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Research Article

Linearizations of Cubic Two-Parameter Eigenvalue Problems and Its Vector Space

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Abstract: The paper considers the study of linearization techniques of cubic two-parameter matrix polynomial (CTMP). We analyze vector spaces of linearization of CTMP, namely ansatz vector space and double ansatz vector space. We also consider cubic two-parameter eigenvalue problem (CTEP) to study their linearization classes. A unified framework on linearization of CTMP will be established. The conditions under which a matrix pencil in the ansatz spaces is a linearization of CTMP will also be derived. Moreover, using these linearization techniques, CTEP is first converted into a singular linear two-parameter eigenvalue problem (2PEP) of larger size so that existing numerical method for (2PEP) can be applied.

Keywords: 2PEP, CTMP, CTEP, kronecker product, linearization, matrix polynomial

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1. Introduction

Consider, the following CTMP

$$E(\lambda,\ \mu) = P_0 + \mu P_1 + \lambda P_2 + \mu^2 P_3 + \lambda\mu P_4 + \lambda^2 P_5 + \mu^3 P_6 + \lambda\mu^2 P_7 + \lambda^2\mu P_8 + \lambda^3 P_9 \tag{1}$$

where, P_i are $n \times n$; $i := 0 : 9$ matrices over C such that at least one of the matrices P_6, P_7, P_8 or P_9 is nonzero, $x \in C^n$ is a nonzero vector and λ, $\mu \in C$ are spectral parameters. If for some $(\lambda, \mu) \in C \times C$ there exist $0 \neq x \in C^n$ such that $E(\lambda, \mu)x = 0$, then x is called eigenvector corresponding to eigenvalue (λ, μ) of CTMP. Our focus is on examining the linear two-parameter matrix polynomial of the form

$$L(\lambda,\ \mu) := L_0 + \lambda L_1 + \mu L_2, \tag{2}$$

which agrees with the eigenvalues (λ, μ) of (1).

The standard form of CTEP is given by

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$$E_i(\lambda, \mu)x_i := 0, \text{ for } i = 1, 2; \tag{3}$$

where

$$E_i(\lambda, \mu) = P_{i0} + \mu P_{i1} + \lambda P_{i2} + \mu^2 P_{i3} + \lambda\mu P_{i4} + \lambda^2 P_{i5} + \mu^3 P_{i6} + \lambda\mu^2 P_{i7} + \lambda^2\mu P_{i8} + \lambda^3 P_{i9},$$

for $i = 1, 2$ and P_{ij} are $n_i \times n_i$ matrices over C; $i := 1 : 2, j := 0 : 9$ such that at least one of the matrices P_{i6}, P_{i7}, P_{i8}, P_{i9}; $i := 1 : 2$ is nonzero, $x_i \in C^{n_i}$ is a nonzero vector for $i := 1 : 2$ and $\lambda, \mu \in C$ are spectral parameters. Here, the problem is to find the scalars $\lambda, \mu \in C$ and the corresponding non zero vectors $x_i \in C^{n_i}$, $i := 1 : 2$ that satisfy the equations (3). The pair (λ, μ) is called an eigenvalue of the CTEP, if $E_i(\lambda, \mu)x_i = 0$ for some $0 \neq x_i \in C^{n_i}$, $i := 1 : 2$ and the tensor product $x_1 \otimes x_2$ is called the corresponding right eigenvector. Similarly, a tensor product $v_1 \otimes v_2$ is called a left eigenvector of the CTEP if $v_i \neq 0$; $i := 1 : 2$ satisfy $v_i^* E_i(\lambda, \mu) = 0$.

There are two types of numerical approaches for solving the polynomial two-parameter eigenvalue problems. Those that deal directly with the problem and those that compute the eigenvalues of the linearized forms. Literature on the direct methods to find complete solutions of the CTEPs, is very limited. Unlike the quadratic two-parameter eigenvalue problem (QTEP) [1], there is no direct method to solve these CTEPs. The CTEP appears in the paper [2] (Example 20), where the problem is linearized into a linear 2PEP. But the authors did not provided a brief discussion on the Kronecker structure involved in theory similar to quadratic case, due to their complex structures. These motivate to search possible linearizations of CTEPs, that can be used to solve the eigenvalue problem effectively.

Two-parameter matrix polynomial is the generalization of one parameter matrix polynomial of the form $P(\lambda) := \sum_{j=1}^{k} \lambda^j A_j$; $A_j \in C^{n \times n}$. One parameter matrix polynomials arise in many physical applications and they received attention from the researchers [3-5]. Limited research works are found in the literature of two-parameter matrix polynomials. Linearization is a classical approach to investigate polynomial eigenvalue problem of the form $P(\lambda)x = 0$. It is the process which converts a polynomial eigenvalue problem into a generalised eigenvalue problem (GEP) of the form $Ax = \lambda Bx$ of high dimension, where A and B are any matrices over C, x is non zero vector and λ is the spectral parameter. More details on linearization of matrix polynomials are found in the works [6-10], and the references therein. Explicit constructions of various linearization classes using different polynomial bases are reported in [11]. Few Literature on linearizations of QTEPs are found in the works of [2, 12] and for quadratic matrix polynomials in [13-14]. The usual method to solve the CTEP defined in (3) is by linearizing it into a 2PEP of larger dimension, which is being singular. It is found that, the linearization of the one parameter matrix polynomials induces a GEP, whereas the linearization of (3) induces a singular 2PEP and can be solved by adopting the method proposed in [15].

For a given CTMP defined in (1) with $n_1 = n$, our area of interest is the vector spaces created when the form is linearized to (2). These linearization classes help us to show the singularity of the associated linear 2PEP of the CTEP (1) with $n_1 = n_2 = n$, which is of the form

$$\begin{aligned} (L_0^{(1)} + \lambda L_1^{(1)} + \mu L_2^{(1)})w_1 &= 0 \\ (L_0^{(2)} + \lambda L_1^{(2)} + \mu L_2^{(2)})w_2 &= 0 \end{aligned} \tag{4}$$

where, $w_i \in C^{6n} / \{0\}$; $L_i^{(j)} \in C^{6n \times 6n}$, $i = 0 : 2$, $j = 1, 2$ such that (4) agree with the eigenvalues of (3). The linearization influences the sensitivity of the eigenvalues. Therefore, it is important to identify potential linearizations and study their constructions.

The paper is designed as follows: Section 2 contains some basic preliminaries which are used throughout the paper. Section 3 contains a unified framework of vector space of linearization of CTMP. Similarly, in section 4, linearizations of CTMP are discussed. Finally, in section 5 a conclusion is drawn on the whole work.

2. Preliminaries

The following basic definitions and results will be used throughout the paper.

Theorem 2.1 (Bézout's theorem, [16]) Two projective curves of orders n and m with no common component have precisely nm points of intersection counting multiplicities.

Definition 2.2 [12] A $ln \times ln$ linear matrix polynomial $L(\lambda, \mu) = L_0 + \lambda L_1 + \mu L_2$ is a linearization of an $n \times n$ matrix polynomial $Q(\lambda, \mu)$ if there exist polynomials $P(\lambda, \mu)$ and $R(\lambda, \mu)$, whose determinant is a non-zero constant independent of λ and μ, such that

$$\begin{bmatrix} Q(\lambda, \mu) & 0 \\ 0 & I_{(l-1)n} \end{bmatrix} = P(\lambda, \mu)L(\lambda, \mu)R(\lambda, \mu)$$

Theorem 2.3 The determinant of a block-triangular matrix is the product of the determinants of the diagonal blocks.

Definition 2.4 [17] The Kronecker Product ($\otimes$) for two matrices A and B is defined as $A \otimes B = \{a_{ij}B\}$, where a_{ij} are the elements in i^{th} row and j^{th} column of the matrix A.

Definition 2.5 (Column shifted sum) [6] Let $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z} \in C^{6n \times 6n}$ be any three block matrices of the form $\widetilde{X} = [X_{ij}]$, $\widetilde{Y} = [Y_{ij}]$, $\widetilde{Z} = [Z_{ij}]$, where each of the X_{ij}, Y_{ij}, $Z_{ij} \in C^{n \times n}$, for $i, j := 1 : 6$. Then, the box-addition for these three block matrices, $\widetilde{X}\,\widetilde{Y}$ and $\widetilde{Z}$ is defined as

$$\widetilde{X} \boxplus \widetilde{Y} \boxplus \widetilde{Z} = \begin{bmatrix} X_{11} & X_{12} & X_{13} & 0 & X_{14} & X_{15} & 0 & X_{16} & 0 & 0 \\ X_{21} & X_{22} & X_{23} & 0 & X_{24} & X_{25} & 0 & X_{26} & 0 & 0 \\ X_{31} & X_{32} & X_{33} & 0 & X_{34} & X_{35} & 0 & X_{36} & 0 & 0 \\ X_{41} & X_{42} & X_{43} & 0 & X_{44} & X_{45} & 0 & X_{46} & 0 & 0 \\ X_{51} & X_{52} & X_{53} & 0 & X_{54} & X_{55} & 0 & X_{56} & 0 & 0 \\ X_{61} & X_{62} & X_{63} & 0 & X_{64} & X_{65} & 0 & X_{66} & 0 & 0 \end{bmatrix}$$

$$+\begin{bmatrix} 0 & Y_{11} & Y_{12} & Y_{13} & 0 & Y_{14} & Y_{15} & 0 & Y_{16} & 0 \\ 0 & Y_{21} & Y_{22} & Y_{23} & 0 & Y_{24} & Y_{25} & 0 & Y_{26} & 0 \\ 0 & Y_{31} & Y_{32} & Y_{33} & 0 & Y_{34} & Y_{35} & 0 & Y_{36} & 0 \\ 0 & Y_{41} & Y_{42} & Y_{43} & 0 & Y_{44} & Y_{45} & 0 & Y_{46} & 0 \\ 0 & Y_{51} & Y_{52} & Y_{53} & 0 & Y_{54} & Y_{55} & 0 & Y_{56} & 0 \\ 0 & Y_{61} & Y_{62} & Y_{63} & 0 & Y_{64} & Y_{65} & 0 & Y_{66} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} \\ 0 & 0 & 0 & 0 & Z_{21} & Z_{22} & Z_{23} & Z_{24} & Z_{25} & Z_{26} \\ 0 & 0 & 0 & 0 & Z_{31} & Z_{32} & Z_{33} & Z_{34} & Z_{35} & Z_{36} \\ 0 & 0 & 0 & 0 & Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} & Z_{46} \\ 0 & 0 & 0 & 0 & Z_{51} & Z_{52} & Z_{53} & Z_{54} & Z_{55} & Z_{56} \\ 0 & 0 & 0 & 0 & Z_{61} & Z_{62} & Z_{63} & Z_{64} & Z_{65} & Z_{66} \end{bmatrix};$$

where, "+" is the usual addition of matrices.

