets to bounded sets.

In fact, the following statement suffices to demonstrate that $\exists$ a positive constant Λ, such that $\forall\ \Upsilon \in \Upsilon_2(\varkappa)$, $\varkappa \in B_p = \{\varkappa \in \mathcal{G}_g'' : \|\varkappa\|_c^2 \le p\}$ one has $\|\Upsilon\|_c^2 \le \Lambda$.

If $\Upsilon \in \Upsilon_2(\varkappa)$, then $\exists\ g \in T_{\mathscr{G},\xi}$ such that $\forall\ \iota \in \mathcal{V}$,

$$\Upsilon(\iota) = \int_0^\iota \mathcal{S}(\iota, \varpi)g(\varpi)dW(\varpi) + \int_0^\iota \mathcal{S}(\iota, \varpi)Bu_\delta(\varpi, \xi)d\varpi$$

$$+ \sum_{0<\iota_\iota<\iota} \mathcal{N}(\iota, \iota_\iota)\chi_\iota\left(\varkappa(\iota_\iota^-) + \hat{\beta}(\iota_\iota^-)\right) + \sum_{0<\iota_\iota<\iota} \mathcal{S}(\iota, \iota_\iota)\overline{\chi_\iota}\left(\varkappa(\iota_\iota^-) + \hat{\beta}(\iota_\iota^-)\right).$$

Therefore, by hypothesis, $\forall\ \iota \in \mathcal{V}$, we get

$$E\|\Upsilon\|^2 \le 4E\|\int_0^\iota \mathcal{S}(\iota, \varpi)g(\varpi)dW(\varpi)\|^2 + 4E\|\int_0^\iota \mathcal{S}(\iota, \varpi)Bu_\delta(\varpi, \xi)d\varpi\|^2$$

$$+4E\|\sum_{0<\iota_\iota<\iota} \mathcal{N}(\iota, \iota_\iota)\chi_\iota\left(\varkappa(\iota_\iota^-) + \hat{\beta}(\iota_\iota^-)\right)\|^2$$

$$+4E\|\sum_{0<\iota_\iota<\iota} \mathcal{S}(\iota, \iota_\iota)\overline{\chi_\iota}\left(\varkappa(\iota_\iota^-) + \hat{\beta}(\iota_\iota^-)\right)\|^2$$

$$\le 4P_2 Tr(\mathcal{Q})\int_0^\iota \varrho_{p'}(\varpi)d\varpi + \frac{28}{\delta}P_2^2 P_B^2 c\Big[2E\|\xi_c\|^2 + 2\int_0^c E\|\Upsilon(\varpi)\|_{L_2^0}^2 d\varpi + P_1 E\|\alpha(0)\|^2$$

$$+2P_2 E(\|\xi_1\|^2 + \|\mathfrak{H}(c, \alpha(0))\|^2) + P_1\int_0^c \widetilde{L_h}[1+p']d\varpi + P_2 Tr(\mathcal{Q})\int_0^c \varrho_{p'}(\varpi)d\varpi$$

$$+P_1 n^2\sum_{\iota=1}^n P_m(j^{-2}p') + P_2 n^2\sum_{\iota=1}^n \overline{P_m}(j^{-2}p')\Big]$$

$$+4P_1 n^2\sum_{\iota=1}^n P_m(j^{-2}p') + 4P_2 n^2\sum_{\iota=1}^n \overline{P_m}(j^{-2}p')$$

$$\le 4P_2 Tr(\mathcal{Q})\|\varrho_{p'}\|_{L^1} + \frac{28}{\delta}P_2^2 P_B^2 c\Big[2E\|\xi_c\|^2 + 2\int_0^c E\|\Upsilon(\varpi)\|_{L_2^0}^2 d\varpi + P_1 E\|\alpha(0)\|^2$$

$$+2P_2 E(\|\xi_1\|^2 + \|\mathfrak{H}(c, \alpha(0))\|^2) + P_1\int_0^c \widetilde{L_h}[1+p']d\varpi + P_2 Tr(\mathcal{Q})\left\|\varrho_{p'}\right\|_{L^1}$$

$$+P_1 n^2 \sum_{\iota=1}^{n} P_m (j^{-2} p^{'}) + P_2 n^2 \sum_{\iota=1}^{n} \overline{P_m} (j^{-2} p^{'})\Big]$$

$$+4P_1 n^2 \sum_{\iota=1}^{n} P_m (j^{-2} p^{'}) + 4P_2 n^2 \sum_{\iota=1}^{n} \overline{P_m} (j^{-2} p^{'})$$

$$\leq \Lambda.$$

Then, $\forall\ \Upsilon \in \Upsilon_2(\varkappa)$, we get $\|\Upsilon\|_c^2 \leq \Lambda$.

CLAIM 3: Bounded sets are converted into $\mathcal{G}_g''$ equicontinuous sets via Υ_2.

$$\Upsilon_2(\iota) = \int_0^{\iota} \mathcal{S}(\iota,\ \varpi) g(\varpi) dW(\varpi) + \int_0^{\iota} \mathcal{S}(\iota,\ \varpi) Bu_{\delta}(\varpi,\ \xi) d\varpi$$

$$+ \sum_{0<\iota_\iota<\iota} \mathcal{N}(\iota,\ \iota_\iota)\chi_\iota\Big(\varkappa(\iota_\iota^-) + \hat{\beta}(\iota_\iota^-)\Big) \sum_{0<\iota_\iota<\iota} \mathcal{S}(\iota,\ \iota_\iota)\overline{\chi_\iota}\Big(\varkappa(\iota_\iota^-) + \hat{\beta}(\iota_\iota^-)\Big).$$

Let $0 < \iota_1 < \iota_2 \leq c$. Then, we obtain $\forall\ \varkappa \in B_p$ and $\Upsilon \in \Upsilon_2(\varkappa)$, $\exists\ g \in T_{\mathscr{G},\xi}$ such that

$$E\left\|\Upsilon(\iota_2) - \Upsilon(\iota_1)\right\|^2 \leq E\Bigg\|\Big[\int_0^{\iota_2} \mathcal{S}(\iota_2,\ \varpi) g(\varpi) dW(\varpi) + \int_0^{\iota_2} \mathcal{S}(\iota_2,\ \varpi) Bu_{\delta}(\varpi,\ \xi) d\varpi$$

$$+ \sum_{0<\iota_\iota<\iota_2} \mathcal{N}(\iota_2,\ \iota_\iota)\chi_\iota\Big(\varkappa(\iota_\iota^-) + \hat{\beta}(\iota_\iota^-)\Big) + \sum_{0<\iota_\iota<\iota_2} \mathcal{S}(\iota_2,\ \iota_\iota)\overline{\chi_\iota}\Big(\varkappa(\iota_\iota^-) + \hat{\beta}(\iota_\iota^-)\Big)\Big]$$

$$-\Big[\int_0^{\iota_1} \mathcal{S}(\iota_1,\ \varpi) g(\varpi) dW(\varpi) + \int_0^{\iota-1} \mathcal{S}(\iota_1,\ \varpi) Bu_{\delta}(\varpi,\ \xi) d\varpi$$

$$+ \sum_{0<\iota_\iota<\iota_1} \mathcal{N}(\iota_1,\ \iota_\iota)\chi_\iota\Big(\varkappa(\iota_\iota^-) + \hat{\beta}(\iota_\iota^-)\Big) + \sum_{0<\iota_\iota<\iota_1} \mathcal{S}(\iota_1,\ \iota_\iota)\overline{\chi_\iota}\Big(\varkappa(\iota_\iota^-) + \hat{\beta}(\iota_\iota^-)\Big)\Big]\Bigg\|^2$$

$$\leq 10E\left\|\int_0^{\iota_1-\varrho} \big[\mathcal{S}(\iota_2,\ \varpi) - \mathcal{S}(\iota_1,\ \varpi)\big] g(\varpi) dW(\varpi)\right\|^2$$

$$+10E\left\|\int_{\iota_1-\varrho}^{\iota_1} \big[\mathcal{S}(\iota_2,\ \varpi) - \mathcal{S}(\iota_1,\ \varpi)\big] g(\varpi) dW(\varpi)\right\|^2$$

$$+10E\left\|\int_{\iota_1}^{\iota_2} \mathcal{S}(\iota_2,\ \varpi) g(\varpi) dW(\varpi)\right\|^2$$

$$+10E\left\|\int_0^{\iota_1-\varrho} \big[\mathcal{S}(\iota_2,\ \varpi) - \mathcal{S}(\iota_1,\ \varpi)\big] Bu_{\delta}(\varpi,\ \xi) d\varpi\right\|^2$$

$$+10E\left\|\int_{\iota_1-\varrho}^{\iota_1}\left[\mathcal{S}(\iota_2,\varpi)-\mathcal{S}(\iota_1,\varpi)\right]Bu_\delta(\varpi,\xi)d\varpi\right\|^2$$

$$+10E\left\|\int_{\iota_1}^{\iota_2}\mathcal{S}(\iota_2,\varpi)Bu_\delta(\varpi,\xi)d\varpi\right\|^2$$

$$+10E\left\|\sum_{0<\iota_\iota<\iota_2}\left[\mathcal{N}(\iota_2,\iota_\iota)-\mathcal{N}(\iota_1,\iota_\iota)\right]\chi_\iota\left(\varkappa(\iota_\iota^-)+\hat{\beta}(\iota_\iota^-)\right)\right\|^2$$

$$+10E\left\|\sum_{0<\iota_\iota<\iota_2}\mathcal{N}(\iota_2,\iota_\iota)\chi_\iota\left(\varkappa(\iota_\iota^-)+\hat{\beta}(\iota_\iota^-)\right)\right\|^2$$

$$+10E\left\|\sum_{0<\iota_\iota<\iota_1}\left[\mathcal{S}(\iota_2,\iota_\iota)-\mathcal{S}(\iota_1,\iota_\iota)\right]\overline{\chi_\iota}\left(\varkappa(\iota_\iota^-)+\hat{\beta}(\iota_\iota^-)\right)\right\|^2$$

