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Research Article

Families of Graceful Spiders with 3ℓ, $3\ell+2$ and $3\ell-1$ Legs

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Abstract: We say that a tree is a spider if has at most one vertex of degree greater than two. We prove the existence of families of graceful spiders with 3ℓ, $3\ell+2$ and $3\ell-1$ legs. We provide specific labels for each spider graph, these labels are constructed from graceful path graphs that have a particular label, so there is a correspondence between some paths and graceful spiders that we are studying, this correspondence is described in an algorithm outlined in the preliminaries.

***Keywords*:** graceful labelling, graph labeling, trees, spider

MSC: 05C78

1. Introduction

A graceful labeling f of a tree $T := (E(T), V(T))$, is a bijective function from the set of vertices $V(T)$ of T to the set $\{0, 1, 2, \cdots, |E(T)|\}$ such that the set $\{|f(u)-f(v)|: \{u, v\} \in E(T)\}$ is equal to $\{1, 2, \cdots, |E(T)|\}$, where $E(T)$ is the set of edges of T and $|E(T)|$ is its cardinality.

A tree T is graceful if there is some graceful labeling for T. In 1964, Ringel and Rosa [1, 2] proposed the famous and still unsolved *graceful tree conjecture*, which states that all trees are graceful.

A tree T is a *spider* if it has at most on branch point, that is, at most one vertex v such that its degree $d(v)$ satisfies $d(v) > 2$. Let v^* be the unique branch point of a spider T. It's worth mentioning that a certain type of spider graph, known as *sun graphs* have been studied from another perspective in [3].

Gallian in [4] observed that conjecture for the case of spider graphs is still open; regarding this, there are the following advances. Huang et al. [5] proved that all spider graphics with three or four legs are graceful. Poljak et al. and Bahls et al. [6, 7] also proved that every spider in which lengths of any two its legs differ by at most one is graceful. Jampachon et al. and Panpa et al. [8, 9] proved that spiders with three legs (four legs in [9]) of any length and arbitrary legs of length one are graceful.

The most important advances and results in an effort to prove the conjecture for spider graphs, are those mentioned in the previous paragraph. This is why any significant progress is crucial because each researcher contributes new techniques to tackle this problem, in that direction after studying some properties and applications of labeled graphs by Huamaní et al. [10]. In this paper, three new results are tested, which are in the Theorems 2, 3 and 4.

We will also give an alternative proof of Theorem 1 proved in [6, 7].

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2. Preliminaries

Definition 1 A path is a spider with only one leg. A caterpillar is a tree such that if one removes all of its leaves, the remaining graph is a path.

Rosa proved in [2] that all caterpillars are graceful. Since paths are also caterpillars, it will follow that paths are also graceful. From the proof for caterpillars, we can write the following lemma for paths.

Lemma 1 Every path L of length m is graceful.

Let's describe the label that make L graceful. For this, let us denote v^* one of the vertices of degree 1 from path L, and let us denote by v_j the vertex in L of distance j from v^*.

Let f be the labeling defined as follows:

i) $f(v^*) = 0$;

ii) if j is odd, $f(v_j) = m - \frac{j-1}{2}$;

iii) if j is even, $f(v_j) = \frac{j}{2}$.

where $j = 1, 2, \cdots, m$.

From the label f for paths of Lemma 1, it is possible to build graceful spider. We will describe this way of building with example in which an algorithm will be given, which can be extended to find several families of graceful spiders.

2.1 *Algorithm for the construction of a graceful labeling for a spider with legs of the same length*