3. Linearization of CTMP

Let CTMP be as given in (1). In this section, we discuss the standard linearization for it, and extend the idea of the vector space of linearization for one-parameter matrix polynomial reported in [6] to the system (1).

Let x be an eigenvector of $E(\lambda, \mu)$ as defined in (1) corresponding to the eigenvalue (λ, μ); i.e. $E(\lambda, \mu)x = 0$.

Denote

$$x = x_{00},$$

$$\lambda x_{00} = \lambda x = x_{10},$$

$$\mu x_{00} = \mu x = x_{01},$$

$$\mu^2 x_{00} = \mu^2 x = x_{02},$$

$$\lambda\mu x_{00} = \lambda\mu x = x_{11},$$

$$\lambda^2 x_{00} = \lambda^2 x = x_{20}.$$

Then, we get

$$E(\lambda, \mu)x = (P_0 + \mu P_1 + \lambda P_2 + \mu^2 P_3 + \lambda\mu P_4 + \lambda^2 P_5 + \mu^3 P_6 + \lambda\mu^2 P_7 + \lambda^2 \mu P_8 + \lambda^3 P_9)x$$

$$= 0$$

$$\Rightarrow P_0(x_{00}) + P_1(x_{01}) + P_2(x_{10}) + P_3(x_{02}) + P_4(x_{11}) + P_5(x_{20}) + P_6(\mu x_{02}) + P_7(\lambda x_{02}) + P_8(\lambda x_{11}) + P_9(\lambda x_{20})$$

$$= 0. \tag{5}$$

The equation (5) can be represented in the following matrix form

$$\left(\begin{bmatrix} P_5 & P_4 & P_3 & P_2 & P_1 & P_0 \\ 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} P_9 & P_8 & P_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & P_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x_{20} \\ x_{11} \\ x_{02} \\ x_{10} \\ x_{01} \\ x_{00} \end{bmatrix} = 0 \tag{6}$$

Assume that,

$$w = \begin{bmatrix} x_{20} \\ x_{11} \\ x_{02} \\ x_{10} \\ x_{01} \\ x_{00} \end{bmatrix} = \begin{bmatrix} \lambda^2 x \\ \lambda\mu x \\ \mu^2 x \\ \lambda x \\ \mu x \\ x \end{bmatrix} = \begin{bmatrix} \lambda^2 \\ \lambda\mu \\ \mu^2 \\ \lambda \\ \mu \\ 1 \end{bmatrix} \otimes x \tag{7}$$

Denote,

$$L_0 = \begin{bmatrix} P_5 & P_4 & P_3 & P_2 & P_1 & P_0 \\ 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; L_1 = \begin{bmatrix} P_9 & P_8 & P_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{bmatrix}; L_2 = \begin{bmatrix} 0 & 0 & P_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \Lambda = \begin{bmatrix} \lambda^2 \\ \lambda\mu \\ \mu^2 \\ \lambda \\ \mu \\ 1 \end{bmatrix}$$

$$L(\lambda, \mu) := L_0 + \lambda L_1 + \mu L_2 \tag{8}$$

Here each $L_i \in C^{6n \times 6n}$, $i := 0 : 2$. Then, the equation (6) can be represented in the following matrix form

$$L(\lambda, \mu)w := (L_0 + \lambda L_1 + \mu L_2)w = 0 \tag{9}$$

Thus x is the eigenvector corresponding to the eigenvalue (λ, μ) of $E(\lambda, \mu)$ if and only if $L(\lambda, \mu)w = 0$, i.e. w is the eigenvector corresponding to an eigenvalue (λ, μ) of $L(\lambda, \mu)$.

Theorem 3.1 Let $E(\lambda, \mu)$ be as given in (1) and $L(\lambda, \mu)$ be defined by (8). Then, $L(\lambda, \mu)$ is a linearization of $E(\lambda, \mu)$.
Proof. Define,

$$R(\lambda, \mu) = \begin{bmatrix} \lambda^2 I_n & I_n & 0 & 0 & 0 & 0 \\ \lambda\mu I_n & 0 & I_n & 0 & 0 & 0 \\ \mu^2 I_n & 0 & 0 & I_n & 0 & 0 \\ \lambda I_n & 0 & 0 & 0 & I_n & 0 \\ \mu I_n & 0 & 0 & 0 & 0 & I_n \\ I_n & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

$$P(\lambda, \mu) = \begin{bmatrix} I_n & W_{12} & W_{13} & W_{14} & W_{15} & W_{16} \\ 0 & 0 & \lambda I_n & 0 & 0 & I_n \\ 0 & \lambda I_n & 0 & 0 & I_n & 0 \\ 0 & \mu I_n & 0 & I_n & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

where, $W_{12} = P_1 + \lambda\mu P_7 + \mu^2 P_6 + \mu P_3 + \lambda^2 P_8 + \lambda P_4$; $W_{13} = \lambda^2 P_9 + \lambda P_5 + P_2$; $W_{14} = \lambda P_7 + \mu P_6 + P_3$; $W_{15} = \lambda P_8 + P_4$; $W_{16} = \lambda P_9 + P_5$. Here E and F are unimodular CTMPs.

Then we have

$$P(\lambda, \mu) L(\lambda, \mu) R(\lambda, \mu) = \begin{bmatrix} E(\lambda, \mu) & 0 \\ 0 & I_{5n} \end{bmatrix} \tag{12}$$

Thus $\det E(\lambda, \mu) = \gamma \det L(\lambda, \mu)$ for some $\gamma \neq 0$. Thus $L(\lambda, \mu)$ is the linearization of $E(\lambda, \mu)$ with order $6n \times 6n$. □

The linearization $L(\lambda, \mu)$ is the standard linearization of $E(\lambda, \mu)$. For any $x \in C^n$, from (9), we have

$$L(\lambda, \mu) \times (\Lambda \otimes x) = [(E(\lambda, \mu)x)^T \ 0 \ 0 \ 0 \ldots 0]^T \tag{13}$$

Thus the solution of (6) agree with the solution of $E(\lambda, \mu)x = 0$.

Following the idea of (13), if we replace x by I_n, we get

$$L(\lambda, \mu) \times (\Lambda \otimes I_n) = L(\lambda, \mu) \times \begin{bmatrix} \lambda^2 I_n \\ \lambda\mu I_n \\ \mu^2 I_n \\ \lambda I_n \\ \mu I_n \\ I_n \end{bmatrix} = \begin{bmatrix} E(\lambda, \mu) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = e_1 \otimes E(\lambda, \mu), \ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

Our aim is to find linear two-parameter matrix polynomials $L(\lambda, \mu)$ that satisfy,

$$L(\lambda, \mu) \times (\Lambda \otimes I_n) = v \otimes E(\lambda, \mu) \tag{15}$$

for some $v \in C^6$. We denote

$$V_Q = \{v \otimes E(\lambda, \mu) : v \in C^6\} \tag{16}$$

and define

$$L(E(\lambda, \mu)) = \{L(\lambda, \mu) : L(\lambda, \mu) \times (\Lambda \otimes I_n) \in V_Q\} \tag{17}$$

which agree with the Definition 3.1 from [6]. It is to be noted that $L(E(\lambda, \mu)) \neq \phi$ as the standard linearization $L(\lambda, \mu) \in L(E(\lambda, \mu))$. Again, $L(E(\lambda, \mu))$ is a vector space. The study on the structures of $L(\lambda, \mu)$ in this space is done in the following subsection.

3.1 *Ansatz space*

Consider $E(\lambda, \mu)$ defined in (1). We refer the definition of ansatz vector form [6].

Definition 3.2 [6] If $L(\lambda, \mu) \in L(E(\lambda, \mu))$ for some $v \in C^6$, then v is called an Ansatz vector associated with $L(\lambda, \mu)$ and $L(E(\lambda, \mu))$ is called the Ansatz space.

To investigate the structure of each $L(\lambda, \mu) \in L(E(\lambda, \mu))$ we recall the Definition 2.5 of box-addition for the three $6n \times 6n$ block matrices which is inspired by the column shifted sum for block matrices. For the standard linearization $L(\lambda, \mu) \in L(E(\lambda, \mu))$, we have

$$L_1 \boxplus L_2 \boxplus L_0 = \begin{bmatrix} P_9 & P_8 & P_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & P_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & P_5 & P_4 & P_3 & P_2 & P_1 & P_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} P_9 & P_8 & P_7 & P_6 & P_5 & P_4 & P_3 & P_2 & P_1 & P_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= e_1 \otimes \begin{bmatrix} P_9 & P_8 & P_7 & P_6 & P_5 & P_4 & P_3 & P_2 & P_1 & P_0 \end{bmatrix}.$$

Thus, for a linear two-parameter matrix polynomial $L(\lambda, \mu)$, we can draw a relation between the box addition of the coefficient matrices and product of $L(\lambda, \mu)$ with $v \otimes E(\lambda, \mu)$ by the following Lemma.