$$+10E\left\|\sum_{\iota_1<\iota_\iota<\iota_2}\mathcal{S}(\iota_2,\iota_\iota)\overline{\chi_\iota}\left(\varkappa(\iota_\iota^-)+\hat{\alpha}(\iota_\iota^-)\right)\right\|^2$$

$$\leq 10Tr(\mathcal{Q})E\int_0^{\iota_1-\varrho}\left\|\mathcal{S}(\iota_2,\varpi)-\mathcal{S}(\iota_1,\varpi)\right\|^2\left\|g(\varpi)\right\|^2d\varpi$$

$$+10Tr(\mathcal{Q})E\int_{\iota_1-\varrho}^{\iota_1}\left\|\mathcal{S}(\iota_2,\varpi)-\mathcal{S}(\iota_1,\varpi)\right\|^2\left\|g(\varpi)\right\|^2d\varpi$$

$$+10Tr(\mathcal{Q})E\int_{\iota_1}^{\iota_2}\left\|\mathcal{S}(\iota_2,\varpi)\right\|^2\left\|g(\varpi)\right\|^2d\varpi$$

$$+70\int_0^{\iota_1-\varrho}\left\|\mathcal{S}(\iota_2,\varpi)-\mathcal{S}(\iota_1,\varpi)\right\|^2\|B\|^2\left\{2E\|\xi_c\|^2+2\int_0^c E\|\Upsilon(\varsigma)\|_{L_2^0}^2d\varsigma\right.$$

$$+\|\mathcal{N}(c,0)\|^2E\|\alpha(0)\|^2+2\|\mathcal{S}(c,0)\|^2E\left[\|\xi_1\|^2+\|\mathfrak{H}(c,0)\|^2\right]$$

$$+\int_0^c\|\mathcal{N}(c,\varsigma)\|^2E\left\|\mathfrak{H}(\varsigma,\varkappa_\varsigma+\hat{\beta}_\varsigma)\right\|^2d\varsigma+Tr(\mathcal{Q})\int_0^c\|\mathcal{S}(c,\varsigma)\|^2E\|g(\varsigma)\|^2d\varsigma$$

$$+n^2\sum_{\iota=1}^n\|\mathcal{N}(c,\iota_\iota)\|^2E\left\|\chi_\iota\left(\varkappa(\iota_\iota^-)+\hat{\beta}(\iota_\iota^-)\right)\right\|^2$$

$$+n^2\sum_{\iota=1}^{n}\left\|\mathcal{S}(c,\,\iota_\iota)\right\|^2 E\left\|\overline{\chi_\iota}\left(\varkappa(\iota_\iota^-)+\hat{\beta}(\iota_\iota^-)\right)\right\|^2\Bigg\}(\varpi)d\varpi$$

$$+70\int_{\iota_1-\varrho}^{\iota_1}\left\|\mathcal{S}(\iota_2,\,\varpi)-\mathcal{S}(\iota_1,\,\varpi)\right\|^2\left\|B\right\|^2\Bigg\{2E\left\|\xi_c\right\|^2+2\int_0^c E\left\|\Upsilon(\varsigma)\right\|_{L_2^0}^2 d\varsigma$$

$$+\left\|\mathcal{N}(c,\,0)\right\|^2 E\left\|\alpha(0)\right\|^2+2\left\|\mathcal{S}(c,\,0)\right\|^2 E\left[\left\|\xi_1\right\|^2+\left\|\mathfrak{H}(c,\,0)\right\|^2\right]$$

$$+\int_0^c\left\|\mathcal{N}(c,\,\varsigma)\right\|^2 E\left\|\mathfrak{H}(\varsigma,\,\varkappa_\varsigma+\hat{\beta}_\varsigma)\right\|^2 d\varsigma+Tr(\mathcal{Q})\int_0^c\left\|\mathcal{S}(c,\,\varsigma)\right\|^2 E\left\|g(\varsigma)\right\|^2 d\varsigma$$

$$+n^2\sum_{\iota=1}^{n}\left\|\mathcal{N}(c,\,\iota_\iota)\right\|^2 E\left\|\chi_\iota\left(\varkappa(\iota_\iota^-)+\hat{\beta}(\iota_\iota^-)\right)\right\|^2$$

$$+n^2\sum_{\iota=1}^{n}\left\|\mathcal{S}(c,\,\iota_\iota)\right\|^2 E\left\|\overline{\chi_\iota}\left(\varkappa(\iota_\iota^-)+\hat{\beta}(\iota_\iota^-)\right)\right\|^2\Bigg\}(\varpi)d\varpi$$

$$+70\int_{\iota_1}^{\iota_2}\left\|\mathcal{S}(\iota_2,\,\varpi)\right\|^2\left\|B\right\|^2\Bigg\{2E\left\|\xi_c\right\|^2+2\int_0^c E\left\|\Upsilon(\varsigma)\right\|_{L_2^0}^2 d\varsigma$$

$$+\left\|\mathcal{N}(c,\,0)\right\|^2 E\left\|\alpha(0)\right\|^2+2\left\|\mathcal{S}(c,\,0)\right\|^2 E\left[\left\|\xi_1\right\|^2+\left\|\mathfrak{H}(c,\,0)\right\|^2\right]$$

$$+\int_0^c\left\|\mathcal{N}(c,\,\varsigma)\right\|^2 E\left\|\mathfrak{H}(\varsigma,\,\varkappa_\varsigma+\hat{\beta}_\varsigma)\right\|^2 d\varsigma+Tr(\mathcal{Q})\int_0^c\left\|\mathcal{S}(c,\,\varsigma)\right\|^2 E\left\|g(\varsigma)\right\|^2 d\varsigma$$

$$+n^2\sum_{\iota=1}^{n}\left\|\mathcal{N}(c,\,\iota_\iota)\right\|^2 E\left\|\chi_\iota\left(\varkappa(\iota_\iota^-)+\hat{\beta}(\iota_\iota^-)\right)\right\|^2$$

$$+n^2\sum_{\iota=1}^{n}\left\|\mathcal{S}(c,\,\iota_\iota)\right\|^2 E\left\|\overline{\chi_\iota}\left(\varkappa(\iota_\iota^-)+\hat{\beta}(\iota_\iota^-)\right)\right\|^2\Bigg\}(\varpi)d\varpi$$

$$+10n^2\sum_{\iota=1}^{n}\left\|\mathcal{N}(\iota_2,\,\varpi)-\mathcal{N}(\iota_1,\,\varpi)\right\|^2 E\left\|\chi_\iota\left(\varkappa(\iota_\iota^-)+\hat{\beta}(\iota_\iota^-)\right)\right\|^2$$

$$+10n^2\sum_{\iota=1}^{n}\left\|\mathcal{N}(\iota_2,\,\varpi)\right\|^2 E\left\|\chi_\iota\left(\varkappa(\iota_\iota^-)+\hat{\beta}(\iota_\iota^-)\right)\right\|^2$$

$$+10n^2\sum_{\iota=1}^{n}\left\|\mathcal{S}(\iota_2,\,\varpi)-\mathcal{S}(\iota_1,\,\varpi)\right\|^2 E\left\|\overline{\chi_\iota}\left(\varkappa(\iota_\iota^-)+\hat{\beta}(\iota_\iota^-)\right)\right\|^2$$

$$+10n^2\sum_{\iota=1}^{n}\left\|\mathcal{S}(\iota_2,\ \varpi)\right\|^2 E\left\|\overline{\chi_\iota}\left(\varkappa(\iota_\iota^-)+\hat{\beta}(\iota_\iota^-)\right)\right\|^2$$

$$\leq 10Tr(\mathcal{Q})\int_0^{\iota_1-\varrho}\left\|\mathcal{S}(\iota_2,\ \varpi)-\mathcal{S}(\iota_1,\ \varpi)\right\|^2\varrho_{p'}(\varpi)d\varpi$$

$$+10Tr(\mathcal{Q})\int_{\iota_1-\varrho}^{\iota_1}\left\|\mathcal{S}(\iota_2,\ \varpi)-\mathcal{S}(\iota_1,\ \varpi)\right\|^2\varrho_{p'}(\varpi)d\varpi$$

$$+10P_2Tr(\mathcal{Q})\int_{\iota_1}^{\iota_2}\varrho_{p'}(\varpi)d\varpi$$

$$+70P_B\int_0^{\iota_1-\varrho}\left\|\mathcal{S}(\iota_2,\ \varpi)-\mathcal{S}(\iota_1,\ \varpi)\right\|^2\left[2E\left\|\xi_c\right\|^2+2\int_0^c E\left\|\Upsilon(\varsigma)\right\|_{L_2^0}^2 d\varsigma\right.$$

$$+P_1E\left\|\alpha(0)\right\|^2+2P_2E\left(\left\|\xi_1\right\|^2+\left\|\mathfrak{H}(c,\ \alpha(0))\right\|^2\right)$$

$$+P_1\int_0^c\widetilde{L_h}\left[1+p'\right]d\varsigma+P_2Tr(\mathcal{Q})\int_0^c\varrho_{p'}(\varsigma)d\varsigma$$

$$\left.+P_1n^2\sum_{\iota=1}^{n}P_m\left(j^{-2}p'\right)+P_2n^2\sum_{\iota=1}^{n}\overline{P_m}\left(j^{-2}p'\right)\right](\varpi)d\varpi$$

$$+70P_B\int_{\iota_1-\varrho}^{\iota_1}\left\|\mathcal{S}(\iota_2,\ \varpi)-\mathcal{S}(\iota_1,\ \varpi)\right\|^2\left[2E\left\|\xi_c\right\|^2+2\int_0^c E\left\|\Upsilon(\varsigma)\right\|_{L_2^0}^2 d\varsigma\right.$$