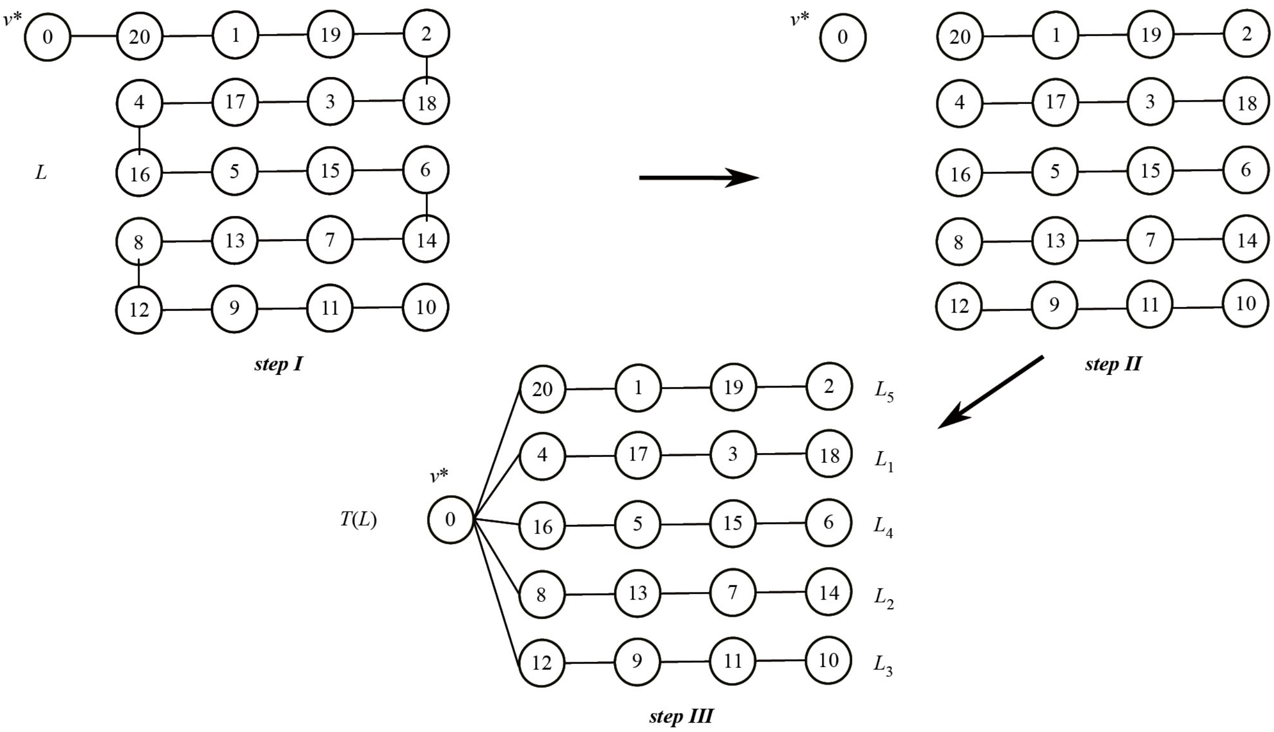

Figure 1. Steps of the construction of a graceful label, for a spider with $\ell = 5$ legs, all with length $m = 4$

Let T be a spider of ℓ legs all of length m, let us build a graceful label for this spider.

Step I: Let's build a path P from length ℓm, we have that this path P is graceful with the label f described in the Lemma 1.

Step II: Considering the label f from path L, all its edges are labeled with the set $\{1, 2, \cdots, \ell m\}$. Let's remove all edges that have label im where $i = 1, 2, \cdots, \ell$. Doing this we get $\ell + 1$ disjoint labeled paths, of which ℓ paths have length m and one path (from zero length) that vertex v^* with label 0.

Step III: From step II in each of the ℓ disjoint paths, one of the extreme vertices has label im and we will denote this path by L_i, where $i = 1, 2, \cdots, \ell$. Then, we connect with an edge the vertices of label im of L_i with the vertex v^* for each $i = 1, 2, \cdots, \ell$. When finished, we obtain an graceful spider graph that we will denote by $T(L)$ and this spider has ℓ legs each of length m. Figure 1 shows an example for $\ell = 5$ and $m = 4$.

Remark 1 Taking into account the previous algorithm, it is valid to ask:

Given a graceful path L with the Lemma label 1, What kind of graceful spiders can be obtained by removing edges from L and rejoining them with v^*, similar to the previous algorithm?

We give a partial answer to this question, for the moment, we find three families of graceful spiders, which are given in Theorems 1, 2, 3 and 4.

3. Main results

Theorem 1 Let T be a spider with ℓ legs, each of which has length m, for some $m > 1$. Then T is graceful.

Proof. Since ℓ is the number of legs of length m. Note that T has $n+1 = \ell m + 1$ vertices, to be labeled by the set $\{0, 1, 2, \cdots, n\}$. Label the legs by $L_1, L_2, \cdots, L_\ell$ each of length m. Let v^* denote the branch point of T and denote by $v_{i,\,j}$ the vertex en L_i of distance j from v^*.

Let ψ be the labeling defined as follows:

I) $\psi(v^*) = 0$;

II) if i is any and j is odd,

$$\psi(v_{i,\,j}) = im - \frac{j-1}{2};$$

III) if i is any and j is even,

$$\psi(v_{i,\,j}) = (\ell - i)m + \frac{j}{2}.$$

To help compute the edge labels, we note that the local maxima of ψ occur at $v_{i,\,j}$ for which i y j have the same parity, that is,

$$i \equiv \begin{cases} \left[j + \frac{(-1)^{i+1}+1}{2}\right](\mathrm{mod}\,2), & \text{if } i \leq \left\lfloor \frac{\ell}{2} \right\rfloor \\ \left[j + \frac{(-1)^{i}+1}{2}\right](\mathrm{mod}\,2), & \text{if } i > \left\lfloor \frac{\ell}{2} \right\rfloor, \end{cases}$$

For such i and j, with $i \leq \left\lfloor \frac{\ell}{2} \right\rfloor$, we have

$$\psi(v_{i,\,j}) - \psi(v_{i,\,j+1}) = \ell m - 2mi + j > 0, \tag{1}$$

$$\psi(v_{i,\,j}) - \psi(v_{i,\,j-1}) = \ell m - 2mi + j - 1 > 0, \tag{2}$$

and for $i > \left\lfloor \frac{\ell}{2} \right\rfloor$ we have

$$\psi(v_{i,\,j}) - \psi(v_{i,\,j+1}) = 2mi - \ell m - j > 0, \tag{3}$$

$$\psi(v_{i,\,j}) - \psi(v_{i,\,j-1}) = 2mi - \ell m - j + 1 > 0. \tag{4}$$

Suppose, to obtain a contradiction, there are two distinct edges that share the same label. By considering the indexes of the vertices at both ends end of these edges, we see that we can choose distinct pairs of indexes $(i,\ j)$ and $(i',\ j')$ such that i and j have the same parity, i' and j' likewise have the same parity, and a edge incident on $v_{i,\,j}$ shares the same label as a different edge incident on $v_{i',\,j'}$, that is, one of these three cases occur:

$$\psi(v_{i,\,j}) - \psi(v_{i,\,j+1}) = \psi(v_{i',\,j'}) - \psi(v_{i',\,j'+1}), \tag{5}$$

$$\psi(v_{i,\,j}) - \psi(v_{i,\,j+1}) = \psi(v_{i',\,j'}) - \psi(v_{i',\,j'-1}), \tag{6}$$

$$\psi(v_{i,\,j}) - \psi(v_{i,\,j-1}) = \psi(v_{i',\,j'}) - \psi(v_{i',\,j'-1}). \tag{7}$$

When $i,\, i' \leq \left\lfloor \frac{\ell}{2} \right\rfloor$ or $i,\, i' > \left\lfloor \frac{\ell}{2} \right\rfloor$ or $i \leq \left\lfloor \frac{\ell}{2} \right\rfloor,\, i' > \left\lfloor \frac{\ell}{2} \right\rfloor$.

Consider first the case where (5) and $i,\, i' \leq \left\lfloor \frac{\ell}{2} \right\rfloor$ hold. From (1), we obtain $2m(i - i') + (j' - j) = 0$, which shows that $j \neq j'$, since otherwise $i = i'$ as well, contrary to the assumption that $(i,\ j) \neq (i',\ j')$. We therefore can write

$$2m = \frac{j - j'}{i - i'}.$$

Thus $|i - i'| \geq 1$ and $|j - j'| \leq m - 1,$ and

$$2m = \frac{|j - j'|}{|i - i'|} \leq \frac{m-1}{1} = m - 1,$$

a contradiction.

Similar contradictions arise when (5), (6), (7) and $i,\, i' \leq \left\lfloor \frac{\ell}{2} \right\rfloor$ or $i,\, i' > \left\lfloor \frac{\ell}{2} \right\rfloor$ or $i \leq \left\lfloor \frac{\ell}{2} \right\rfloor,\, i' > \left\lfloor \frac{\ell}{2} \right\rfloor$ hold. Thus, no two distinct edges bear the same label, and ψ is graceful. □

The labeling ψ place 0 at the center of the spider and notice that the difference between the labels at $v_{i,\,j}$ and $v_{i+1,\,j}$ are multiples of m. This is illustrated in the Figure 2, where $\ell = 5$ and $m = 4$.

The general idea of this proof was adopted from the paper [6], and a similar approach was addressed in [11].

Theorem 2 Let T be a spider with 3ℓ legs, where ℓ legs have a length of $2m+1$ and 2ℓ legs have a length of $m+1$, for $\ell,\, m > 1$. Thus, T is graceful.

Proof. Let ℓ be the number of legs, where each has a length of $2m+1$ and 2ℓ legs have a length of $m+1$. Note that T has $n+1 = \ell(4m+3)+1$ vertices, to be labeled by the set $\{0,\ 1,\ 2,\ \cdots,\ n\}$. Label the legs by $L_1,\, L_2,\, \cdots,\, L_{3\ell}$, where the legs $L_{3\left\lfloor \frac{\ell}{2} \right\rfloor - 3p - 1}$ and $L_{3\left\lfloor \frac{\ell}{2} \right\rfloor + 3p + 3}$ have length $2m+1$, for $p = 0,\ 1,\ 2,\ \cdots,\ \ell - \left\lfloor \frac{\ell}{2} \right\rfloor - 1$; the rest of the legs have length $m+1$. Let v^* denote the branch point of T and denote $v_{i,\,j}$ the vertex in L_i of distance j from v^*.

Let φ the labeling defined as follows:

I) $\varphi(v^*) = 0$;

II) if $i \leq 3\ell - 3\left\lfloor \frac{\ell}{2} \right\rfloor$, $i \equiv 1(\mathrm{mod}\,3)$ or $i \equiv 2(\mathrm{mod}\,3)$ and j is odd. Also, if $i > 3\ell - 3\left\lfloor \frac{\ell}{2} \right\rfloor$, $i \equiv 2(\mathrm{mod}\,3)$ and j is odd.

$$\varphi(v_{i,\,j}) = i(m+1) + \left\lfloor \frac{i}{3} \right\rfloor m + \frac{j-1}{2}; \tag{8}$$

III) if $i \leq 3\ell - 3\left\lfloor \frac{\ell}{2} \right\rfloor$, $i \equiv 1(\mathrm{mod}\,3)$ or $i \equiv 2(\mathrm{mod}\,3)$ and j is even. Also, if $i > 3\ell - 3\left\lfloor \frac{\ell}{2} \right\rfloor$, $i \equiv 2(\mathrm{mod}\,3)$ and j is even.

$$\varphi(v_{i,\,j}) = \ell(4m+3) - i(m+1) - \left\lfloor \frac{i}{3} \right\rfloor m - \frac{j-2}{2};$$