Lemma 3.3 Let $E(\lambda, \mu)$ be as defined in (1) with coefficient matrices of order $n \times n$, and $v \in C^6$ and $L(\lambda, \mu)$ be any linearization of the form (2) with $L_0, L_1, L_2 \in C^{6n \times 6n}$, then $L(\lambda, \mu) \times (\Lambda \otimes I_n) = v \otimes E(\lambda, \mu)$ if and only if

$$L_1 \boxplus L_2 \boxplus L_0 = v \otimes [P_9 \ P_8 \ P_7 \ P_6 \ P_5 \ P_4 \ P_3 \ P_2 \ P_1 \ P_0].$$

Proof. Consider, $L(\lambda, \mu) = L_0 + \lambda L_1 + \mu L_2$, where

$$L_1 = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} \end{bmatrix}, L_2 = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & B_{16} \\ B_{21} & B_{22} & B_{23} & B_{24} & B_{25} & B_{26} \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} & B_{36} \\ B_{41} & B_{42} & B_{43} & B_{44} & B_{45} & B_{46} \\ B_{51} & B_{52} & B_{53} & B_{54} & B_{55} & B_{56} \\ B_{61} & B_{62} & B_{63} & B_{64} & B_{65} & B_{66} \end{bmatrix}, L_0 = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix};$$

with each $A_{ij}, B_{ij}, C_{ij} \in C^{n \times n}$ for $i, j := 1 : 6$.

First, we consider,

$$L(\lambda, \mu) \times (\Lambda \otimes I_n) = v \otimes E(\lambda, \mu), \text{ for } v = [v_1\ v_2 \ldots v_6]^T. \tag{18}$$

To show,

$$L_1 \boxplus L_2 \boxplus L_0 = v \otimes [P_9\ P_8\ P_7\ P_6\ P_5\ P_4\ P_3\ P_2\ P_1\ P_0]. \tag{19}$$

From (18), we have

$$\begin{bmatrix} C_{11} + \lambda A_{11} + \mu B_{11} & C_{12} + \lambda A_{12} + \mu B_{12} & \ldots & C_{16} + \lambda A_{16} + \mu B_{16} \\ \vdots & \vdots & \ldots \vdots & \\ C_{61} + \lambda A_{61} + \mu B_{61} & C_{62} + \lambda A_{62} + \mu B_{62} & \ldots & C_{66} + \lambda A_{66} + \mu B_{66} \end{bmatrix} \times \begin{bmatrix} \lambda^2 I_n \\ \vdots \\ I_n \end{bmatrix}$$

$$= \begin{bmatrix} v_1 E(\lambda,\ \mu) \\ \vdots \\ v_6 E(\lambda,\ \mu) \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} \lambda^2 C_{11} + \lambda^3 A_{11} + \lambda^2 \mu B_{11} + \lambda\mu C_{12} + \lambda^2 \mu A_{12} + \lambda\mu^2 B_{12} + \ldots C_{16} + \lambda A_{16} + \mu B_{16} \\ \vdots \\ \lambda^2 C_{61} + \lambda^3 A_{61} + \lambda^2 \mu B_{61} + \lambda\mu C_{62} + \lambda^2 \mu A_{62} + \lambda\mu^2 B_{62} + \ldots C_{66} + \lambda A_{66} + \mu B_{66} \end{bmatrix}$$

$$= \begin{bmatrix} v_1 P_0 + v_1 \mu P_1 + v_1 \lambda P_2 + v_1 \mu^2 P_3 + v_1 \lambda\mu P_4 + v_1 \lambda^2 P_5 + v_1 \mu^3 P_6 + v_1 \lambda\mu^2 P_7 + v_1 \lambda^2 \mu P_8 + v_1 \lambda^3 P_9 \\ \vdots \\ v_1 P_0 + v_1 \mu P_1 + v_6 \lambda P_2 + v_6 \mu^2 P_3 + v_6 \lambda\mu P_4 + v_6 \lambda^2 P_5 + v_6 \mu^3 P_6 + v_6 \lambda\mu^2 P_7 + v_6 \lambda^2 \mu P_8 + v_6 \lambda^3 P_9 \end{bmatrix}$$

Now equating matrix entries on both sides we have,

$$\lambda^2 C_{i1} + \lambda^3 A_{i1} + \lambda^2 \mu B_{i1} + \lambda\mu C_{i2} + \lambda^2 \mu A_{i2} + \lambda\mu^2 B_{i2} + \ldots + C_{i6} + \lambda A i_6 + \mu B i_6$$

$$= v_i P_0 + v_i \mu P_1 + v_i \lambda P_2 + v_i \mu^2 P_3 + v_i \lambda\mu P_4 + v_i \lambda^2 P_5 + v_i \mu^3 P_6 + v_i \lambda\mu^2 P_7 + v_i \lambda^2 \mu P_8 + v_i \lambda^3 P_9$$

for $i := 1 : 6$.

For each of these equations if we equate the coefficient matrices for λ^3, $\lambda^2\mu$, $\lambda\mu^2$, λ^2, $\lambda\mu$, μ^2, λ, μ and the constant terms we get

$$A_{i1} = v_i P_9$$

$$B_{i1} + A_{i2} = v_i P_8$$

$$B_{i2} + A_{i3} = v_i P_7$$

$$B_{i3} = v_i P_6$$

$$C_{i1} + A_{i4} = v_i P_5$$

$$C_{i2} + B_{i4} + A_{i5} = v_i P_4$$

$$C_{i3} + B_{i5} = v_i P_3$$

$$C_{i4} + A_{i6} = v_i P_2$$

$$C_{i5} + B_{i6} = v_i P_1$$

$$C_{i6} = v_i P_0 \tag{20}$$

for $i = 1 : 6$.

Then, by using Definition 2.5, we have

$$L_1 \boxplus L_2 \boxplus L_0 = \begin{bmatrix} A_{11} & A_{12}+B_{11} & A_{13}+B_{12} & B_{13} & A_{14}+C_{11} & A_{15}+B_{14}+C_{12} & B_{15}+C_{13} & A_{16}+C_{14} & B_{16}+C_{15} & C_{16} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{61} & A_{62}+B_{61} & A_{63}+B_{62} & B_{63} & A_{64}+C_{61} & A_{65}+B_{64}+C_{62} & B_{65}+C_{63} & A_{66}+C_{64} & B_{66}+C_{65} & C_{66} \end{bmatrix}$$

Now, by using equations in (20), we get

$$\begin{bmatrix} v_1P_9 & v_1P_8 & v_1P_7 & v_1P_6 & v_1P_5 & v_1P_4 & v_1P_3 & v_1P_2 & v_1P_1 & v_1P_0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_6P_9 & v_6P_8 & v_6P_7 & v_6P_6 & v_6P_5 & v_1P_4 & v_6P_3 & v_6P_2 & v_6P_1 & v_6P_0 \end{bmatrix}$$

$$= \begin{bmatrix} v_1 \\ \vdots \\ v_6 \end{bmatrix} \otimes \begin{bmatrix} P_9 & P_8 & P_7 & P_6 & P_5 & P_4 & P_3 & P_2 & P_1 & P_0 \end{bmatrix}$$

$$= v \otimes \begin{bmatrix} P_9 & P_8 & P_7 & P_6 & P_5 & P_4 & P_3 & P_2 & P_1 & P_0 \end{bmatrix},$$

which is (19). Thus (18) implies the (19).

Conversely, we consider that (19) holds. Then

$$L_1 \boxplus L_2 \boxplus L_0 = v \otimes [P_9 \; P_8 \; P_7 \; P_6 \; P_5 \; P_4 \; P_3 \; P_2 \; P_1 \; P_0],$$

i.e.,

$$\begin{bmatrix} A_{11} & A_{12}+B_{11} & A_{13}+B_{12} & B_{13} & A_{14}+C_{11} & A_{15}+B_{14}+C_{12} & B_{15}+C_{13} & A_{16}+C_{14} & B_{16}+C_{15} & C_{16} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{61} & A_{62}+B_{61} & A_{63}+B_{62} & B_{63} & A_{64}+C_{61} & A_{65}+B_{64}+C_{62} & B_{65}+C_{63} & A_{66}+C_{64} & B_{66}+C_{65} & C_{66} \end{bmatrix}$$

$$= \begin{bmatrix} v_1P_9 & v_1P_8 & v_1P_7 & v_1P_6 & v_1P_5 & v_1P_4 & v_1P_3 & v_1P_2 & v_1P_1 & v_1P_0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_6P_9 & v_6P_8 & v_6P_7 & v_6P_6 & v_6P_5 & v_1P_4 & v_6P_3 & v_6P_2 & v_6P_1 & v_6P_0 \end{bmatrix}.$$

Equating matrix entries from both side we again have the system of equations as in (20) for $i := 1 : 6$.