$$+P_1E\left\|\alpha(0)\right\|^2+2P_2E\left(\left\|\xi_1\right\|^2+\left\|\mathfrak{H}(c,\ \alpha(0))\right\|^2\right)$$

$$+P_1\int_0^c\widetilde{L_h}\left[1+p'\right]d\varsigma+P_2Tr(\mathcal{Q})\int_0^c\varrho_{p'}(\varsigma)d\varsigma$$

$$\left.+P_1n^2\sum_{\iota=1}^{n}P_m\left(j^{-2}p'\right)+P_2n^2\sum_{\iota=1}^{n}\overline{P_m}\left(j^{-2}p'\right)\right](\varpi)d\varpi$$

$$+70P_BP_2\int_{\iota_1}^{\iota_2}\left[2E\left\|\xi_c\right\|^2+2\int_0^c E\left\|\Upsilon(\varsigma)\right\|_{L_2^0}^2 d\varsigma\right.$$

$$+P_1E\left\|\alpha(0)\right\|^2+2P_2E\left(\left\|\xi_1\right\|^2+\left\|\mathfrak{H}(c,\ \alpha(0))\right\|^2\right)$$

$$+P_1\int_0^c\widetilde{L_h}\left[1+p'\right]d\varsigma+P_2Tr(\mathcal{Q})\int_0^c\varrho_{p'}(\varsigma)d\varsigma$$

$$+P_1 n^2 \sum_{\imath=1}^{n} P_m\left(j^{-2} p'\right) + P_2 n^2 \sum_{\imath=1}^{n} \overline{P_m}\left(j^{-2} p'\right)\Bigg](\varpi) d\varpi$$

$$+10 n^2 \sum_{\imath=1}^{n} \left\| \mathcal{N}(\imath_2, \varpi) - \mathcal{N}(\imath_1, \varpi) \right\|^2 P_m\left(j^{-2} p'\right) + 10 P_1 n^2 \sum_{\imath=1}^{n} P_m\left(j^{-2} p'\right)$$

$$+10 n^2 \sum_{\imath=1}^{n} \left\| \mathcal{S}(\imath_2, \varpi) - \mathcal{S}(\imath_1, \varpi) \right\|^2 \overline{P_m}\left(j^{-2} p'\right) + 10 P_2 n^2 \sum_{\imath=1}^{n} \overline{P_m}\left(j^{-2} p'\right).$$

The right hand side of the inequality aforementioned is isolated of $\varkappa \in B_p$ and $\|\Upsilon(\imath_2) - \Upsilon(\imath_1)\|^2 \to 0$ as $\imath_2 - \imath_1 \to 0$, $\forall\ \varkappa \in B_p$. Thus, the compactness of $\mathcal{S}(\imath, \varpi)$ and $\mathcal{N}(\imath, \varpi)$ for $\imath > 0$ provides the continuity belongs to the uniform operator topology. As a result, $\{\Upsilon_2(\varkappa) : \varkappa \in B_p\}$ is equicontinuous.

CLAIM 4: Υ_2 is a compact multi-valued map.

Through the claim, we conclude that $\Upsilon_2 \in B_p$ is uniformly bounded and equicontinuous. Hence, it is enough to verify by using Arzela-Ascoli theorem Υ_2 maps B_p into a precompact set belongs to $\mathcal{G}_g''$, i.e., for every fixed $\imath \in \mathcal{V}$, $\Pi(\imath) = \{\Upsilon_2 \varkappa(\imath) : x \in B_p\}$ is precompact belongs to $\mathbf{Z}$.

Clearly, $\Pi(0) = \{\Upsilon(0)\}$. Assume that $\imath > 0$ be fixed and for $0 < \varrho < \imath$, determine

$$\Upsilon_2^{\varrho}(\imath) = \int_0^{\imath-\varrho} \mathcal{S}(\imath, \varpi) g(\varpi) dW(\varpi) + \int_0^{\imath-\varrho} \mathcal{S}(\imath, \varpi) B u_\delta(\varpi, \xi) d\varpi$$

$$+ \sum_{0<\imath_\imath<\imath} \mathcal{N}(\imath, \imath_\imath) \chi_\imath\left(\varkappa(\imath_\imath^-) + \hat{\beta}(\imath_\imath^-)\right) + \sum_{0<\imath_\imath<\imath} \mathcal{S}(\imath, \imath_\imath) \overline{\chi_\imath}\left(\varkappa(\imath_\imath^-) + \hat{\beta}(\imath_\imath^-)\right).$$

Since $\mathcal{S}(\imath, \varpi)$ is a compact operator, the family $\Pi_\varrho(\imath) = \{\Upsilon_2^{\varrho}\varkappa(\imath) : \varkappa \in B_p\}$ is precompact in $\varkappa\ \forall\ \varrho$, $0 < \varrho < \imath$. Moreover,

$$E\left\|\Upsilon_2 \varkappa(\imath) - \Upsilon_2^{\varrho}\varkappa(\imath)\right\|^2 \le 2E\left\|\int_{\imath-\varrho}^{\imath} \mathcal{S}(\imath, \varpi) g(\varpi) dW(\varpi)\right\|^2 + 2E\left\|\int_{\imath-\varrho}^{\imath} \mathcal{S}(\imath, \varpi) B u_\delta(\varpi, \xi) d\varpi\right\|^2$$

$$\le 2E\left\|\int_{\imath-\varrho}^{\imath} \mathcal{S}(\imath, \varpi) g(\varpi) dW(\varpi)\right\|^2$$

$$+14\int_{\imath-\varrho}^{\imath} \|\mathcal{S}(\imath, \varpi)\|^2 \|B\|^2 \left\|B^*\right\|^2 \left\|\mathcal{S}^*(c, \varpi)\right\|^2 (\delta I + \aleph_0^c)^{-1} \Big[2E\|\xi_c\|^2$$

$$+2\int_0^c E\|\Upsilon(\varsigma)\|_{L_2^0}^2 d\varsigma + \|\mathcal{N}(c, 0)\|^2 E\|\alpha(0)\|^2$$

$$+2\|\mathcal{S}(c, 0)\|^2 E\left[\|\xi_1\|^2 + \|\mathfrak{H}(c, 0)\|^2\right]$$

$$+\int_0^c \|\mathcal{N}(c,\varsigma)\|^2 E\left\|\mathfrak{H}(\varsigma, \varkappa_\varsigma + \hat{\beta}_\varsigma)\right\|^2 d\varsigma$$

$$+\int_0^c \left\|\mathcal{S}(c,\varsigma)^2\right\| E\|g(\varsigma)\|^2 dW(\varsigma)$$

$$+\sum_{0<\iota_\imath<c} \|\mathcal{N}(c,\iota_\imath)\|^2 E\left\|\chi_\imath\left(\varkappa(\iota_\imath^-)+\hat{\beta}(\iota_\imath^-)\right)\right\|^2$$

$$\left.+\sum_{0<\iota_\imath<c} \|\mathcal{S}(c,\iota_\imath)\|^2 E\left\|\overline{\chi_\imath}\left(\varkappa(\iota_\imath^-)+\hat{\beta}(\iota_\imath^-)\right)\right\|^2\right](\varpi)d\varpi.$$

Therefore, $E\|\Upsilon_2\varkappa(\iota) - \Upsilon_2^{\varrho}\varkappa(\iota)\|^2 \to 0$, as $\varrho \to 0^+$, in addition there are precompact sets are arbitrary closed to the family $\{\Upsilon_2\varkappa(\iota) : x \in B_p\}$. Since Υ_2 is compact.

CLAIM 5: To prove Υ_2 has a closed graph.

Suppose that $\varkappa_n$ tends to $\varkappa_*$ as n tends to ∞, $\overline{\xi}_n \in \Upsilon_2 \varkappa_n\ \forall\ \varkappa_n$ belongs to B_p, and $\overline{\xi}_n \to \overline{\xi}_*$ as $n \to \infty$. We derive $\overline{\xi}_* \in \Upsilon_2 \varkappa_*$. Since $\overline{\xi}_n \in \Upsilon_2 \varkappa_n$, $\exists\ g_n \in T_{\mathscr{G},\varkappa_n}$ such that

$$\overline{\xi_n}(\iota) = \int_0^\iota \mathcal{S}(\iota,\varpi) g_n(\varpi) dw(\varpi) + \int_0^\iota \mathcal{S}(\iota,\varpi) BB^* \mathcal{S}^*(c,\varpi)(\delta I + \aleph_0^c)^{-1}\left[E\xi_c + \int_0^c \Upsilon(\varsigma) dW(\varsigma)\right.$$

$$-\mathcal{N}(c,0)\alpha(0) - \mathcal{S}(c,0)[\xi_1 + \mathfrak{H}(c,0)] + \int_0^c \mathcal{N}(c,\varsigma)\mathfrak{H}(\varsigma,(\varkappa_n)_\varsigma + \hat{\beta}_\varsigma) d\varsigma$$

$$-\int_0^c \mathcal{S}(c,\varsigma) g_n(\varsigma) dW(\varsigma) - \sum_{0<\iota_\imath<c} \mathcal{N}(c,\iota_\imath)\chi_\imath\left(\varkappa_n(\iota_\imath^-)+\hat{\beta}(\iota_\imath^-)\right)$$

$$\left.-\sum_{0<\iota_\imath<c} \mathcal{S}(c,\iota_\imath)\overline{\chi_\imath}\left(\varkappa_n(\iota_\imath^-)+\hat{\beta}(\iota_\imath^-)\right)\right](\varpi)d\varpi$$

$$+\sum_{0<\iota_\imath<\iota} \mathcal{N}(\iota,\iota_\imath)\chi_\imath\left(\varkappa_n(\iota_\imath^-)+\hat{\beta}(\iota_\imath^-)\right)$$

$$+\sum_{0<\iota_\imath<\iota} \mathcal{S}(\iota,\iota_\imath)\overline{\chi_\imath}\left(\varkappa_n(\iota_\imath^-)+\hat{\beta}(\iota_\imath^-)\right),\ \iota \in \mathcal{V}.$$