IV) if $i \leq 3\ell - 3\left\lfloor \frac{\ell}{2} \right\rfloor$, $i \equiv 0(\mathrm{mod}\,3)$ and j is odd. Also, if $i > 3\ell - 3\left\lfloor \frac{\ell}{2} \right\rfloor$, $i \equiv 0(\mathrm{mod}\,3)$ or $i \equiv 1(\mathrm{mod}\,3)$ and j is odd.

$$\varphi(v_{i,\,j}) = i(m+1) + \left\lfloor \frac{i}{3} \right\rfloor m - \frac{j-1}{2};$$

V) if $i \leq 3\ell - 3\left\lfloor \frac{\ell}{2} \right\rfloor$, $i \equiv 0(\mathrm{mod}\,3)$ and j is even. Also, if $i > 3\ell - 3\left\lfloor \frac{\ell}{2} \right\rfloor$, $i \equiv 0(\mathrm{mod}\,3)$ or $i \equiv 1(\mathrm{mod}\,3)$ and j is even.

$$\varphi(v_{i,\,j}) = \ell(4m+3) - i(m+1) - \left\lfloor \frac{i}{3} \right\rfloor m + \frac{j}{2}.$$

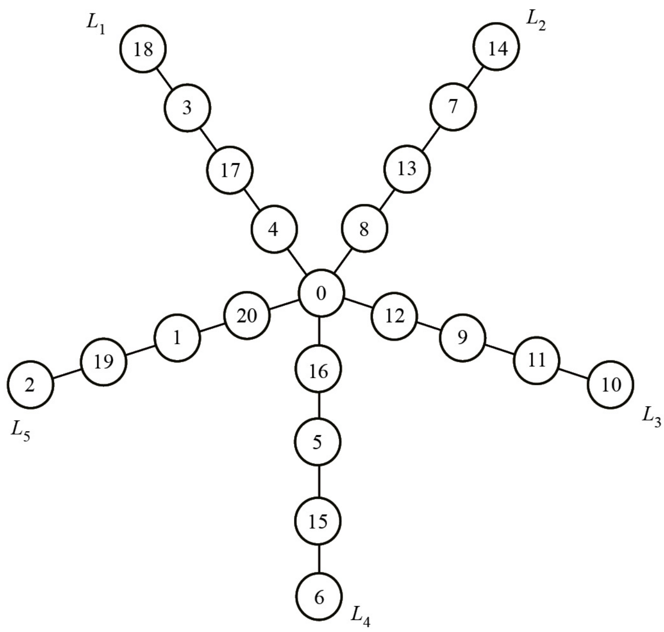

Figure 2. The labeling ψ for $\ell = 5$ and $m = 4$

The proof of this theorem will be done following the steps of the proof of Theorem (1). Thus, to help compute the edge labels, note that the local maximum of φ occurs at $v_{i,\, j}$ for which i and j have the same parity, that is,

$$i \equiv \begin{cases} \left[j+\frac{(-1)^{i+1}+1}{2}\right](\bmod 2), & \text{if } i \leq \left\lfloor \frac{3\ell}{2} \right\rfloor \\ \\ \left[j+\frac{(-1)^{i}+1}{2}\right](\bmod 2), & \text{if } i > \left\lfloor \frac{3\ell}{2} \right\rfloor \end{cases} \tag{9}$$

Now let's calculate all the differences between the labels of vertex pairs $v_{i,\, j}$; $v_{i,\, j+1}$ and $v_{i,\, j}$; $v_{i,\, j-1}$ where $v_{i,\, j}$ is the local maximum of φ. Thus:

$$\varphi(v_{i,\, j}) - \varphi(v_{i,\, j+1}) > 0 \quad \text{and} \quad \varphi(v_{i,\, j}) - \varphi(v_{i,\, j-1}) > 0. \tag{10}$$

Next, for i and j with the same parity, we list all possible cases where the equations in 10 hold for φ. We will only provide conditions for the "i" since "j" is determined by their parity.

For $i \leq 3\ell - 3\left\lfloor \frac{\ell}{2} \right\rfloor$, $i \equiv 1(\bmod 3)$ or $i \equiv 2(\bmod 3)$ we have:

$$\varphi(v_{i,\, j}) - \varphi(v_{i,\, j+1}) = \ell(4m+3) - 2i(m+1) - 2m\left\lfloor \frac{i}{3} \right\rfloor - j + 1 > 0, \tag{11}$$

$$\varphi(v_{i,\, j}) - \varphi(v_{i,\, j-1}) = \ell(4m+3) - 2i(m+1) - 2m\left\lfloor \frac{i}{3} \right\rfloor - j + 2 > 0. \tag{12}$$

For $i \leq 3\ell - 3\left\lfloor \frac{\ell}{2} \right\rfloor$, $i \equiv 0(\bmod 3)$ we have:

$$\varphi(v_{i,\, j}) - \varphi(v_{i,\, j+1}) = \ell(4m+3) - 2i(m+1) - 2m\left\lfloor \frac{i}{3} \right\rfloor + j > 0, \tag{13}$$

$$\varphi(v_{i,\, j}) - \varphi(v_{i,\, j-1}) = \ell(4m+3) - 2i(m+1) - 2m\left\lfloor \frac{i}{3} \right\rfloor + j - 1 > 0. \tag{14}$$