From left side of (18), we have

$$\begin{bmatrix} C_{11}+\lambda A_{11}+\mu B_{11} & C_{12}+\lambda A_{12}+\mu B_{12} & \dots & C_{16}+\lambda A_{16}+\mu B_{16} \\ \vdots & \vdots & \dots & \vdots \\ C_{61}+\lambda A_{61}+\mu B_{61} & C_{62}+\lambda A_{62}+\mu B_{62} & \dots & C_{66}+\lambda A_{66}+\mu B_{66} \end{bmatrix} \times \begin{bmatrix} \lambda^2 I_n \\ \vdots \\ I_n \end{bmatrix}.$$

$$= \begin{bmatrix} \lambda^2 C_{11} + \lambda^3 A_{11} + \lambda^2 \mu B_{11} + \lambda\mu C_{12} + \lambda^2 \mu A_{12} + \lambda \mu^2 B_{12} + \dots C_{16} + \lambda A_{16} + \mu B_{16} \\ \vdots \\ \lambda^2 C_{61} + \lambda^3 A_{61} + \lambda^2 \mu B_{61} + \lambda\mu C_{62} + \lambda^2 \mu A_{62} + \lambda \mu^2 B_{62} + \dots C_{66} + \lambda A_{66} + \mu B_{66} \end{bmatrix}.$$

Now, by separating the coefficients of λ^3, $\lambda^2\mu$, $\lambda\mu^2$, μ^3, λ^2, $\lambda\mu$, μ^2, λ, μ and the costant terms in each of the entries of the above matrix we get,

$$= \begin{bmatrix} \lambda^3 A_{11} + \lambda^2\mu(B_{11} + A_{12}) + \lambda\mu^2(B_{12} + A_{13}) + \mu^3(B_{13}) + \lambda^2(C_{11} + A_{14}) + \lambda\mu(C_{12} + B_{14} + A_{15}) \\ +\mu^2(C_{13} + B_{15}) + \lambda(C_{14} + A_{16}) + \mu(C_{15} + B_{16}) + C_{16} \\ \vdots \\ \lambda^3 A_{61} + \lambda^2\mu(B_{61} + A_{62}) + \lambda\mu^2(B_{62} + A_{63}) + \mu^3(B_{63}) + \lambda^2(C_{61} + A_{64}) + \lambda\mu(C_{62} + B_{64} + A_{65}) \\ +\mu^2(C_{63} + B_{65}) + \lambda(C_{64} + A_{66}) + \mu(C_{65} + B_{66}) + C_{66} \end{bmatrix}$$

Now, by using equations from (20), we have

$$\begin{bmatrix} \lambda^3 v_1 P_9 + \lambda^2\mu v_1 P_8 + \lambda\mu^2 v_1 P_7 + \mu^3 v_1 P_6 + \lambda^2 v_1 P_5 + \lambda\mu v_1 P_4 + \mu^2 v_1 P_3 + \lambda v_1 P_2 + \mu v_1 P_1 + v_1 P_0 \\ \vdots \\ \lambda^3 v_1 P_9 + \lambda^2\mu v_6 P_8 + \lambda\mu^2 v_6 P_7 + \mu^3 v_6 P_6 + \lambda^2 v_6 P_5 + \lambda\mu v_6 P_4 + \mu^2 v_6 P_3 + \lambda v_6 P_2 + \mu v_6 P_1 + v_6 P_0 \end{bmatrix}$$

$$= \begin{bmatrix} v_1 E(\lambda, \mu) \\ \vdots \\ v_6 E(\lambda, \mu) \end{bmatrix}.$$

This is the required (18). Hence, (19) also implies the (18). □

Now, based on this Lemma 3.3, we can investigate the structure of any $L(\lambda, \mu)$, which belongs to the ansatz space $L(E(\lambda, \mu))$. Thus, we have the following Theorem.

Theorem 3.4 Let $E(\lambda, \mu)$ be as defined in (1) with coefficient matrices of order $n \times n$, and $L(\lambda, \mu)$ be as defined in (2). Let $v \in C^6$. Then, $L(\lambda, \mu) \in L(E(\lambda, \mu))$ corresponding to the ansatz vector v is of the form $\hat{L}_v(\lambda, \mu) := \hat{L}_0 + \lambda\hat{L}_1 + \mu\hat{L}_2$, such that

$$\hat{L}_0 = \begin{bmatrix} Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & v \otimes P_0 \end{bmatrix},$$

$$\hat{L}_1 = \begin{bmatrix} v \otimes P_9 & -Y_1 + v \otimes P_8 & -Y_2 + v \otimes P_7 & -Z_1 + v \otimes P_5 & -Z_2 + v \otimes P_4 & -Z_4 + v \otimes P_2 \end{bmatrix},$$

$$\hat{L}_2 = \begin{bmatrix} Y_1 & Y_2 & v \otimes P_6 & 0 & -Z_3 + v \otimes P_3 & -Z_5 + v \otimes P_1 \end{bmatrix}.$$

where

$$Y_p = \begin{bmatrix} Y_{1p} \\ Y_{2p} \\ Y_{3p} \\ Y_{4p} \\ Y_{5p} \\ Y_{6p} \end{bmatrix}, Z_j = \begin{bmatrix} Z_{1j} \\ Z_{2j} \\ Z_{3j} \\ Z_{4j} \\ Z_{5j} \\ Z_{6j} \end{bmatrix} \in C^{6n \times n},$$

for $p := 1 : 2, j := 1 : 5$ are arbitrary.

Proof. Let $F : L(E(\lambda, \mu)) \to V_Q$ be any linear map defined by

$$F(L(\lambda, \mu)) = L(\lambda, \mu)(\Lambda \otimes I_n).$$

To show that F is surjective. Let $v \otimes E(\lambda, \mu) \in V_Q$ be any element. Consider a two-parameter matrix pencil $\hat{L}_v(\lambda, \mu)$ $:= \hat{L}_0 + \lambda\hat{L}_1 + \mu\hat{L}_2$, where

$$\hat{L}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & v \otimes P_0 \end{bmatrix}$$

$$\hat{L}_1 = \begin{bmatrix} v \otimes P_9 & v \otimes P_8 & v \otimes P_7 & v \otimes P_5 & v \otimes P_4 & v \otimes P_2 \end{bmatrix}$$

$$\hat{L}_2 = \begin{bmatrix} 0 & 0 & v \otimes P_6 & 0 & v \times P_3 & v \otimes P_1 \end{bmatrix}$$

Then after simplification, we get

$$\hat{L}_1 \boxplus \hat{L}_2 \boxplus \hat{L}_0 = v \otimes [P_9 \ P_8 \ P_7 \ P_6 \ P_5 \ P_4 \ P_3 \ P_2 \ P_1 \ P_0],$$

So by Lemma 3.3, $\hat{L}_v(\lambda, \mu)$ is a F-preimage of $v \otimes E(\lambda, \mu)$. Then we have,

$$\hat{L}_v(\lambda, \mu) \times (\Lambda \otimes I_n) = v \otimes E(\lambda, \mu).$$

Therefore, the set of all F-preimages of $v \otimes E(\lambda, \mu)$ is $\hat{L}_v(\lambda, \mu) + \text{Ker}F$.

Again, $\text{Ker}F = \{L(\lambda, \mu) \in L(E(\lambda, \mu)) : L(\lambda, \mu) \times (\Lambda \otimes I_n) = 0\}$. By Lemma 3.3 it follows that, if $(\hat{L}_0 + \lambda\hat{L}_1 + \mu\hat{L}_2) \times (\Lambda \otimes I_n) = 0$, if and only if $\hat{L}_1 \boxplus \hat{L}_2 \boxplus \hat{L}_0 = 0$. The definition of $\boxplus$ ensures that $\hat{L}_i$, $i := 0 : 2$, are of the form

$$\hat{L}_0 = \begin{bmatrix} Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & 0 \end{bmatrix},$$

$$\hat{L}_1 = \begin{bmatrix} 0 & -Y_1 & -Y_2 & -Z_1 & -Z_2 & -Z_4 \end{bmatrix},$$

$$\hat{L}_2 = \begin{bmatrix} Y_1 & Y_2 & 0 & 0 & -Z_3 & -Z_5 \end{bmatrix};$$

where $Y_i, Z_j \in \mathbb{C}^{6n \times n}$, for $i := 1 : 2, j := 1 : 5$ are arbitrary.

Therefore, the set of all preimages of $v \otimes E(\lambda, \mu)$ is of the form $\hat{L}_v(\lambda, \mu) = \hat{L}_0 + \lambda\hat{L}_1 + \mu\hat{L}_2$, with

$$\hat{L}_0 = \begin{bmatrix} Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & v \otimes P_0 \end{bmatrix},$$

$$\hat{L}_1 = \begin{bmatrix} v \otimes P_9 & -Y_1 + v \otimes P_8 & -Y_2 + v \otimes P_7 & -Z_1 + v \otimes P_5 & -Z_2 + v \otimes P_4 & -Z_4 + v \otimes P_2 \end{bmatrix},$$

$$\hat{L}_2 = \begin{bmatrix} Y_1 & Y_2 & v \otimes P_6 & 0 & -Z_3 + v \otimes P_3 & -Z_5 + v \otimes P_1 \end{bmatrix}.$$

This completes the proof. □

The dimension of the ansatz space $L(E(\lambda, \mu))$ can be calculated easily by the following corollary.

Corollary 3.5 Dimension of $L(E(\lambda, \mu)) = 42n^2 + 6$.

Proof. Since the map F is surjective, we have,

$$\dim F = \dim(\text{Range}F) + \dim(\text{Ker}F) = \dim V_Q + \dim(\text{Ker}F) = 6 + 42n^2.$$ □

3.2 *Construction for linearization*

It is worth mentioning that not all the two-parameter matrix polynomials $L(\lambda, \mu) \in L(E(\lambda, \mu))$ are linearization of $E(\lambda, \mu)$. For example, for the ansatz vector $v = 0$, we don't have any $L(\lambda, \mu) \in L(E(\lambda, \mu))$ which will be a linearization of $E(\lambda, \mu)$. Therefore, it becomes necessary to classify which $L(\lambda, \mu)$ in $L(E(\lambda, \mu))$ are linearizations.