We must demonstrate that $\exists\ g_* \in T_{\mathscr{G},\varkappa_*}$ such that

$$\overline{\xi_*}(\iota) = \int_0^\iota \mathcal{S}(\iota, \varpi) g_*(\varpi) dW(\varpi) + \int_0^\iota \mathcal{S}(\iota, \varpi) BB^* \mathcal{S}^*(c, \varpi)(\delta I + \aleph_0^c)^{-1} \Big[E\xi_c + \int_0^c \Upsilon(\varsigma) dW(\varsigma)$$

$$-\mathcal{N}(c, 0)\alpha(0) - \mathcal{S}(c, 0)[\xi_1 + \mathfrak{H}(c, 0)] + \int_0^c \mathcal{N}(c, \varsigma)\mathfrak{H}(\varsigma, (\varkappa_*)_\varsigma + \hat{\beta}_\varsigma) d\varsigma$$

$$-\int_0^c \mathcal{S}(c, \varsigma) g_*(\varsigma) dW(\varsigma) - \sum_{0<\iota_l<c} \mathcal{N}(c, \iota_l)\chi_l \Big(\varkappa_*(\iota_l^-) + \hat{\beta}(\iota_l^-) \Big)$$

$$- \sum_{0<\iota_l<c} \mathcal{S}(c, \iota_l)\overline{\chi_l} \Big(\varkappa_*(\iota_l^-) + \hat{\beta}(\iota_l^-) \Big) \Big] (\varpi) d\varpi$$

$$+ \sum_{0<\iota_l<\iota} \mathcal{N}(\iota, \iota_l)\chi_l \Big(\varkappa_*(\iota_l^-) + \hat{\beta}(\iota_l^-) \Big)$$

$$+ \sum_{0<\iota_l<\iota} \mathcal{S}(\iota, \iota_l)\overline{\chi_l} \Big(\varkappa_*(\iota_l^-) + \hat{\beta}(\iota_l^-) \Big), \ \iota \in \mathcal{V}.$$

Now, $\forall \ \iota \in \mathcal{V}$, because $\mathscr{G}$ is continuous and from $\varkappa^\varrho$, we get

$$\Bigg\| \Bigg(\overline{\xi_n}(\iota) - \sum_{0<\iota_l<\iota} \mathcal{N}(\iota, \iota_l)\chi_l \Big(\varkappa_n(\iota_l^-) + \hat{\beta}(\iota_l^-) \Big) - \sum_{0<\iota_l<\iota} \mathcal{S}(\iota, \iota_l)\overline{\chi_l} \Big(\varkappa_n(\iota_l^-) + \hat{\beta}(\iota_l^-) \Big)$$

$$-\int_0^\iota \mathcal{S}(\iota, \varpi) BB^* \mathcal{S}^*(c, \varpi)(\delta I + \aleph_0^c)^{-1} \Big[E\xi_c + \int_0^c \Upsilon(\varsigma) dW(\varsigma) - \mathcal{N}(c, 0)\alpha(0)$$

$$-\mathcal{S}(c, 0)[\xi_1 + \mathfrak{H}(c, 0)] + \int_0^c \mathcal{N}(c, \varsigma)\mathfrak{H}(\varsigma, (\varkappa_n)_\varsigma + \hat{\beta}_\varsigma) d\varsigma$$

$$- \sum_{0<\iota_l<c} \mathcal{N}(c, \iota_l)\chi_l \Big(\varkappa_n(\iota_l^-) + \hat{\beta}(\iota_l^-) \Big)$$

$$- \sum_{0<\iota_l<c} \mathcal{S}(c, \iota_l)\overline{\chi_l} \Big(\varkappa_n(\iota_l^-) + \hat{\beta}(\iota_l^-) \Big) \Big] (\varpi) d\varpi \Bigg)$$

$$- \Bigg(\overline{\xi_*}(\iota) - \sum_{0<\iota_l<\iota} \mathcal{N}(\iota, \iota_l)\chi_l \Big(\varkappa_*(\iota_l^-) + \hat{\beta}(\iota_l^-) \Big)$$

$$- \sum_{0<\iota_l<\iota} \mathcal{S}(\iota, \iota_l)\overline{\chi_l} \Big(\varkappa_*(\iota_l^-) + \hat{\beta}(\iota_l^-) \Big)$$

$$-\int_0^{\iota}\mathcal{S}(\iota,\varpi)BB^*\mathcal{S}^*(c,\varpi)(\delta I+\aleph_0^c)^{-1}\Big[E\xi_c+\int_0^c\Upsilon(\varsigma)dW(\varsigma)-\mathcal{N}(c,0)\alpha(0)$$

$$-\mathcal{S}(c,0)[\xi_1+\mathfrak{H}(c,0)]+\int_0^c\mathcal{N}(c,\varsigma)\mathfrak{H}(\varsigma,(\varkappa_*)_\varsigma+\hat{\beta}_\varsigma)d\varsigma$$

$$-\sum_{0<\iota_l<c}\mathcal{N}(c,\iota_l)\chi_l\left(\varkappa_*(\iota_l^-)+\hat{\beta}(\iota_l^-)\right)$$

$$-\sum_{0<\iota_l<c}\mathcal{S}(c,\iota_l)\overline{\chi_l}\left(\varkappa_*(\iota_l^-)+\hat{\beta}(\iota_l^-)\right)\Big](\varpi)d\varpi\Bigg\|\to 0 \text{ as } n\to\infty.$$

Suppose $\Theta: L^2(\mathcal{V},\mathbf{Z})\to\mathcal{C}(\mathcal{V},\mathbf{Z})$ which is continuous,

$$(\Theta g)(\iota)=\int_0^{\iota}\mathcal{S}(\iota,\varpi)g(\varpi)dW(\varpi)$$

$$-\int_0^{\iota}\mathcal{S}(\iota,\varpi)BB^*\mathcal{S}^*(c,\varpi)(\delta I+\aleph_0^c)^{-1}\left(\int_0^c\mathcal{S}(c,\varsigma)g(\varsigma)dW(\varsigma)\right)d\varpi.$$

Therefore, by Lemma 2.4, $\Theta\circ T_{\mathscr{G}}$ is a closed graph operator. Then, by referring Θ, we get

$$\overline{\xi_n}(\iota)-\sum_{0<\iota_l<\iota}\mathcal{N}(\iota,\iota_l)\chi_l\left((\varkappa_n)_{\iota_l}+\hat{\beta}_{\iota_l}\right)-\sum_{0<\iota_l<\iota}\mathcal{S}(\iota,\iota_l)\overline{\chi_l}\left((\varkappa_n)_{\iota_l}+\hat{\beta}_{\iota_l}\right)$$

$$-\int_0^{\iota}\mathcal{S}(\iota,\varpi)BB^*\mathcal{S}^*(c,\varpi)(\delta I+\aleph_0^c)^{-1}\Big[E\xi_c+\int_0^c\Upsilon(\varsigma)dW(\varsigma)-\mathcal{N}(c,0)\alpha(0)$$

$$-\mathcal{S}(c,0)[\xi_1+\mathfrak{H}(c,0)]+\int_0^c\mathcal{N}(c,\varsigma)\mathfrak{H}(\varsigma,(\varkappa_n)_\varsigma+\hat{\beta}_\varsigma)d\varsigma$$

$$-\sum_{0<\iota_l<c}\mathcal{N}(c,\iota_l)\chi_l\left((\varkappa_n)_{\iota_l}+\hat{\beta}_{\iota_l}\right)$$

$$-\sum_{0<\iota_l<c}\mathcal{S}(c,\iota_l)\overline{\chi_l}\left((\varkappa_n)_{\iota_l}+\hat{\beta}_{\iota_l}\right)\Big](\varpi)d\varpi\in\Theta(T_{\mathscr{G},\varkappa_n}).$$

Because $\varkappa_n\to\varkappa_*$, for some $\varkappa^*\in T_{\mathscr{G},\varkappa_*}$. From Lemma 2.4,

$$\overline{\xi_*}(\iota)-\sum_{0<\iota_l<\iota}\mathcal{N}(\iota,\iota_l)\chi_l\left((\varkappa_*)_{\iota_l}+\hat{\beta}_{\iota_l}\right)-\sum_{0<\iota_l<\iota}\mathcal{S}(\iota,\iota_l)\overline{\chi_l}\left((\varkappa_*)_{\iota_l}+\hat{\beta}_{\iota_l}\right)$$

$$-\int_0^\iota\mathcal{S}(\iota,\varpi)BB^*\mathcal{S}^*(c,\varpi)(\delta I+\aleph_0^c)^{-1}\Bigg[E\xi_c+\int_0^c\Upsilon(\varsigma)dW(\varsigma)-\mathcal{N}(c,0)\alpha(0)$$

$$-\mathcal{S}(c,0)[\xi_1+\mathfrak{H}(c,0)]+\int_0^c\mathcal{N}(c,\varsigma)\mathfrak{H}(\varsigma,(\varkappa_*)_\varsigma+\hat{\beta}_\varsigma)d\varsigma$$

$$-\sum_{0<\iota_l<c}\mathcal{N}(c,\iota_l)\chi_l\left((\varkappa_*)_{\iota_l}+\hat{\beta}_{\iota_l}\right)$$

$$-\sum_{0<\iota_l<c}\mathcal{S}(c,\iota_l)\overline{\chi_l}\left((\varkappa_*)_{\iota_l}+\hat{\beta}_{\iota_l}\right)\Bigg](\varpi)d\varpi$$

$$=\int_0^\iota\mathcal{S}(\iota,\varsigma)\left[g_*(\varsigma)+BB^*\mathcal{S}^*(c,\varpi)(\delta I+\aleph_0^c)^{-1}\left(\int_0^c\mathcal{S}(c,\varpi)g_*(\varpi)dW(\varpi)\right)(\varsigma)\right]d\varsigma,$$

for some $g_*\in(T_{\mathscr{G},\varkappa_*})$. Hence Υ_2 has a closed graph.