For $i > 3\ell - 3\left\lfloor \frac{\ell}{2} \right\rfloor$, $i \equiv 2(\bmod 3)$ we have:

$$\varphi(v_{i,\, j}) - \varphi(v_{i,\, j+1}) = 2i(m+1) + 2m\left\lfloor \frac{i}{3} \right\rfloor - \ell(4m+3) + j - 1 > 0, \tag{15}$$

$$\varphi(v_{i,\, j}) - \varphi(v_{i,\, j-1}) = 2i(m+1) + 2m\left\lfloor \frac{i}{3} \right\rfloor - \ell(4m+3) + j - 2 > 0. \tag{16}$$

For $i > 3\ell - 3\left\lfloor \frac{\ell}{2} \right\rfloor$, $i \equiv 0(\bmod 3)$ or $i \equiv 1(\bmod 3)$ we have:

$$\varphi(v_{i,\,j})-\varphi(v_{i,\,j+1})=2i(m+1)+2m\left\lfloor\frac{i}{3}\right\rfloor-\ell(4m+3)-j>0, \tag{17}$$

$$\varphi(v_{i,\,j})-\varphi(v_{i,\,j-1})=2i(m+1)+2m\left\lfloor\frac{i}{3}\right\rfloor-\ell(4m+3)-j+1>0. \tag{18}$$

For $\left\lfloor\frac{3\ell}{2}\right\rfloor<i\leq 3\ell-3\left\lfloor\frac{\ell}{2}\right\rfloor$, $i\equiv 2(\mathrm{mod}\,3)$ we have:

$$\varphi(v_{i,\,j})-\varphi(v_{i,\,j+1})=2i(m+1)+2m\left\lfloor\frac{i}{3}\right\rfloor-\ell(4m+3)+j-1>0, \tag{19}$$

$$\varphi(v_{i,\,j})-\varphi(v_{i,\,j-1})=2i(m+1)+2m\left\lfloor\frac{i}{3}\right\rfloor-\ell(4m+3)+j-2>0. \tag{20}$$

And for $\left\lfloor\frac{3\ell}{2}\right\rfloor<i\leq 3\ell-3\left\lfloor\frac{\ell}{2}\right\rfloor$, $i\equiv 0(\mathrm{mod}\,3)$ we have:

$$\varphi(v_{i,\,j})-\varphi(v_{i,\,j+1})=2i(m+1)+2m\left\lfloor\frac{i}{3}\right\rfloor-\ell(4m+3)-j>0, \tag{21}$$

$$\varphi(v_{i,\,j})-\varphi(v_{i,\,j-1})=2i(m+1)+2m\left\lfloor\frac{i}{3}\right\rfloor-\ell(4m+3)-j+1>0. \tag{22}$$

Remark 2 The equations from 11 to 22 provides us the labels on the edges of graph T, which must not be repeated (definition of a graceful graph), and to prove this, we will follow the following reasoning.

Suppose, to obtain a contradiction, that there are distinct edges that share the same label. By considering the indexes of the vertices at both ends end of these edges, we see that we can choose distinct pairs of indexes $(i,\,j)$ and $(i',\,j')$ such that i and j have the same parity, i' and j' likewise have the same parity, and an edge incident on $v_{i,\,j}$ shares the same label as a different edge incident on $v_{i',\,j'}$, that is, one of these three cases occur:

$$\varphi(v_{i,\,j})-\varphi(v_{i,\,j+1})=\varphi(v_{i',\,j'})-\varphi(v_{i',\,j'+1}), \tag{23}$$

$$\varphi(v_{i,\,j})-\varphi(v_{i,\,j+1})=\varphi(v_{i',\,j'})-\varphi(v_{i',\,j'-1}), \tag{24}$$

$$\varphi(v_{i,\,j})-\varphi(v_{i,\,j-1})=\varphi(v_{i',\,j'})-\varphi(v_{i',\,j'-1}), \tag{25}$$

when $i,\,i'\leq 3\ell-3\left\lfloor\frac{\ell}{2}\right\rfloor$ or $i,\,i'>3\ell-3\left\lfloor\frac{\ell}{2}\right\rfloor$ or $i\leq 3\ell-3\left\lfloor\frac{\ell}{2}\right\rfloor$, $i'>3\ell-3\left\lfloor\frac{\ell}{2}\right\rfloor$.

Consider the case where (23) and $i,\,i'\leq 3\ell-3\left\lfloor\frac{\ell}{2}\right\rfloor$ hold. From (11), we obtain $2(m+1)(i-i')+2m\left(\left\lfloor\frac{i}{3}\right\rfloor-\left\lfloor\frac{i'}{3}\right\rfloor\right)+(j-j')=0$, which shows that $j\neq j'$, since it is contrary to the assumption that $(i,\,j)\neq(i',\,j')$. Consequently, we can write

$$2(m+1)+2m\left(\frac{\left\lfloor \frac{i}{3}\right\rfloor-\left\lfloor \frac{i'}{3}\right\rfloor}{i-i'}\right)=\frac{j'-j}{i-i'}.$$

Thus $|i-i'|\geq 1$ and $|j-j'|<2m,$ and $0\leq \dfrac{\left\lfloor \frac{i}{3}\right\rfloor-\left\lfloor \frac{i'}{3}\right\rfloor}{i-i'}<1$, and

$$2m+2\leq\left|2(m+1)+2m\left(\frac{\left\lfloor \frac{i}{3}\right\rfloor-\left\lfloor \frac{i'}{3}\right\rfloor}{i-i'}\right)\right|=\left|\frac{j'-j}{i-i'}\right|<2m,$$

a contradiction.