From the Standard linearization case we have obtained (14), it shows the possibility of constructing potential linearizations of $E(\lambda, \mu)$ with the ansatz vectors which are scalar multiples of e_1. In the following Theorem, we derive a linearization condition for $L(\lambda, \mu) \in L(E(\lambda, \mu))$ corresponding to the ansatz vector $\alpha e_1 \in C^n$ for some scalar $\alpha \neq 0$. These linearizations are easy to construct with the help of Theorem 3.4.

Theorem 3.6 Let $E(\lambda, \mu)$ be a CTMP as given in (1) with real or complex coefficient matrices of order $n \times n$. Let $L(\lambda, \mu) = L_0 + \lambda L_1 + \mu L_2 \in L(E(\lambda, \mu))$ corresponding to the ansatz vector $v = \alpha e_1$ for some $0 \neq \alpha \in C$, where

$$L_0 = \begin{bmatrix} Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & \alpha e_1 \otimes P_0 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} \alpha e_1 \otimes P_9 & -Y_1 + \alpha e_1 \otimes P_8 & -Y_2 + \alpha e_1 \otimes P_7 & -Z_1 + \alpha e_1 \otimes P_5 & -Z_2 + \alpha e_1 \otimes P_4 & -Z_4 + \alpha e_1 \otimes P_2 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} Y_1 & Y_2 & \alpha e_1 \otimes P_6 & 0 & -Z_3 + \alpha e_1 \times P_3 & -Z_5 + \alpha e_1 \otimes P_1 \end{bmatrix}.$$

where

$$Y_1 = \begin{bmatrix} Y_{11} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, Y_2 = \begin{bmatrix} Y_{12} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, Z_1 = \begin{bmatrix} Z_{11} \\ Z_{21} \\ Z_{31} \\ Z_{41} \\ Z_{51} \\ Z_{61} \end{bmatrix}, Z_2 = \begin{bmatrix} Z_{12} \\ Z_{22} \\ Z_{32} \\ Z_{42} \\ Z_{52} \\ Z_{62} \end{bmatrix}, Z_3 = \begin{bmatrix} Z_{13} \\ Z_{23} \\ Z_{33} \\ Z_{43} \\ Z_{53} \\ Z_{63} \end{bmatrix}, Z_4 = \begin{bmatrix} Z_{14} \\ Z_{24} \\ Z_{34} \\ Z_{44} \\ Z_{54} \\ Z_{64} \end{bmatrix}, Z_5 = \begin{bmatrix} Z_{15} \\ Z_{25} \\ Z_{35} \\ Z_{45} \\ Z_{55} \\ Z_{65} \end{bmatrix} \in C^{6n \times n};$$

with

$$det \begin{bmatrix} Z_{25} & Z_{24} & Z_{23} & Z_{22} & Z_{21} \\ Z_{35} & Z_{34} & Z_{33} & Z_{32} & Z_{31} \\ Z_{45} & Z_{44} & Z_{43} & Z_{42} & Z_{41} \\ Z_{55} & Z_{54} & Z_{53} & Z_{52} & Z_{51} \\ Z_{65} & Z_{64} & Z_{63} & Z_{62} & Z_{61} \end{bmatrix} \neq 0.$$

Then $L(\lambda, \mu)$ is a linearization of $E(\lambda, \mu)$.

Proof. By Theorem 3.4, for any two-parameter cubic polynomial $E(\lambda, \mu)$ corresponding to the ansatz vector $v = \alpha e_1$, there exists a linear two-parameter matrix polynomial $L(\lambda, \mu) = L_0 + \lambda L_1 + \mu L_2 \in L(E(\lambda, \mu))$, such that

$$L(\lambda, \mu) = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & \alpha P_0 \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} & Z_{25} & 0 \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} & Z_{35} & 0 \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} & 0 \\ Z_{51} & Z_{52} & Z_{53} & Z_{54} & Z_{55} & 0 \\ Z_{61} & Z_{62} & Z_{63} & Z_{64} & Z_{65} & 0 \end{bmatrix} + \lambda \begin{bmatrix} \alpha P_9 & -Y_{11} + \alpha P_8 & -Y_{12} + \alpha P_7 & -Z_{11} + \alpha P_5 & -Z_{12} + \alpha P_4 & -Z_{14} + \alpha P_2 \\ 0 & -Y_{21} & -Y_{22} & -Z_{21} & -Z_{22} & -Z_{24} \\ 0 & -Y_{31} & -Y_{32} & -Z_{31} & -Z_{32} & -Z_{34} \\ 0 & -Y_{41} & -Y_{42} & -Z_{41} & -Z_{42} & -Z_{44} \\ 0 & -Y_{51} & -Y_{52} & -Z_{51} & -Z_{52} & -Z_{54} \\ 0 & -Y_{61} & -Y_{62} & -Z_{61} & -Z_{62} & -Z_{64} \end{bmatrix}$$

$$+ \mu \begin{bmatrix} Y_{11} & Y_{12} & \alpha P_6 & 0 & -Z_{13} + \alpha P_3 & -Z_{15} + \alpha P_1 \\ Y_{21} & Y_{22} & 0 & 0 & -Z_{23} & -Z_{25} \\ Y_{31} & Y_{32} & 0 & 0 & -Z_{33} & -Z_{35} \\ Y_{41} & Y_{42} & 0 & 0 & -Z_{43} & -Z_{45} \\ Y_{51} & Y_{52} & 0 & 0 & -Z_{53} & -Z_{55} \\ Y_{61} & Y_{62} & 0 & 0 & -Z_{63} & -Z_{65} \end{bmatrix}$$

Thus we have,

$$L(\lambda,\ \mu)=\begin{bmatrix} T_1(\lambda,\ \mu) & T_2(\lambda,\ \mu) & T_3(\lambda,\ \mu) & T_4(\lambda,\ \mu) & T_5(\lambda,\ \mu) & T_6(\lambda,\ \mu) \\ \mu Y_{21}+Z_{21} & -\lambda Y_{21}+\mu Y_{22}+Z_{22} & -\lambda Y_{22}+Z_{23} & -\lambda Z_{21}+Z_{24} & -\lambda Z_{22}-\mu Z_{23}+Z_{25} & -\lambda Z_{24}-\mu Z_{25} \\ \mu Y_{31}+Z_{31} & -\lambda Y_{31}+\mu Y_{32}+Z_{32} & -\lambda Y_{32}+Z_{33} & -\lambda Z_{31}+Z_{34} & -\lambda Z_{32}-\mu Z_{33}+Z_{35} & -\lambda Z_{34}-\mu Z_{35} \\ \mu Y_{41}+Z_{41} & -\lambda Y_{41}+\mu Y_{42}+Z_{42} & -\lambda Y_{42}+Z_{43} & -\lambda Z_{41}+Z_{44} & -\lambda Z_{42}-\mu Z_{43}+Z_{45} & -\lambda Z_{44}-\mu Z_{45} \\ \mu Y_{51}+Z_{51} & -\lambda Y_{51}+\mu Y_{52}+Z_{52} & -\lambda Y_{52}+Z_{53} & -\lambda Z_{51}+Z_{54} & -\lambda Z_{52}-\mu Z_{53}+Z_{55} & -\lambda Z_{54}-\mu Z_{55} \\ \mu Y_{61}+Z_{61} & -\lambda Y_{61}+\mu Y_{62}+Z_{62} & -\lambda Y_{62}+Z_{63} & -\lambda Z_{61}+Z_{64} & -\lambda Z_{62}-\mu Z_{63}+Z_{65} & -\lambda Z_{64}-\mu Z_{65} \end{bmatrix}$$

where,

$$T_1(\lambda,\mu)=Z_{11}+\alpha\lambda P_9+\mu Y_{11},$$

$$T_2(\lambda,\mu)=Z_{12}-\lambda Y_{11}+\alpha\lambda P_8+\mu Y_{12},$$

$$T_3(\lambda,\mu)=Z_{13}-\lambda Y_{12}+\alpha\lambda P_7+\alpha P_6,$$

$$T_4(\lambda,\mu)=Z_{14}-\lambda Z_{11}+\alpha\mu P_5,$$

$$T_5(\lambda,\mu)=Z_{15}-\lambda Z_{12}+\alpha\lambda P_4-\mu Z_{13}+\alpha P_3,$$

$$T_6(\lambda,\mu)=\alpha P_0-\lambda Z_{14}+\alpha\lambda P_2-\mu Z_{15}+\alpha\mu P_1.$$

Define,

$$R((\lambda,\ \mu)=\begin{bmatrix} \frac{\lambda^2}{\alpha}I & 0 & \lambda I & 0 & 0 & I \\ \frac{\lambda\mu}{\alpha}I & \lambda I & 0 & 0 & I & 0 \\ \frac{\mu^2}{\alpha}I & \mu I & 0 & I & 0 & 0 \\ \frac{\lambda}{\alpha}I & 0 & I & 0 & 0 & 0 \\ \frac{\mu}{\alpha}I & I & 0 & 0 & 0 & 0 \\ \frac{I}{\alpha} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{21}$$