Step 3:

From the Steps 1 and 2, Υ_1 and Υ_2 satisfied all the assumptions of Lemma 2.4.

Here, we establish that Lemma 2.4 conclusion's (ii) cannot possible. Let $\mathcal{V}=\{\varkappa\in\mathbf{Z}:\lambda\varkappa\in\Upsilon_1\varkappa+\Upsilon_2\varkappa$, for some $\lambda\in(0,1)\}$ is bounded. Later we show that

$$\Upsilon(\mathcal{V})\subset\mathcal{V}.$$

If not, then for all positive number p, $\varkappa^p\in\mathcal{V}$, $\Upsilon(\varkappa^p)\notin\mathcal{V}$, that is $\|\Upsilon(\varkappa^p)\|_c=\sup\{\|\Upsilon_\xi^p\|_c:\Upsilon_\xi^p\in(\Upsilon\varkappa^p)\}>p$ and $\exists$ an $g^p\in T_{\mathscr{G},\varkappa^p}$, such that

$$\xi^p(\iota)=\lambda\mathcal{N}(\iota,0)\alpha(0)+\lambda\mathcal{S}(\iota,0)[\xi_1+\mathfrak{H}(\iota,\alpha(0))]-\lambda\int_0^\iota\mathcal{N}(\iota,\varpi)\mathfrak{H}^p(\varpi,\xi_\varpi)d\varpi$$

$$+\lambda\int_0^\iota\mathcal{S}(\iota,\varpi)g^p(\varpi)dW(\varpi)+\lambda\int_0^\iota\mathcal{S}(\iota,\varpi)Bu_\delta^p(\varpi,\xi)d\varpi$$

$$+\lambda\sum_{0<\iota_l<\iota}\mathcal{N}(\iota,\iota_l)\chi_l(\xi_{\iota l})+\lambda\sum_{0<\iota_l<\iota}\mathcal{S}(\iota,\iota_l)\overline{\chi_l}(\xi_{\iota l}),\ \iota\in\mathcal{V},$$

for some $\lambda\in(0,1)$.

For $\delta>0$, applying $\mathbf{H_4}$, we get

$$E\left\|u_\delta^p(\iota)\right\|^2 = E\left\|B^*\mathcal{S}^*(c,\varpi)(\delta I+\aleph_0^c)^{-1}\left[E\xi_c+\int_0^c \Upsilon(\varsigma)dW(\varsigma)-\mathcal{N}(c,0)\alpha(0)\right.\right.$$

$$-\mathcal{S}(c,0)[\xi_1+\mathfrak{H}(c,0)]+\int_0^c\mathcal{N}(c,\varsigma)\mathfrak{H}(\varsigma,\varkappa_\varsigma+\hat{\beta}_\varsigma)d\varsigma-\int_0^c\mathcal{S}(c,\varsigma)g(\varsigma)dW(\varsigma)$$

$$\left.\left.-\sum_{0<\iota_\iota<c}\mathcal{N}(\iota,\iota_\iota)\chi_\iota(\xi_{\iota_\iota})-\sum_{0<\iota_\iota<c}\mathcal{S}(c,\iota_\iota)\overline{\chi_\iota}(\xi_{\iota\iota})\right]\right\|^2$$

$$\leq 7\left\|B^*\right\|^2\left\|\mathcal{S}^*(c,\varpi)\right\|^2\left\|(\delta I+\aleph_0^c)^{-1}\right\|^2\left[2E\|\xi_c\|^2+2\int_0^c E\|\Upsilon(\varsigma)\|_{L_2^0}^2 d\varsigma\right.$$

$$+\|\mathcal{N}(c,0)\|^2E\|\alpha(0)\|^2+2\|\mathcal{S}(c,0)\|^2E\left[\|\xi_1\|^2+\|\mathfrak{H}(c,0)\|^2\right]$$

$$+\int_0^c\|\mathcal{N}(c,\varsigma)\|^2E\left\|\mathfrak{H}(\varsigma,\varkappa_\varsigma+\hat{\beta}_\varsigma)\right\|^2d\varsigma+Tr(\mathcal{Q})\int_0^c\|\mathcal{S}(c,\varsigma)\|^2E\|g(\varsigma)\|^2d\varsigma$$

$$\left.+n^2\sum_{\iota=1}^n\|\mathcal{N}(c,\iota_\iota)\|^2E\left\|\chi_\iota(\xi_{\iota_\iota})\right\|^2+n^2\sum_{\iota=1}^n\|\mathcal{S}(c,\iota_\iota)\|^2E\left\|\overline{\chi_\iota}(\xi_{\iota\iota})\right\|^2\right]$$

$$\leq\frac{7}{\delta}P_BP_2\left[2E\|\xi_c\|^2+2\int_0^cE\|\Upsilon(\varsigma)\|_{L_2^0}^2d\varsigma+P_1E\|\alpha(0)\|^2\right.$$

$$+2P_2E\left[\|\xi_1\|^2+\|\mathfrak{H}(c,0)\|^2\right]+P_1\int_0^c\widetilde{L_h}\left[1+\left\|\varkappa_\varsigma+\hat{\beta}_\varsigma\right\|^2\right]d\varsigma$$

$$\left.+P_2Tr(\mathcal{Q})\int_0^c\varrho_{p'}(\varsigma)d\varsigma+n^2P_1\sum_{\iota=1}^nP_m(j^{-2}p')+n^2P_2\sum_{\iota=1}^n\overline{P_m}(j^{-2}p')\right].$$

For such $\delta>0$, we determine

$$p\leq E\left\|\Upsilon\varkappa^p(\iota)\right\|^2$$

$$\leq 7E\|\mathcal{N}(\iota,0)\alpha(0)\|^2+7E\|\mathcal{S}(\iota,0)[\xi_1+\mathfrak{H}(c,0)]\|^2+7E\left\|\int_0^\iota\mathcal{N}(\iota,\varpi)\mathfrak{H}^p(\varpi,\xi_\varpi)d\varpi\right\|^2$$

$$+7E\left\|\int_0^\iota\mathcal{S}(\iota,\varpi)g^p(\varpi)dW(\varpi)\right\|^2+7E\left\|\int_0^\iota\mathcal{S}(\iota,\varpi)Bu_\delta^p(\varpi,\xi)d\varpi\right\|^2$$

$$+7E\left\|\sum_{0<\iota_\imath<\iota}\mathcal{N}(\iota,\iota_\imath)\chi_\imath(\xi_{\iota\imath})\right\|^2+7E\left\|\sum_{0<\iota_\imath<\iota}\mathcal{S}(\iota,\iota_\imath)\overline{\chi_\imath}(\xi_{\iota\imath})\right\|^2$$

$$\leq 7\|\mathcal{N}(\iota,0)\|^2 E\|\alpha(0)\|^2+14\|\mathcal{S}(\iota,0)\|^2 E\left[\|\xi_1\|^2+\|\mathfrak{H}(c,0)\|^2\right]$$

$$+7\int_0^\iota\|\mathcal{N}(\iota,\varpi)\|^2 E\left\|\mathfrak{H}^p(\varpi,\xi_\varpi)\right\|^2 d\varpi+7Tr(\mathcal{Q})\int_0^\iota\|\mathcal{S}(\iota,\varpi)\|^2\left\|g^p(\varpi)\right\|^2 d\varpi$$

$$+7\int_0^\iota\|\mathcal{S}(\iota,\varpi)\|^2\|B\|^2 E\left\|u_\delta^p(\varpi,z)\right\|^2 d\varpi+7n^2\sum_{\imath=1}^n\|\mathcal{N}(\iota,\iota_\imath)\|^2 E\|\chi_\imath(\xi_{\iota\imath})\|^2$$

$$+7n^2\sum_{\imath=1}^n\|\mathcal{S}(\iota,\iota_\imath)\|^2 E\left\|\overline{\chi_\imath}(\xi_{\iota\imath})\right\|^2$$

$$\leq 7P_1E\|\alpha(0)\|^2+14P_2E\left[\|\xi_1\|^2+\|\mathfrak{H}(c,0)\|^2\right]+7P_1\int_0^\iota\widetilde{L_h}\left[1+\|\xi_\varpi\|^2\right]d\varpi$$

$$+7P_2Tr(\mathcal{Q})\int_0^\iota\varrho_{p'}d\varpi+\frac{49}{\delta}P_2^2P_B^2c\left[2E\|\xi_c\|^2+2\int_0^c E\|\Upsilon(\varsigma)\|_{L_2^0}^2 d\varsigma\right.$$

$$+P_1E\|\alpha(0)\|^2+2P_2E\left[\|\xi_1\|^2+\|\mathfrak{H}(c,0)\|^2\right]+P_1\int_0^c\widetilde{L_h}\left[1+\left\|\varkappa_\varsigma+\hat{\beta}_\varsigma\right\|^2\right]d\varsigma$$

$$\left.+P_2Tr(\mathcal{Q})\int_0^c\varrho_{p'}d\varsigma+n^2P_1\sum_{\imath=1}^n P_m(j^{-2}p')+n^2P_2\sum_{\imath=1}^n\overline{P_m}(j^{-2}p')\right]$$

$$+7n^2P_1\sum_{\imath=1}^n P_m(j^{-2}p')+7n^2P_2\sum_{\imath=1}^n\overline{P_m}(j^{-2}p')$$

$$\leq\left(7+\frac{49}{\delta}P_2^2P_B^2c\right)\left[4P_1\widetilde{L_h}c+4P_2Tr(\mathcal{Q})\gamma j^2+4P_1n^2\sum_{\imath=1}^n\tau_\imath+4P_2n^2\sum_{\imath=1}^n\overline{\tau_\imath}\right]+\widehat{P_c},$$

where $\widehat{P_c}$ is independent of p and dividing by p when $p\to\infty$, we get

$$\liminf_{p\to+\infty}\frac{\int_0^c\varrho_{p'}(\varpi)d\varpi}{p}=\liminf_{p\to+\infty}\left(\frac{\int_0^c\varrho_{p'}(\varpi)d\varpi}{p'}\cdot\frac{p'}{p}\right)=4\gamma j^2,$$

$$\liminf_{p\to+\infty}\frac{\sum_{\iota=1}^{n}P_m(j^{-2}p')}{p}=\liminf_{p\to+\infty}\left(\frac{\sum_{\iota=1}^{n}P_m(j^{-2}p')}{j^{-2}p'}\cdot\frac{j^{-2}p'}{p}\right)=4\sum_{\iota=1}^{n}\tau_\iota,$$

$$\liminf_{p\to+\infty}\frac{\sum_{\iota=1}^{n}\overline{P_m}(j^{-2}p')}{p}=\liminf_{p\to+\infty}\left(\frac{\sum_{\iota=1}^{n}\overline{P_m}(j^{-2}p')}{j^{-2}p'}\cdot\frac{j^{-2}p'}{p}\right)=4\sum_{\iota=1}^{n}\overline{\tau_\iota}.$$