Similar contradictions arise when (23), (24), (25) and $i,\ i'\leq 3\ell-3\left\lfloor\frac{\ell}{2}\right\rfloor$ or $i,\ i'>3\ell-3\left\lfloor\frac{\ell}{2}\right\rfloor$ or $i\leq 3\ell-3\left\lfloor\frac{\ell}{2}\right\rfloor$, $i'>3\ell-3\left\lfloor\frac{\ell}{2}\right\rfloor$ hold. Thus, no two distinct edges bear the same labels, and φ is graceful. □

The labeling φ place 0 at the center of the spider and notice that the difference between the vertex labels $v_{i,\,j}$ and $v_{i,\,j+1}$ are multiples of $m+1$ or $2m+1$.

Example 1 Considering $\ell=2$ and $m=2$ in Theorem 2, we will construct a graceful label for a spider T with 6 legs, where 2 legs have length 5 and 4 legs have length 3.

Following the proof of the theorem, let us denote by L_i, $i=1,\ 2,\ \cdots,\ 6$ the six legs of T, where the legs of the form

$$L_{3\left\lfloor\frac{\ell}{2}\right\rfloor-3p-1}\quad\text{and}\quad L_{3\left\lfloor\frac{\ell}{2}\right\rfloor+3p+3}\tag{26}$$

have length $2m+1$ for $p=0,\ 1,\ 2,\ \cdots,\ \ell-\left\lfloor\frac{\ell}{2}\right\rfloor-1$. Substituting $\ell=2$, $m=2$ into the equation (26), we have

$$L_{2-3p}\quad\text{and}\quad L_{4+3p},\tag{27}$$

with $p=0$, so from the equation (27) we have that L_i, $i=2,\ 4$ have length 5 and according to the proof, the remaining legs L_i, $i=1,\ 3,\ 5,\ 6$ have length 3.

Now, let's calculate the labels $\varphi(v_{i,\,j})$ for the vertices of T, to do this, let's find the pairs $(i,\ j)$ that satisfy the conditions in *II*) of the proof, and then we evaluate them at $\varphi(v_{i,\,j})$.

Replacing $\ell=2$ in *II*) we obtain that:

$i\leq 3$, $i\equiv 1(\bmod 3)$ or $i\equiv 2(\bmod 3)$ and j is odd. Also, if $i>3$, $i\equiv 2(\bmod 3)$ and j is odd.

This implies that $i=1,\ 2,\ 5$ and j is odd. Therefore, we can calculate the labels of the vertices $v_{1,\,1}$, $v_{1,\,3}$ in L_1 and $v_{2,\,1}$, $v_{2,\,3}$, $v_{2,\,5}$ in L_2 and $v_{5,\,1}$, $v_{5,\,3}$ in L_5. Replacing these vertices into equation (8), we have $\varphi(v_{1,\,1})=3$, $\varphi(v_{1,\,3})=4$; $\varphi(v_{2,\,1})=6$, $\varphi(v_{2,\,3})=7$, $\varphi(v_{2,\,5})=8$; $\varphi(v_{5,\,1})=17$, $\varphi(v_{5,\,3})=18$, we can see these labels highlighted in dark in Figure 3.

Thus, by calculating the remaining labels for the vertices of T. We can successfully construct a graceful labeling for T as shown in Figure 3.

Now, according Observation 2, let's calculate the differences that occur in the equations (11) to (22). Thus, to calculate the differences in equations (11) and (12), we must first calculate the pairs $(i,\ j)$, where i and j must have the same parity, that is, verify the equation (9) and "i" must satisfy:

$$i \leq 3\ell - 3\left\lfloor \frac{\ell}{2} \right\rfloor,\ i \equiv 1(\bmod 3)\ \text{ or } i \equiv 2(\bmod 3)$$

replacing $\ell = 2$ in this last equation, we have $i = 1, 2$.

Next, given $i = 1$ let's calculate the values of j such that i, j have the same parity. To do this, we replace $i = 1$ in equation (9), which give us

$$1 = (j+1)\bmod 2$$

then $j = 2$, therefore the pair $(1, 2)$ has the same parity. Continuing with $i = 2$, we have that $j = 2, 4$ and the pairs $(2, 2)$ and $(2, 4)$ have the same parity. Finally, let's substitute these (i, j) in equations (11) and (12). Thus, we obtain:

$$\varphi(v_{1,2}) - \varphi(v_{1,3}) = 15 \quad \text{and} \quad \varphi(v_{1,2}) - \varphi(v_{1,1}) = 17,$$

$$\varphi(v_{2,2}) - \varphi(v_{2,3}) = 1 \quad \text{and} \quad \varphi(v_{2,2}) - \varphi(v_{2,1}) = 10,$$

$$\varphi(v_{2,4}) - \varphi(v_{2,5}) = 7 \quad \text{and} \quad \varphi(v_{2,4}) - \varphi(v_{2,3}) = 8,$$

the remaining differences are calculated using the equations from 13 to 22. As we mentioned earlier, these differences represent the labels on the edges between the corresponding vertices, so that T is a graceful graph, see these labels in Figure 3.