Then, we have

$$L(\lambda,\ \mu)R(\lambda,\ \mu)=\begin{bmatrix} E(\lambda,\ \mu) & \hat{T}_1 & \hat{T}_2 & \hat{T}_3 & \hat{T}_4 & \hat{T}_5 \\ 0 & -\lambda^2 Y_{21}+Z_{25} & \lambda\mu Y_{21}+Z_{24} & -\lambda Y_{22}+Z_{23} & -\lambda Y_{21}+\mu Y_{22}+Z_{22} & \mu Y_{21}+Z_{21} \\ 0 & -\lambda^2 Y_{31}+Z_{35} & \lambda\mu Y_{31}+Z_{34} & -\lambda Y_{32}+Z_{33} & -\lambda Y_{31}+\mu Y_{32}+Z_{32} & \mu Y_{31}+Z_{31} \\ 0 & -\lambda^2 Y_{41}+Z_{45} & \lambda\mu Y_{41}+Z_{44} & -\lambda Y_{42}+Z_{43} & -\lambda Y_{41}+\mu Y_{42}+Z_{42} & \mu Y_{41}+Z_{41} \\ 0 & -\lambda^2 Y_{51}+Z_{55} & \lambda\mu Y_{51}+Z_{54} & -\lambda Y_{52}+Z_{53} & -\lambda Y_{51}+\mu Y_{52}+Z_{52} & \mu Y_{51}+Z_{51} \\ 0 & -\lambda^2 Y_{61}+Z_{65} & \lambda\mu Y_{61}+Z_{64} & -\lambda Y_{62}+Z_{63} & -\lambda Y_{61}+\mu Y_{62}+Z_{62} & \mu Y_{61}+Z_{61} \end{bmatrix},$$

where $\hat{T}_1=\lambda T_2(\lambda,\ \mu)+\mu T_3(\lambda,\ \mu)+T_5(\lambda,\ \mu),\ \hat{T}_2=\lambda T_1(\lambda,\ \mu)+T_4(\lambda,\ \mu),\ \hat{T}_3=T_3(\lambda,\ \mu),\ \hat{T}_4=T_2(\lambda,\ \mu),\ \hat{T}_5=T_1(\lambda,\ \mu).$ We want to make the entries in the lower block matrix,

$$\begin{bmatrix} -\lambda^2 Y_{21} + Z_{25} & \lambda\mu Y_{21} + Z_{24} & -\lambda Y_{22} + Z_{23} & -\lambda Y_{21} + \mu Y_{22} + Z_{22} & \mu Y_{21} + Z_{21} \\ -\lambda^2 Y_{31} + Z_{35} & \lambda\mu Y_{31} + Z_{34} & -\lambda Y_{32} + Z_{33} & -\lambda Y_{31} + \mu Y_{32} + Z_{32} & \mu Y_{31} + Z_{31} \\ -\lambda^2 Y_{41} + Z_{45} & \lambda\mu Y_{41} + Z_{44} & -\lambda Y_{42} + Z_{43} & -\lambda Y_{41} + \mu Y_{42} + Z_{42} & \mu Y_{41} + Z_{41} \\ -\lambda^2 Y_{51} + Z_{55} & \lambda\mu Y_{51} + Z_{54} & -\lambda Y_{52} + Z_{53} & -\lambda Y_{51} + \mu Y_{52} + Z_{52} & \mu Y_{51} + Z_{51} \\ -\lambda^2 Y_{61} + Z_{65} & \lambda\mu Y_{61} + Z_{64} & -\lambda Y_{62} + Z_{63} & -\lambda Y_{61} + \mu Y_{62} + Z_{62} & \mu Y_{61} + Z_{61} \end{bmatrix},$$

free from λ^2, $\lambda\mu$ and μ.

Setting

$$Y_{2i} = Y_{3i} = Y_{4i} = Y_{5i} = Y_{6i} = 0,$$

for $i = 1, 2$; we get,

$$L(\lambda, \mu)R(\lambda, \mu) = \begin{bmatrix} E(\lambda, \mu) & \hat{T}_1 & \hat{T}_2 & \hat{T}_3 & \hat{T}_4 & \hat{T}_5 \\ 0 & Z_{25} & Z_{24} & Z_{23} & Z_{22} & Z_{21} \\ 0 & Z_{35} & Z_{34} & Z_{33} & Z_{32} & Z_{31} \\ 0 & Z_{45} & Z_{44} & Z_{43} & Z_{42} & Z_{41} \\ 0 & Z_{55} & Z_{54} & Z_{53} & Z_{52} & Z_{51} \\ 0 & Z_{65} & Z_{64} & Z_{63} & Z_{62} & Z_{61} \end{bmatrix}. \tag{22}$$

Define,

$$T(\lambda, \mu) = \begin{bmatrix} \hat{T}_1 & \hat{T}_2 & \hat{T}_3 & \hat{T}_4 & \hat{T}_5 \end{bmatrix} \in C^{n \times 5n},$$

and

$$Z = \begin{bmatrix} Z_{25} & Z_{24} & Z_{23} & Z_{22} & Z_{21} \\ Z_{35} & Z_{34} & Z_{33} & Z_{32} & Z_{31} \\ Z_{45} & Z_{44} & Z_{43} & Z_{42} & Z_{41} \\ Z_{55} & Z_{54} & Z_{53} & Z_{52} & Z_{51} \\ Z_{65} & Z_{64} & Z_{63} & Z_{62} & Z_{61} \end{bmatrix} \in C^{5n \times 5n}.$$

So, we can rewrite $L(\lambda, \mu)R(\lambda, \mu)$ as,

$$L(\lambda, \mu)R(\lambda, \mu) = \begin{bmatrix} E(\lambda, \mu) & T(\lambda, \mu) \\ 0 & Z \end{bmatrix}$$

Since Z is nonsingular, we define

$$P(\lambda, \mu) = \begin{bmatrix} I & -T(\lambda, \mu)Z^{-1} \\ 0 & Z^{-1} \end{bmatrix}$$

Thus we get,

$$P(\lambda, \mu)L(\lambda, \mu)R(\lambda, \mu) = \begin{bmatrix} E(\lambda, \mu) & 0 \\ 0 & I_{5n} \end{bmatrix}.$$

Since both $P(\lambda, \mu)$ and $R(\lambda, \mu)$ are unimodular matrices we get,

$$\det L(\lambda, \mu) = \gamma \det E(\lambda, \mu)$$

for some nonzero $\gamma \in C$. Thus $L(\lambda, \mu)$ is a lnearization of $E(\lambda, \mu)$. □

Consider, $E(\lambda, \mu)$ in (1) and $L(\lambda, \mu) \in L(E(\lambda, \mu))$ corresponding to the nonzero ansatz vector $v \in C^6$ i.e., $L(\lambda, \mu) \times (\Lambda \otimes I_n) = v \otimes E(\lambda, \mu)$. After that, we may use the following process to find a collection of linearizations for $E(\lambda, \mu)$.

Procedure to determine linearizations of $L(E(\lambda, \mu))$

1. Let $E(\lambda, \mu)$ be defined in (1) and $L(\lambda, \mu) = L_0 + \lambda L_1 + \mu L_2 \in L(E(\lambda, \mu))$ with ansatz vector $v \in C^6$.
2. Consider any nonsingular matrix $M := [m_{ij}] \in C^{6\times 6}$; $i, j := 1 : 6$ such that $Mv = \alpha e_1 \in C^6$, where $0 \neq \alpha \in C$.
3. Apply corresponding block transformation $M \otimes I_n$ to $L(\lambda, \mu)$. Then

$$\tilde{L}(\lambda, \mu) = (M \otimes I_n) \times L(\lambda, \mu) = \tilde{L}_0 + \lambda \tilde{L}_1 + \mu \tilde{L}_2$$

such that,

$$\tilde{L}_0 = \begin{bmatrix} \tilde{Z}_1 & \tilde{Z}_2 & \tilde{Z}_3 & \tilde{Z}_4 & \tilde{Z}_5 & \alpha e_1 \otimes P_0 \end{bmatrix},$$

$$\tilde{L}_1 = \begin{bmatrix} \alpha e_1 \otimes P_9 & -\tilde{Y}_1 + \alpha e_1 \otimes P_8 & -\tilde{Y}_2 + \alpha e_1 \otimes P_7 & -\tilde{Z}_1 + \alpha e_1 \otimes P_5 & -\tilde{Z}_2 + \alpha e_1 \otimes P_4 & -\tilde{Z}_4 + \alpha e_1 \otimes P_2 \end{bmatrix},$$

$$\tilde{L}_2 = \begin{bmatrix} \tilde{Y}_1 & \tilde{Y}_2 & \alpha e_1 \otimes P_6 & 0 & \alpha e_1 \otimes P_5 & -\tilde{Z}_3 + \alpha e_1 \otimes P_3 & -\tilde{Z}_5 + \alpha e_1 \otimes P_1 \end{bmatrix},$$

where

$$\widetilde{Y}_1 = (M \otimes I_n) Y_1 = (M \otimes I_n) \begin{bmatrix} Y_{11} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} m_{11} Y_{11} \\ m_{21} Y_{11} \\ m_{31} Y_{11} \\ m_{41} Y_{11} \\ m_{51} Y_{11} \\ m_{61} Y_{11} \end{bmatrix}, \ \tilde{Y}_2 = (M \otimes I_n) Y_2 = (M \otimes I_n) \begin{bmatrix} Y_{21} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} m_{11} Y_{21} \\ m_{21} Y_{21} \\ m_{31} Y_{21} \\ m_{41} Y_{21} \\ m_{51} Y_{21} \\ m_{61} Y_{21} \end{bmatrix},$$

$$\tilde{Z}_i = (M \otimes I_n) Z_i = (M \otimes I_n) \begin{bmatrix} Z_{1i} \\ Z_{2i} \\ Z_{3i} \\ Z_{4i} \\ Z_{5i} \\ Z_{6i} \end{bmatrix} = \begin{bmatrix} m_{11} Z_{1i} + m_{12} Z_{2i} + m_{13} Z_{3i} + m_{14} Z_{4i} + m_{15} Z_{5i} + m_{16} Z_{6i} \\ m_{21} Z_{1i} + m_{22} Z_{2i} + m_{23} Z_{3i} + m_{24} Z_{4i} + m_{25} Z_{5i} + m_{26} Z_{6i} \\ m_{31} Z_{1i} + m_{32} Z_{2i} + m_{33} Z_{3i} + m_{34} Z_{4i} + m_{35} Z_{5i} + m_{36} Z_{6i} \\ m_{41} Z_{1i} + m_{42} Z_{2i} + m_{43} Z_{3i} + m_{44} Z_{4i} + m_{45} Z_{5i} + m_{46} Z_{6i} \\ m_{51} Z_{1i} + m_{52} Z_{2i} + m_{53} Z_{3i} + m_{54} Z_{4i} + m_{55} Z_{5i} + m_{56} Z_{6i} \\ m_{61} Z_{1i} + m_{62} Z_{2i} + m_{63} Z_{3i} + m_{64} Z_{4i} + m_{65} Z_{5i} + m_{66} Z_{6i} \end{bmatrix},$$

for $i = 1 : 5$ are arbitrary.