Therefore, we get

$$\left(7+\frac{49}{\delta}P_2^2P_B^2c\right)\left[4P_1\widetilde{L_h}c+4P_2Tr(\mathcal{Q})\gamma j^2+4P_1n^2\sum_{\iota=1}^{n}\tau_\iota+4P_2n^2\sum_{\iota=1}^{n}\overline{\tau_\iota}\right]\geq 1,$$

which is a contradiction. Therefore, $\delta>0$, $\exists\, p>0$ such that $\Upsilon:\mathcal{G}_g''\to\mathcal{G}_g''$. This implies $\mathcal{V}$ is bounded. Hence Υ has a fixed point $\varkappa(\cdot)$ on $\mathcal{G}_g''$ by Lemma 2.4. Consider $\xi(\iota)=\varkappa(\iota)+\hat{\beta}(\iota)$, $-\infty<\iota\leq c$.

The mild solution of (1)-(3) is Υ, where ξ is a fixed point. □

Definition 3.4 The differential system (1)-(3) is approximately controllable on $[0, c]$ if $\overline{R(c,\alpha,u)}=L^2(\mathscr{F}_c,\mathbf{Z})$, where

$$R(c,\alpha,u)=\{\xi(\alpha,u)(c):\xi\text{ is the solution of the system (1)-(3) and }u\in L^2(\mathcal{V},\mathbb{U})\}.$$

Theorem 3.5 Let the presumptions of Theorem 3.3 hold. Moreover, $\mathbf{H_0}$-$\mathbf{H_7}$ are fulfilled, $\{\mathcal{S}(\iota,0):\iota\geq 0\}$ is compact and g is uniformly bounded, then (1)-(3) is approximately controllable on $\mathcal{V}$.

Proof. Consider $\hat{\xi}^\delta(\cdot)$ be a fixed point of Ψ^ϱ in B_p. According to Theorem 3.3, $\forall$ fixed point of Ψ^ϱ, which is the mild solution of (1)-(3).

$\Rightarrow\hat{\xi}^\delta\in\Psi^\varrho(\hat{\xi}^\delta)$; i.e. from Stochastic Fubini theorem, $\exists\, g\in T_{\mathscr{G},\hat{\xi}^\delta}$ such that

$$\hat{\xi}^\delta(c)=\xi_c-\delta(\delta I+\aleph_0^c)^{-1}\big(E\xi_c-\mathcal{N}(c,0)\alpha(0)-\mathcal{S}(c,0)[\xi_1+\mathfrak{H}(c,\alpha(0))]\big)$$

$$+\int_0^c\delta(\delta I+\aleph_0^c)^{-1}\Upsilon(\varpi)dW(\varpi)+\int_0^c\delta(\delta I+\aleph_\varpi^c)^{-1}\mathcal{N}(c,\varpi)\mathfrak{H}(\varpi,\hat{\xi}^\delta)d\varpi$$

$$-\int_0^c\delta(\delta I+\aleph_\varpi^c)^{-1}\mathcal{S}(c,\varpi)g(\varpi)dW(\varpi)-\delta(\delta I+\aleph_\varpi^c)^{-1}\sum_{0<\iota_\iota<c}\mathcal{N}(c,\iota_\iota)\chi_\iota(\hat{\xi}^\delta_{\iota_\iota})$$

$$-\delta(\delta I+\aleph_\varpi^c)^{-1}\sum_{0<\iota_\iota<c}\mathcal{S}(c,\iota_\iota)\overline{\chi_\iota}(\hat{\xi}^\delta_{\iota_\iota}).$$

By assumptions, we have the sequences $\mathfrak{H}(\varpi,\hat{\xi}^\delta_\varpi)$ and $g(\varpi)$ are uniformly bounded on $\mathcal{V}$. As a result, there are subsequences, which are nonetheless indicated by $\mathfrak{H}(\varpi, \hat{\xi}^{\delta}_{\varpi})$ and $g(\varpi)$ that converge weakly to say, $\mathfrak{H}(\varpi)$ and $g''(\varpi)$ sequentially. The compactness of $\mathcal{N}(\imath, 0)$ and $\mathcal{S}(\imath, 0)$, $\imath > 0$, implies that $\mathcal{N}(c, \varpi)[\mathfrak{H}(\varpi, \hat{\xi}^{\delta}_{\varpi}) - \mathfrak{H}(\varpi)] \to 0$ and $\mathcal{S}(c, \varpi)[g(\varpi) - g''(\varpi)] \to 0$.

On the other hand, by lemma, $\forall\ \imath \in \mathcal{V}$, $\delta(\delta I + \aleph_0^c) \to 0$ strongly as $\delta \to 0^+$ and $\|\delta(\delta I + \aleph_0^c)\| \leq 1$. Therefore, from the Lebesgue dominated convergence theorem

$$E\left\|\hat{\xi}^{\delta}(c)-\xi\right\|^2 \leq 8E\left\|\delta(\delta I+\aleph_0^c)^{-1}\left(E\xi_c-\mathcal{N}(c, 0)\alpha(0)-\mathcal{S}(c, 0)[\xi_1+\mathfrak{H}(c, \alpha(0))]\right)\right\|^2$$

$$+8E\left(\int_0^c\left\|\delta(\delta I+\aleph_0^c)^{-1}\Upsilon(\varpi)\right\|^2_{L_2^0}d\varpi\right)$$

$$+8E\left(\int_0^c\left\|\delta(\delta I+\aleph_{\varpi}^c)^{-1}\mathcal{N}(c, \varpi)[\mathfrak{H}(\varpi, \hat{\xi}^{\delta}_{\varpi})-\mathfrak{H}(\varpi)]\right\|d\varpi\right)^2$$

$$+8E\left(\int_0^c\left\|\delta(\delta I+\aleph_{\varpi}^c)^{-1}\mathcal{N}(c, \varpi)\mathfrak{H}(\varpi)\right\|d\varpi\right)^2$$

$$+8E\left(\int_0^c\left\|\delta(\delta I+\aleph_{\varpi}^c)^{-1}\mathcal{S}(c, \varpi)[g(\varpi)-g''(\varpi)]\right\|_{L_2^0}d\varpi\right)^2$$

$$+8E\left(\int_0^c\left\|\delta(\delta I+\aleph_{\varpi}^c)\mathcal{S}(c, \varpi)g''(\varpi)\right\|_{L_2^0}d\varpi\right)^2$$

$$+8E\left(\left\|\delta(\delta I+\aleph_{\varpi}^c)^{-1}\sum_{0<\iota_\iota<c}\mathcal{N}(c, \iota_\iota)\chi_\iota(\hat{\xi}^{\delta}_{\iota_\iota})\right\|\right)^2$$

$$+8E\left(\left\|\delta(\delta I+\aleph_{\varpi}^c)^{-1}\sum_{0<\iota_\iota<c}\mathcal{S}(c, \iota_\iota)\overline{\chi_\iota}(\hat{\xi}^{\delta}_{\iota_\iota})\right\|\right)^2 \to 0 \text{ as } \delta \to 0^+.$$

So $\hat{\xi}^{\delta}(c) \to \xi_c$ as $\delta \to 0^+$ and this proves that system (1)-(3) is approximately controllable and therefore the proof is completed. □

4. Nonlocal conditons

The idea of nonlocal circumstances was driven by neurological ailments. The existence and uniqueness of mild solutions to nonlocal differential equations for existence and result in [45-46], was originally demonstrated by Byszewski. One may refer [6, 7, 18, 45-46] to see how these papers inspired other scholars to finish significant tasks. Take into account the regarding system with nonlocal conditions:

$$\frac{d}{d\iota}\left[\xi'(\iota)+\mathfrak{H}(\iota,\,\xi_\iota)\right]\in A(\iota)\xi(\iota)+\mathscr{G}(\iota,\,\xi_\iota)\frac{dW(\iota)}{d\iota}+Bu(\iota),\ \iota\in\mathcal{V},\ \iota\neq\iota_\iota,\ \iota=1,\,2,\,\cdots,\,n, \tag{11}$$

$$\Delta\xi\,|_{\iota=\iota_\iota}=\chi_\iota(\xi(\iota_\iota^-)),\ \iota=1,\,2,\,\cdots,\,n,\ \Delta\xi'\,|_{\iota=\iota_\iota}=\overline{\chi_\iota}(\xi(\iota_\iota^-)),\ \iota=1,\,\cdots,\,n, \tag{12}$$

$$\xi(\iota)=\alpha(\iota)+h(\xi_{\iota_1},\,\xi_{\iota_2},\,\xi_{\iota_3},\,\cdots,\,\xi_{\iota_n})\in L^2(\Omega,\,\mathcal{G}_g),\ \iota\in(-\infty,\,0],\ \xi'(0)=\xi_1\in\mathbf{Z}, \tag{13}$$

where $0<\iota_1<\iota_2<\iota_3<\cdots<\iota_\iota\leq c,\ h:\mathcal{G}_g^n\to\mathcal{G}_g$, which satisfies the following:

$\mathbf{H_8}$ $h:\mathcal{G}_g^n\to\mathcal{G}_g$ is continuous and $P_\iota(h)>0$ such that

$$E\left\|h(u_1,\,u_2,\,\cdots,\,u_n)-h(v_1,\,v_2,\,\cdots,\,v_n)\right\|^2\leq\sum_{\iota=1}^{n}P_\iota(h)\left\|u-v\right\|_{\mathcal{G}}^2,$$

$$\forall\ u,\,v\in\mathcal{G}_g\ \text{ and } N_h=\sup\{\|\ h(u_{\iota_1},\,u_{\iota_2},\,\cdots,\,u_{\iota_n})\|^2:u\in\mathcal{G}_g\}.$$

Definition 4.1 An $\mathscr{F}_\iota$-adapted stochastic process $\xi:(-\infty,\,c]\to\mathbf{Z}$ is said to be a mild solution of (11)-(13) given that $\xi_0=[\alpha+q(\xi_{\iota_1},\,\xi_{\iota_2},\,\xi_{\iota_3},\,\cdots,\,\xi_{\iota_n})(0)]\in L^2(\Omega,\,\mathcal{G}_g)$, $\xi'(0)=\xi_1\in\mathbf{Z}$ on $(-\infty,\,0]$, and the impulsive conditions $\Delta\xi|_{\iota=\iota_\iota}=\chi_\iota(\xi(\iota_\iota^-))$, $\Delta\xi'|_{\iota=\iota_\iota}=\overline{\chi_\iota}(\xi(\iota_\iota^-))$, $\iota=1,\,2,\,\cdots,\,n$; $\xi(\cdot)$ to χ_ι is continuous and

(i) $\varkappa(\iota)$ is measurable and adapted to $\mathscr{F}_\iota$, $\iota\leq0$.

(ii) $\varkappa(\iota)\in\mathbf{Z}$ has càdlàg paths on $\iota\in\mathcal{V}$ and $\forall\ \iota\in\mathcal{V}$, $\varkappa(\iota)$ satisfies the integral equation

$$\begin{aligned}\xi(\iota)=&\ \mathcal{N}(\iota,\,0)[\alpha(0)+q(\xi_{\iota_1},\,\xi_{\iota_2},\,\xi_{\iota_3},\,\cdots,\,\xi_{\iota_n})(0)]+\mathcal{S}(\iota,\,0)[\xi_1+\mathfrak{H}(\iota,\,\alpha(0))]\\&-\int_0^\iota\mathcal{N}(\iota,\,\varpi)\mathfrak{H}(\varpi,\,\xi_\varpi)d\varpi+\int_0^\iota\mathcal{S}(\iota,\,\varpi)g(\varpi)dW(\varpi)\\&+\int_0^\iota\mathcal{S}(\iota,\,\varpi)Bu(\varpi)d\varpi+\sum_{0<\iota_\iota<\iota}\mathcal{N}(\iota,\,\iota_\iota)\chi_\iota(\xi_{\iota_\iota})\\&+\sum_{0<\iota_\iota<\iota}\mathcal{S}(\iota,\,\iota_\iota)\overline{\chi_\iota}(\xi_{\iota_\iota}),\ \iota\in\mathcal{V}.\end{aligned}$$

Theorem 4.2 If $\mathbf{H_0}$-$\mathbf{H_8}$ are satisfied. Then, the system (11)-(13) is approximately controllable on $\mathcal{V}$ provide that

$$\left(7+\frac{49}{\delta}P_2^2P_B^2c\right)\left[4P_1\widetilde{L_h}c+4P_2Tr(\mathcal{Q})\gamma j^2+4P_1n^2\sum_{\iota=1}^{n}\tau_\iota+4P_2n^2\sum_{\iota=1}^{n}\overline{\tau_\iota}\right]<1.$$

Remark 4.3 The Dhage's fixed point theorem is applicable to a wide range of problems defined in Banach spaces, Hilbert spaces, and more general metric spaces. It provides conditions under which a self-mapping on a closed, convex, and bounded subset of a Banach space has a fixed point. Unlike the Banach fixed point theorem, which requires a contraction mapping, Dhage's theorem applies to more general mappings. It can handle cases where the mapping is not necessarily a contraction but still possesses certain desirable properties, such as compactness or monotonicity. So, in this paper we will use the well-known Dhage's fixed point theory approach for solving the considered system.

5. Examples

Assume $A(\iota) = A + \widetilde{B}(\iota)$, where A and $\widetilde{B}(\iota)$ are the infinitesimal generator of a sine function $\mathcal{S}(\iota)$ with the cosine function $\mathcal{N}(\iota)$ and $\widetilde{B}(\iota)$ is a closed with $D \subseteq D(\widetilde{B}(\iota)) \to \mathbf{Z}$, $\forall\ \iota \in \mathcal{V}$ referring (4)-(5) respectively.

The family $\mathbb{W}$ is classified as $\mathbb{R}/2\pi\mathbf{Z}$, and we are now framing our issue as belonging to $\mathbf{Z} = L^2(\mathbb{W}, \mathbf{Z})$. It is used to recognise functions on $\mathbb{W}$ and 2π-periodic functions on $\mathbb{R}$. As described below, let $L^2(\mathbb{W}, \mathbf{Z})$ be the space of 2π-periodic 2-integrable functions.

Assume $A\xi(\varpi) = \xi''(\varpi)$ with domain $D(A) = H^2(\mathbb{W}, \mathbf{Z})$, $A : D(A) \subset \mathbf{Z} \to \mathbf{Z}$ given as $K\xi = \xi - \xi''$ where $D(A)$ is provided by

$$D(A) = \{\xi \in \mathbf{Z},\ \xi'' \in \mathbf{Z} : \xi,\ \xi' \text{ are absolutely ontinuous, } \xi(0) = \xi(\pi) = 0\}.$$

Additionally, A having discrete spectrum, spectrum of A holding the eigenvalues $-k^2$, k belongs to $\mathbb{Z}$, along eigenvectors

$$w_p(\varpi) = \frac{1}{\sqrt{2\pi}} e^{ik\varpi},\ k \in \mathbb{Z},$$

w_k for k belongs to $\mathbb{Z}$ is an orthonormal basis of $\mathbf{Z}$. Especially,

$$A\xi = -\sum_{k=1}^{\infty} k^2 \langle \xi, w_k \rangle w_k,$$

$\forall\ \xi$ belongs to $D(A)$. The cosine operator $\mathcal{N}(\iota)$ is given as

$$\mathcal{N}(\iota)\xi = \sum_{k=1}^{\infty} \cos(k\iota) \langle \xi, w_k \rangle w_k,\ \iota \in \mathbb{R},$$

connected along with sine operator

$$\mathcal{S}(\iota)\xi = \sum_{k=1}^{\infty} \frac{\sin(k\iota)}{k} \langle \xi, w_k \rangle w_k,\ \iota \in \mathbb{R}.$$

Undoubtedly, $\|\mathcal{N}(\iota)\|^2 \le 1$, $\forall\ \iota \in \mathbb{R}$. As a result, $\mathcal{N}(\cdot)$ is uniformly bounded on $\mathbb{R}$. Now identify the most important system

$$\frac{\partial}{\partial \iota}\Big[\varkappa'(\iota, \xi) + \widehat{\mathfrak{H}}(\iota, \varkappa(\iota - m, \xi))\Big] \in \varkappa_{\xi\xi}(\iota, \xi) + b(\iota)\frac{\partial}{\partial \iota}\varkappa(\iota, \xi) + \hat{\mu}(\iota, \xi)$$

$$+\widehat{\mathscr{G}}(\iota, \varkappa(\iota - m, \xi))\frac{\partial w(\iota)}{\partial \iota},\ \iota \in \mathcal{V},\ m > 0,\ \xi \in [0, \pi], \tag{14}$$

$$\varkappa(\iota, 0) = \varkappa(\iota, \pi) = 0, \ \iota \in \mathcal{V}, \tag{15}$$

$$\varkappa(\iota_\iota^+, \xi) - \varkappa(\iota_\iota^-, \xi) = \chi_\iota\big(\varkappa(\iota_\iota^-, \xi)\big), \ \varkappa'(\iota_\iota^+, \xi) - \varkappa'(\iota_\iota^-, \xi) = \overline{\chi_\iota}\big(\varkappa(\iota_\iota^-, \xi)\big), \ \iota = 1, \cdots, n, \tag{16}$$

$$\varkappa(\iota, \xi) = \alpha(\iota, \xi), \ \xi \in [0, \pi], \ \iota \in (-\infty, 0], \ \frac{\partial}{\partial \iota}\varkappa(0, \xi) = \varkappa_1, \ \xi \in [0, \pi]. \tag{17}$$

In the above b mapping from $\mathbb{R}$ into $\mathbb{R}$ is continuous. Consider $\iota > 0$ and fix $\lambda = \sup_{0 \le \iota \le c} |b(\iota)|$.