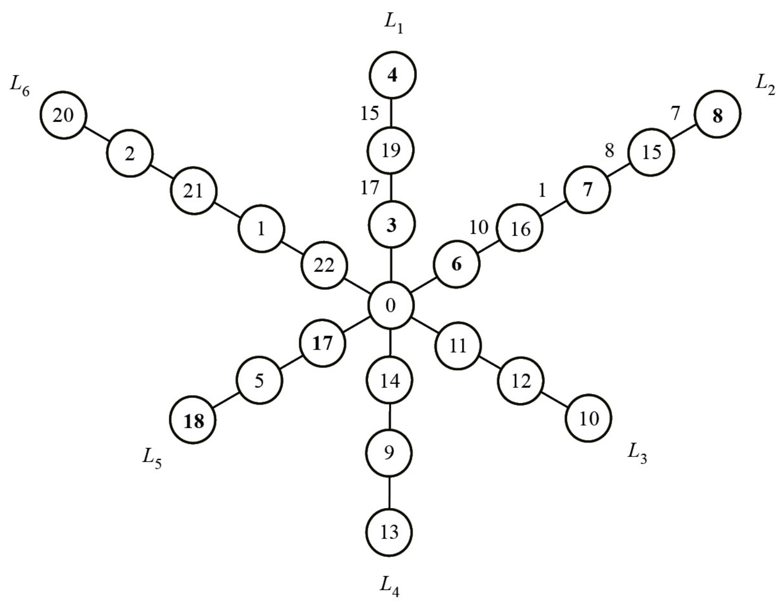

Figure 3. The labeling φ for $\ell = 2$ and $m = 2$

Theorem 3 Let T be a spider with $3\ell+2$ legs, where a leg has length $2m+1$, ℓ legs have length $2m+2$ and $2\ell+1$ legs have length $m+1$, for $\ell,\, m>1$. Then T is graceful.

Proof. Let $3\ell+2$ be the number of legs of T, where a leg has length $2m+1$, ℓ legs have length $2m+2$ and $2\ell+1$ legs have length $m+1$. Note that T has $n+1=4\ell m+4\ell+3m+3$ vertices, to be labeled by the set $\{0,\,1,\,2,\,\cdots,\,n\}$. Label the legs by $L_1,\,L_2,\,\cdots,\,L_{3\ell+2}$, where the legs $L_{1+3(k-1)}$ and $L_{2+3(p-1)}$ have length $m+1$, for $k=1,\,2,\,\cdots,\,\ell+1$ and $p=1,\,2,\,\cdots,\,\ell$; the legs L_{3k} have length $2m+2$, for $k=1,\,2,\,\cdots,\,\ell$ and the legs $L_{3\ell+2}$ have length $2m+1$. Let v^* denote the branch point of T and denote $v_{i,\,j}$ the vertex in L_i of distance j from v^*.

Let ϕ the labeling defined as follows:

I) $\phi(v^*)=0$;

II) if i is any, $i<3\ell+2$ and j is odd,

$$\phi(v_{i,\,j})=\left(i+\left\lceil\frac{i-1}{3}\right\rceil\right)(m+1)+\frac{j-1}{2};$$

III) if i is any, $i<3\ell+2$ and j is even,

$$\phi(v_{i,\,j})=4\ell m+4\ell+3m+2-\left(i+\left\lceil\frac{i-1}{3}\right\rceil\right)(m+1)-\frac{j-2}{2};$$

IV) if $i=3\ell+2$ and j is odd,

$$\phi(v_{i,\,j})=\left(i+\left\lceil\frac{i-1}{3}\right\rceil\right)(m+1)-\frac{j-1}{2}-1;$$

V) if $i=3\ell+2$ y j is even,

$$\phi(v_{i,\,j})=4\ell m+4\ell+3m+2-\left(i+\left\lceil\frac{i-1}{3}\right\rceil\right)(m+1)+\frac{j}{2}+1.$$

In this case, we note that the local maximum of φ occurs at $v_{i,\,j}$ for which i and j have the same parity, that is,

$$i\equiv\begin{cases}\left[j+\frac{(-1)^{i+1}+1}{2}\right](\mathrm{mod}\,2), & \text{if } i\leq\left\lfloor\frac{3\ell+2}{2}\right\rfloor\\[2ex] \left[j+\frac{(-1)^{i}+1}{2}\right](\mathrm{mod}\,2), & \text{if } i>\left\lfloor\frac{3\ell+2}{2}\right\rfloor,\end{cases}$$

the rest of the proof follows the same technique of the proofs of the Theorems (1) and (2). □

The labeling ϕ place 0 at the center of the spider and notice that the difference between the labels at $v_{i,\,j}$ and $v_{i+1,\,j}$, $i<3\ell-2$ are multiples of $m+1$. This is illustrated in the Figure 4, where $\ell=2$ and $m=2$.