4. For $\tilde{L}(\lambda, \mu)$ to be linearization, Y_1, Y_2, Z_1, Z_2, Z_3, Z_4 and Z_5 have to be choosen as follows.

If $m_{21} = m_{31} = m_{41} = m_{51} = m_{61} = 0$ then choose Y_{11} arbitrary, otherwise choose $Y_{11} = 0$. Similarly if $m_{21} = m_{31} = m_{41} = m_{51} = m_{61} = 0$ then choose Y_{21} arbitrary, otherwise choose $Y_{21} = 0$.

Further, we need to choose,

$$Z_1 = \begin{bmatrix} Z_{11} \\ Z_{21} \\ Z_{31} \\ Z_{41} \\ Z_{51} \\ Z_{61} \end{bmatrix}, \ Z_2 = \begin{bmatrix} Z_{12} \\ Z_{22} \\ Z_{32} \\ Z_{42} \\ Z_{52} \\ Z_{62} \end{bmatrix}, \ Z_3 = \begin{bmatrix} Z_{13} \\ Z_{23} \\ Z_{33} \\ Z_{43} \\ Z_{53} \\ Z_{63} \end{bmatrix}, \ Z_4 = \begin{bmatrix} Z_{14} \\ Z_{24} \\ Z_{34} \\ Z_{44} \\ Z_{54} \\ Z_{64} \end{bmatrix}, \ Z_5 = \begin{bmatrix} Z_{15} \\ Z_{25} \\ Z_{35} \\ Z_{45} \\ Z_{55} \\ Z_{65} \end{bmatrix}$$

such that

$$\det\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} \end{bmatrix} \neq 0,$$

where

$$K_{i1} = m_{(i+1)1}Z_{15} + m_{(i+1)2}Z_{25} + m_{(i+1)3}Z_{35} + m_{(i+1)4}Z_{45} + m_{(i+1)3}Z_{55} + m_{(i+1)6}Z_{65},$$

$$K_{i2} = m_{(i+1)1}Z_{14} + m_{(i+1)2}Z_{24} + m_{(i+1)3}Z_{34} + m_{(i+1)4}Z_{44} + m_{(i+1)3}Z_{54} + m_{(i+1)6}Z_{64},$$

$$K_{i3} = m_{(i+1)1}Z_{13} + m_{(i+1)2}Z_{23} + m_{(i+1)3}Z_{33} + m_{(i+1)4}Z_{43} + m_{(i+1)3}Z_{53} + m_{(i+1)6}Z_{63},$$

$$K_{i4} = m_{(i+1)1}Z_{12} + m_{(i+1)2}Z_{22} + m_{(i+1)3}Z_{32} + m_{(i+1)4}Z_{42} + m_{(i+1)3}Z_{52} + m_{(i+1)6}Z_{62},$$

$$K_{i5} = m_{(i+1)1}Z_{11} + m_{(i+1)2}Z_{21} + m_{(i+1)3}Z_{31} + m_{(i+1)4}Z_{41} + m_{(i+1)3}Z_{51} + m_{(i+1)6}Z_{61},$$

for $i = 1 : 5$.

From the construction of M, we can always choose suitable Z_1, Z_2, Z_3, Z_4 and Z_5.

3.3 *Double ansatz space*

Let $E(\lambda, \mu)$ be CTMP as given in (5). We proved that $L(E(\lambda, \mu)) = \{L(\lambda, \mu) : L(\lambda, \mu) \times (\Lambda \otimes I_n) \in V_Q\}$ is vector space. We replace the notation $L(E(\lambda, \mu)))$ by $L_1(E(\lambda, \mu))$ i.e. $L_1(E(\lambda, \mu)) = \{L(\lambda, \mu) : L(\lambda, \mu) \times (\Lambda \otimes I_n) \in V_Q\}$ and refer it as right ansatz space.

Define $L_2(E(\lambda, \mu))) = \{(\Lambda^T \otimes I_n) \times L(\lambda, \mu) : L(\lambda, \mu) \in V_Q\}$ and refer it as left ansatz space. Then we define double ansatz space as the intersection of $L_1(E(\lambda, \mu))$ and $L_2(E(\lambda, \mu))$ as follows:

$$DL(E(\lambda, \mu)) = L_1(E(\lambda, \mu)) \cap L_2(E(\lambda, \mu)) \tag{23}$$

It is easy to see that $L(\lambda, \mu) \in DL(E(\lambda, \mu)) \Leftrightarrow L(\lambda, \mu) \times (\Lambda \otimes I_n) = v \otimes E(\lambda, \mu)$ and $(\Lambda^T \otimes I_n) \times L(\lambda, \mu) = v^T \otimes E(\lambda, \mu)$. To characterize $DL(E(\lambda, \mu))$ we use the Lemma 3.3. Consider

$$L(\lambda, \mu) := L_0 + \lambda L_1 + \mu L_2$$

where each L_i; $i := 0 : 2$ is defined as,

$$L_0 = \begin{bmatrix} 0 & 0 & 0 & -\alpha P_9 & Z & 0 \\ 0 & 0 & 0 & -\alpha P_8 - Z & Z & 0 \\ 0 & 0 & 0 & -\alpha P_7 - Z & -\alpha P_6 & 0 \\ -\alpha P_9 & Z & Z & -\alpha P_5 & Z & 0 \\ -\alpha P_8 - Z & -\alpha P_7 - Z & -\alpha P_6 & -\alpha P_4 - Z & -\alpha P_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha P_0 \end{bmatrix}, L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \alpha P_9 \\ 0 & 0 & 0 & 0 & 0 & \alpha P_8 + Z \\ 0 & 0 & 0 & 0 & 0 & \alpha P_7 + Z \\ 0 & 0 & 0 & \alpha P_9 & -Z & \alpha P_5 \\ 0 & 0 & 0 & \alpha P_8 + Z & 0 & \alpha P_4 + Z \\ \alpha P_9 & -Z & -Z & \alpha P_5 & -Z & \alpha P_2 \end{bmatrix},$$

and

$$L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -Z \\ 0 & 0 & 0 & 0 & 0 & -Z \\ 0 & 0 & 0 & 0 & 0 & \alpha P_6 \\ 0 & 0 & 0 & 0 & -Z & -Z \\ 0 & 0 & 0 & \alpha P_7 + Z & \alpha P_6 & \alpha P_3 \\ \alpha P_8 + Z & \alpha P_7 + Z & \alpha P_6 & \alpha P_4 + Z & \alpha P_3 & \alpha P_1 \end{bmatrix}.$$

It is easy to see that $L(\lambda, \mu) \times (\Lambda \otimes I_n) = v \otimes E(\lambda, \mu)$ and $(\Lambda^T \otimes I_n) \times L(\lambda, \mu) = v^T \otimes I_n$, where $v = [0\ 0\ 0\ 0\ 0\ \alpha]^T \in C^6$ and $\Lambda = [\lambda^2\ \lambda\mu\ \mu^2\ \lambda\ \mu\ 1]^T \in C^6$. Therefore, $L(\lambda, \mu) \in DL(E(\lambda, \mu))$.

4. Linearization of CTEP

The eigenvalues of CTEP are the roots of the system of bivariate polynomials $\det(E_i(\lambda, \mu)) = 0$; $i := 1 : 2$. The following lemma ensures that it has $9n^2$ eigenvalues.

Lemma 4.1 In the generic case, the CTEP defined in (3) has $9n^2$ eigenvalues.

Proof. This lemma can be proved easily using Bézout's theorem. The bivariate polynomials $\det(E_1(\lambda, \mu))$ and $\det(E_2(\lambda, \mu))$ are of order $3n$. Thus, by Bézout's theorem 2.1, this polynomials has $3n \times 3n = 9n^2$ solutions. □

For spectral analysis of CTEP, the usual approach adopted is to linearize it into a 2PEP of the form (4). The de-facto way [18], to solve the problem (4) is by converting it into a coupled GEP given by

$$\Delta_1 z = \lambda \Delta_0 z$$

$$\Delta_2 z = \mu \Delta_0 z \tag{24}$$

where $z = w_1 \otimes w_2$ is decomposable tensor and

$$w_i = \Lambda \otimes x_i = [\lambda^2 x_i\ \ \lambda\mu x_i\ \ \mu^2 x_i\ \ \lambda x_i\ \ \mu x_i\ \ x_i]^T;\ i := 1 : 2$$

Each operator matrices Δ_i, $i := 0 : 2$ is defined as follows

$$\Delta_0 = L_1^{(1)} \otimes L_2^{(2)} - L_2^{(1)} \otimes L_1^{(2)} \tag{25}$$

$$\Delta_1 = L_2^{(1)} \otimes L_0^{(2)} - L_0^{(1)} \otimes L_2^{(2)};\ \Delta_2 = L_0^{(1)} \otimes L_1^{(2)} - L_1^{(1)} \otimes L_0^{(2)} \tag{26}$$

The system (24) is referred to as singular or nonsingular according to the operator matrix Δ_0 specified in equation (25) is singular or nonsingular. The operator matrices $\Gamma_i = \Delta_0^{-1}\Delta_i$, $i := 1 : 2$ commute for nonsingular problem and the eigenvalues of (4) agree with the eigenvalues of joint GEPs of the type (24). Using the conventional numerical method for GEPs [19], we can find the numerical solution for nonsingular problems using this relation. However, solving the problem with low-order matrices is more convenient. The major computational drawbacks are the cost of computing the operator matrices Δ_i, $i = 0 : 2$ of size $36n \times 36n$. Thus, it is necessary to adopt numerical algorithm to find the solution of the problem. Again, the matrices in the linearized version of the problem are not of full rank and therefore, the operator determinant Δ_0 is singular having rank $20n^2$. Such a problem cannot be transformed into the joint GEP system of the kind (24). For the singular case, there are infinitely many eigenvalues that satisfy the equivalent systems of joint GEPs of the type (24), which makes computing appropriate eigenvalues of the problem challenging. The relationship between equations (4) and the joint GEP specified in equations (24) is less investigated for singular case. In the extant literature, there are numerical techniques to find the some part of the numeric of singular 2PEP [2, 15, 19-21] and the references therein. It is easy to verify that $x_1 \otimes x_2$ is an eigenvector corresponding to an eigenvalue (λ, μ) of a CTEP if and only if $w_1 \otimes w_2$ is an eigenvector corresponding to the eigenvalue (λ, μ) of the linearization.