The one-dimensional Brownian motion $w(\iota)$ is defined as having two sides that are defined on the filtered probability space $(\Omega, \mathscr{F}, P)$. Let us $\widetilde{B}(\iota)\xi(\iota) = A(\iota)\xi'(\iota)$. We prove that the equation $A(\iota) = A + \widetilde{B}(\iota)$ is closed before asserting that the expression $A + \widetilde{B}(\iota)$ generates an evolution operator. Therefore, the scalar IVP solution

$$u''(\iota) = -k^2 u(\iota) + v(\iota),$$

$$u(\zeta) = 0, \ u'(\zeta) = u_1,$$

is given by

$$u(\iota) = \frac{u_1}{k}\sin k(\iota - \zeta) + \frac{1}{k}\int_\zeta^\iota \sin k(\iota - \sigma) v(\sigma) d\sigma.$$

Therefore,

$$u''(\iota) = -k^2 u(\iota) + inb(\iota)u(\iota), \tag{18}$$

$$u(\zeta) = 0, \ u'(\zeta) = u_1, \tag{19}$$

fulfilled

$$u(\iota) = \frac{u_1}{k}\sin k(\iota - \zeta) + i\int_\zeta^\iota \sin k(\iota - \sigma) b(\sigma) u(\sigma) d\sigma.$$

The Gronwall-Bellman lemma yields

$$|u(\iota)| \le \frac{|u_1|}{k} e^{\beta(\iota - \zeta)}, \tag{20}$$

for $\zeta \le \iota$. We stands for $u_k(\iota, \zeta)$ the solution of (18)-(19). Define

$$\mathcal{S}(\iota, \zeta)\xi = \sum_{k=1}^{\infty} u_k(\iota, \zeta)\langle \xi, w_k \rangle w_k.$$

Thus, the value (20) that $\mathcal{S}(\iota, \varpi)$ mapping from $\mathbf{Z}$ into $\mathbf{Z}$ is simplified and fulfilled the Definition 2.2.

Now we determine

$$\mathbb{U} := \left\{ \xi : \xi = \sum_{k=2}^{\infty} \xi_k e_k,\ \sum_{k=2}^{\infty} \xi_k^2 < \infty \right\},$$

the infinite dimensional Hilbert space $\mathbb{U}$ and specify the norm of $\mathbb{U}$ as

$$\|\xi\|_{\mathbb{U}} = \Big(\sum_{k=2}^{\infty} \xi_k^2\Big)^{\frac{1}{2}}.$$

Determine $\xi(\sigma)(\eta) = \xi(\sigma, \eta)$, $\mathcal{Q}(\iota, \xi)(\sigma) = \mathcal{Q}(\iota, \xi(\sigma))$ and it satisfies the assumptions $\mathbf{H_2}$. Determine $B \in \mathcal{L}(\mathbb{U}, \mathbf{Z})$ is in the following way:

$$B\xi = 2\xi_2 k_3 + \sum_{k=2}^{\infty} \xi_k e_k, \text{ for } \xi = \sum_{k=2}^{\infty} \xi_k e_k \in \mathbb{U},$$

also $v = \sum_{k=1}^{\infty} v_k e_k \in \mathbf{Z}$, $\langle B\xi, v\rangle = \langle \xi, B^* v\rangle$, thus

$$B^* v = (2v_1 + v_2)k_4 + \sum_{k=3}^{\infty} v_k e_k.$$

Further $\mathcal{G}_g$ with the norm is given as

$$\|\varphi\|_{\mathcal{G}_g} = \int_{-\infty}^{0} g(\hbar) \sup_{\hbar \le \theta \le 0} \left(E\|\alpha(\theta)\|^2 \right)_{L^2}^{\frac{1}{2}} d\hbar,$$

where $g(\hbar) = e^{2\hbar}$, $\hbar < 0$ and $j = \int_{-\infty}^{0} g(\hbar) d\hbar = \frac{1}{2}$.

Assume $\varkappa(\iota)(\xi) = \varkappa(\iota, \xi)$. Define $\mathscr{G}(\iota, \varkappa)(\cdot) = \hat{\mathscr{G}}(\iota, \varkappa(\cdot))$, $\mathfrak{H}(\iota, \varkappa)(\cdot) = \hat{\mathfrak{H}}(\iota, \varkappa(\cdot))$ and $B : \mathbb{U} \to \mathbf{Z}$ is classified as $Bu(\iota)(\xi) = \hat{\mu}(\iota, \xi)$. Thus, clearly $\|\mathcal{N}(\iota, \varpi)\|^2 \le 1$ and $\|\mathcal{S}(\iota, \varpi)\|^2 \le 1$, $\forall\ \iota \in \mathbb{R}$ and $\mathcal{S}(\iota, \varpi)$ is compact $\forall\ \iota \in \mathbb{R}$. Hence the following options, $A(\iota)$, B, the system (14)-(17) signified to (1)-(3). Therefore, every conditions of Theorem 3.5 are satisfied and then the considered system (14)-(17) is approximate controllable on $[0, c]$.

Next, we verify that the Hypothesis $\mathbf{H_1}$-$\mathbf{H_7}$ for the above system (1)-(3) one by one.

Verification of $\mathbf{H_1}$:

The operator $\mathcal{N}(\iota, 0)$, $\iota > 0$ is compact. Thus, clearly $\|\mathcal{N}(\iota, \zeta)\|^2 \le 1$ and $\|\mathcal{S}(\iota, \zeta)\|^2 \le 1$, for $\iota \in \mathbb{R}$ and $\mathcal{N}(\iota, \zeta)$ is compact for all $\iota \in \mathbb{R}$.

From the above conditions are satisfied the assumption $\mathbf{H_1}$.

Verification of $\mathbf{H_2}$:

Assume that $\mathfrak{H}(\iota, \varkappa)(\cdot) = \hat{\mathfrak{H}}(\iota, \varkappa(\cdot))$.

The function $\mathfrak{H} : \mathcal{V} \times \mathcal{G}_g \to \mathbf{Z}$ is give by

$$\mathfrak{H}(\iota, \xi) = \widehat{\mathfrak{H}}(\iota, \varkappa)(\cdot). \tag{21}$$

Therefore,

$$E\left\|\widehat{\mathfrak{H}}\left(\iota, \varkappa(\iota - m, \zeta)\right)\right\|^2 \leq \widetilde{L_h}\left(1 + \|\xi(\iota - m, \zeta)\|_{\mathcal{G}_g}^2\right), \tag{22}$$

where $\widetilde{L_h}$ is a positive constant.

From the equations (21) and (22), we notice that the assumptions $\mathbf{H_2}$ is verified.

Verification of $\mathbf{H_3}$, $\mathbf{H_4}$ and $\mathbf{H_5}$:

Set

$$\mathscr{G}(\iota, \xi) = \widehat{\mathscr{G}}\left(\iota, \varkappa(\iota - m, \xi)\right) = \left\{\mathscr{G} \in \mathbf{Z};\ g_1\left(\iota, \varkappa(\iota - m, \zeta)\right) \leq \mathscr{G} \leq g_2\left(\iota, \varkappa(\iota - m, \zeta)\right)\right\}, \tag{23}$$

where $g_1, g_2 : \mathcal{V} \times \mathcal{G}_g \to BCC(L\mathbb{U}, \mathbf{Z}))$. We assume that for each $\iota \in \mathcal{V}$, g_1 is lower semicontinuous and g_2 is upper semi-continuous. Assume that there exists $p > 0$ function such that

$$\max\left\{E\left\|\mathscr{G}_1\left(\iota, \varkappa(\iota - m, \zeta)\right)\right\|^2, E\left\|\mathscr{G}_2\left(\iota, \varkappa(\iota - m, \zeta)\right)\right\|^2\right\} \leq \varrho_p(\iota). \tag{24}$$

From the equations (23) and (24) along with the above discussion, we observe that $\hat{\mathscr{G}}$ is nonempty and further clear that $\hat{\mathscr{G}}$ is compact and convex valued, and is upper semi-continuous. Therefore, $\hat{\mathscr{G}}$ satisfies the condition $\mathbf{H_3}$, $\mathbf{H_4}$ and $\mathbf{H_5}$.

Verification of $\mathbf{H_6}$ and $\mathbf{H_7}$:

From the equation (14)-(17),

$$\Delta\xi|_\iota = \varkappa(\iota_\iota^+, \xi) - \varkappa(\iota_\iota^-, \xi) = \chi_\iota\left(\varkappa(\iota_\iota^-, \xi)\right), \iota = 1, 2, \cdots, n,$$

and we consider the function $\chi_\iota \in \mathcal{C}(\mathbf{Z}, \mathbf{Z})$ and there exists $P_m : [0, \infty) \to (0, \infty)$ is given by

$$\widehat{\chi_\iota}(\xi) = \chi_\iota\left(\varkappa(\iota_\iota^-, \xi)\right)$$

$$E\left\|\widehat{\chi_\iota}(\xi)\right\|^2 \leq P_m\left(E\|\xi\|^2\right). \tag{25}$$

Similarly,

$$\Delta\xi'|_\iota = \varkappa'(\iota_\iota^+, \xi) - \varkappa'(\iota_\iota^-, \xi) = \overline{\chi_\iota}\left(\varkappa(\iota_\iota^-, \xi)\right), \iota = 1, 2, \cdots, n,$$

and we consider the function $\overline{\chi_\iota} \in \mathcal{C}(\mathbf{Z}, \mathbf{Z})$ and there exists $\overline{P_m} : [0, \infty) \to (0, \infty)$ is given by

$$\widehat{\chi_\iota}(\xi) = \widehat{\chi_\iota}\left(\varkappa(\iota_\iota^-, \xi)\right)$$

$$E\left\|\widehat{\chi_\iota}(\xi)\right\|^2 \le \overline{P_m}\left(E\|\xi\|^2\right). \tag{26}$$

From the equations (25) and (26), we observe that the condition $\mathbf{H_6}$ and $\mathbf{H_7}$ is satisfied.

Clearly, all the assumptions of the Theorem 3.3 are satisfied. Hence, by the conclusion of the Theorem 3.3, it follows that the system (1)-(3) has a solution.

6. Conclusion

In this work, a set of pertinent characteristics is used to assess the approximate controllability of impulsive second-order stochastic neutral differential evolution systems. We apply the ideas of sine function, cosine function, and fixed point strategy to validate our findings. We'll discuss the second-order impulsive stochastic differential system with nonlocal conditions later. At last, we give a speculative illustration of the effectiveness of our work. In the future, we will investigate the Sobolev-type of approximate controllability results for second-order non-instantaneous impulsive stochastic integro-differential systems.

Conflict of interest

The authors declare no competing financial interest.

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