Theorem 4.2 Let CTEP defined in (3). A class of linearizations of (3) is given by

$$L_i(\lambda,\ \mu)w_i := (L_0^{(i)} + \lambda L_1^{(i)} + \mu L_2^{(i)})w_i = 0;\ i := 1:2$$

where $w_i = \Lambda \otimes x_i$ and $L_i^{(i)}$ are defined as,

$$L_0^{(i)} = \begin{bmatrix} Z_{11}^{(i)} & Z_{12}^{(i)} & Z_{13}^{(i)} & Z_{14}^{(i)} & Z_{15}^{(i)} & \alpha_i P_{i0} \\ Z_{21}^{(i)} & Z_{22}^{(i)} & Z_{23}^{(i)} & Z_{24}^{(i)} & Z_{25}^{(i)} & 0 \\ 0 & Z_{32}^{(i)} & Z_{33}^{(i)} & Z_{34}^{(i)} & Z_{35}^{(i)} & 0 \\ 0 & 0 & Z_{43}^{(i)} & Z_{44}^{(i)} & Z_{45}^{(i)} & 0 \\ 0 & 0 & 0 & Z_{54}^{(i)} & Z_{55}^{(i)} & 0 \\ 0 & 0 & 0 & 0 & Z_{65}^{(i)} & 0 \end{bmatrix} \tag{27}$$

$$L_1^{(i)} = \begin{bmatrix} \alpha_i P_{i9} & -Y_{11}^{(i)} + \alpha_i P_{i8} & -Y_{12}^{(i)} + \alpha_i P_{i7} & -Z_{11}^{(i)} + \alpha_i P_{i5} & -Z_{12}^{(i)} + \alpha_i P_{i4} & -Z_{14}^{(i)} + \alpha_i P_{i2} \\ 0 & 0 & -Y_{22}^{(i)} & -Z_{21}^{(i)} & -Z_{22}^{(i)} & -Z_{24}^{(i)} \\ 0 & 0 & 0 & 0 & -Z_{32}^{(i)} & -Z_{34}^{(i)} \\ 0 & 0 & 0 & 0 & 0 & -Z_{44}^{(i)} \\ 0 & 0 & 0 & 0 & 0 & -Z_{54}^{(i)} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{28}$$

$$L_2^{(i)} = \begin{bmatrix} Y_{11}^{(i)} & Y_{12}^{(i)} & \alpha_i P_{i6} & 0 & -Z_{13}^{(i)} + \alpha_i P_{i3} & -Z_{15}^{(i)} + \alpha_i P_{i1} \\ 0 & Y_{22}^{(i)} & 0 & 0 & -Z_{23}^{(i)} & -Z_{25}^{(i)} \\ 0 & 0 & 0 & 0 & -Z_{33}^{(i)} & -Z_{35}^{(i)} \\ 0 & 0 & 0 & 0 & -Z_{43}^{(i)} & -Z_{45}^{(i)} \\ 0 & 0 & 0 & 0 & 0 & -Z_{55}^{(i)} \\ 0 & 0 & 0 & 0 & 0 & -Z_{65}^{(i)} \end{bmatrix} \tag{29}$$

Proof. The proof of the theorem follows from the Theorem 3.6, by considering $L_i(\lambda, \mu)w_i := (L_0^{(i)} + \lambda L_1^{(i)} + \mu L_2^{(i)})w_i = 0$; $i := 1 : 2$ of $E_i(\lambda, \mu)$ associated with ansatz vector $0 \neq \alpha_i e_1 \in C^6$; $i := 1 : 2$. □

Theorem 4.3 The linearization of CTEP derived from the equations (3) are singular.

Proof. Consider the linearizations defined by the equation (4) i.e.,

$$L_i(\lambda,\ \mu)w_i := (L_0^{(i)} + \lambda L_1^{(i)} + \mu L_2^{(i)})w_i = 0;\ i := 1:2$$

where $L_i^{(i)}$ are as defined in (27), (28) and (29) for $i = 0 : 2$; $j = 1, 2$.

Then the operator matrix Δ_0 is given by (25) as follows,

$$\Delta_0 = \begin{bmatrix} D_{11}^1 & D_{12}^1 & D_{13}^1 & D_{14}^1 & D_{15}^1 & D_{16}^1 \\ 0 & 0 & 0 & -Z_{21}^{(1)} \otimes L_2^{(2)} & -Z_{22}^{(1)} \otimes L_2^{(2)} & -Z_{24}^{(1)} \otimes L_2^{(2)} \\ 0 & 0 & 0 & 0 & -Z_{32}^{(1)} \otimes L_2^{(2)} & -Z_{34}^{(1)} \otimes L_2^{(2)} \\ 0 & 0 & 0 & 0 & 0 & -Z_{44}^{(1)} \otimes L_2^{(2)} \\ 0 & 0 & 0 & 0 & 0 & -Z_{54}^{(1)} \otimes L_2^{(2)} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} D_{11}^2 & D_{11}^2 & D_{13}^2 & 0 & D_{15}^2 & D_{16}^2 \\ 0 & D_{22}^2 & 0 & 0 & -Z_{23}^{(1)} \otimes L_1^{(2)} & -Z_{25}^{(1)} \otimes L_1^{(2)} \\ 0 & 0 & 0 & 0 & -Z_{33}^{(1)} \otimes L_1^{(2)} & -Z_{35}^{(1)} \otimes L_1^{(2)} \\ 0 & 0 & 0 & 0 & -Z_{43}^{(1)} \otimes L_1^{(2)} & -Z_{45}^{(1)} \otimes L_1^{(2)} \\ 0 & 0 & 0 & 0 & 0 & -Z_{55}^{(1)} \otimes L_1^{(2)} \\ 0 & 0 & 0 & 0 & 0 & -Z_{65}^{(1)} \otimes L_1^{(2)} \end{bmatrix}.$$

where,

$$D_{11}^{1} = \alpha_1 P_{19} \otimes L_2^{(2)},$$

$$D_{12}^{1} = (-Y_{11}^{(1)} + \alpha_1 P_{18}) \otimes L_2^{(2)},$$

$$D_{13}^{1} = (-Y_{12}^{(1)} + \alpha_1 P_7) \otimes L_2^{(2)},$$

$$D_{14}^{1} = -Z_{11}^{(1)} + \alpha_1 P_{15} \otimes L_2^{(2)},$$

$$D_{15}^{1} = (-Z_{12}^{(1)} + \alpha_1 P_{14}) \otimes L_2^{(2)},$$

$$D_{16}^{1} = (-Z_{14}^{(1)} + \alpha_1 P_{12}) \otimes L_2^{(2)},$$

$$D_{23}^{1} = Y_{22}^{(1)} \otimes L_2^{(2)}.$$

$$D_{11}^{2} = Y_{11}^{(1)} \otimes L_1^{(2)},$$

$$D_{12}^{2} = Y_{12}^{(1)} \otimes L_1^{(2)},$$

$$D_{13}^{2} = \alpha_1 P_{16} \otimes L_1^{(2)},$$

$$D_{15}^{2} = (Z_{13}^{(1)} + \alpha_1 P_{13}) \otimes L_1^{(2)},$$

$$D_{16}^{2} = (Z_{15}^{(1)} + \alpha_1 P_{11}) \otimes L_1^{(2)},$$

$$D_{22}^{2} = Y_{22}^{(1)} \otimes L_1^{(2)}.$$

More than one block on the diagonal of the block-triangular matrix Δ_0 is zero. Using Theorem 3.4 we have $\det(\Delta_0) = 0$. So the linearizations are singular and hence the theorem. □

5. Conclusion

We have described a unified framework on vector space of linearization of CTMP. Moreover, for a given CTMP, a vector space of linear 2PEPs has been constructed. A set of linearization classes are also identified which lie in the vector space. We also derived a class of singular linearizations for a CTEP. The linearization process discussed in this paper can be extended to *k*-parameter polynomial eigenvalue problem. However, the associated linear two-parameter forms yields matrices of larger structures. Therefore, the proofs for the aforementioned Lemmas and Theorems become challenging to work due to the involvement of large structures of the matrices. However, by suitably defined block matrices with the coefficient matrices of *k*-parameter matrix polynomials, the linearization framework can be made feasible to work and it can be considered as future avenue of research in this area.

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The authors declare no competing financial interest.

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