Theorem 4 Let T be a spider with $3\ell-1$ legs, where ℓ legs have a length of $2m+1$ and $2\ell-1$ legs have a length of $m+1$, for $\ell,\,m>1$. Thus, T is graceful.

Proof. Let ℓ be the number of legs of T, where ℓ legs has length $2m+1$ and $2\ell-1$ legs have length $m+1$. Note that T has $n+1=\ell(2m+1)+(2\ell-1)(m+1)+1$ vertices, to be labeled by the set $\{0,\,1,\,2,\,\cdots,\,n\}$. Label the legs by $L_1,\,L_2,\,\cdots,\,L_{3\ell-1}$, where the legs $L_{B(q)}$ and $L_{H(r)}$ have length $m+1$; the legs $L_{N(p)}$ have length $2m+1$, where:

$$N(p) = 3p-1,\ p = 1, 2\cdots, \ell; \quad B(q) = 3q,\ q = 1, 2, \cdots, \ell-1; \quad H(r) = 3r-2,\ r = 1, 2, \cdots, \ell.$$

Let v^* denote the branch point of T and denote $v_{i,\,j}$ the vertex in L_i of distance j from v^*.
Let ζ the labeling defined as follows:
I) $\zeta(v^*) = 0$;
II) if $i = N(p)$ or $i = B(q)$ and j is odd.

$$\zeta(v_{i,\,j}) = \left(i + \left\lfloor \frac{i+1}{3} \right\rfloor\right) m + i - \frac{j-1}{2};$$

III) if $i = N(p)$ or $i = B(q)$ and j is even.

$$\zeta(v_{i,\,j}) = 4\ell m + 3\ell - m - 1 - \left(i + \left\lfloor \frac{i+1}{3} \right\rfloor\right) m - i + \frac{j}{2};$$

IV) if $i = H(r)$ and j is odd.

$$\zeta(v_{i,\,j}) = \left(i + \left\lfloor \frac{i+1}{3} \right\rfloor\right) m + i + \frac{j-1}{2};$$

V) if $i = H(r)$ and j is even.

$$\zeta(v_{i,\,j}) = 4\ell m + 3\ell - m - 1 - \left(i + \left\lfloor \frac{i+1}{3} \right\rfloor\right) m - i - \frac{j-2}{2}.$$

In this case, we note that the local maximum of ζ occurs at $v_{i,\,j}$ for which i and j have the same parity, that is,

$$i \equiv \begin{cases} \left[j + \frac{(-1)^{i+1}+1}{2}\right] (\mathrm{mod}\, 2), & \text{if } i \leq \left\lfloor \frac{3\ell-1}{2} \right\rfloor \\ \left[j + \frac{(-1)^{i}+1}{2}\right] (\mathrm{mod}\, 2), & \text{if } i > \left\lfloor \frac{3\ell-1}{2} \right\rfloor, \end{cases}$$

the rest of the proof follows the same technique of the proofs of the Theorems (1) and (2). An example is illustrated in the Figure 5, where $\ell = 3$ and $m = 2$. □

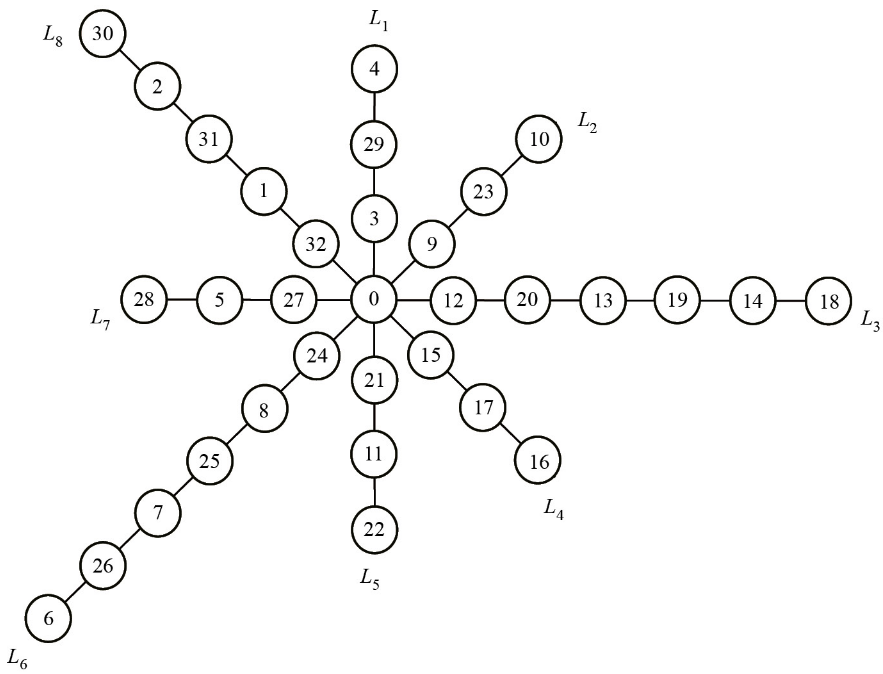

Figure 4. The labeling ϕ for $\ell = 2$ and $m = 2$

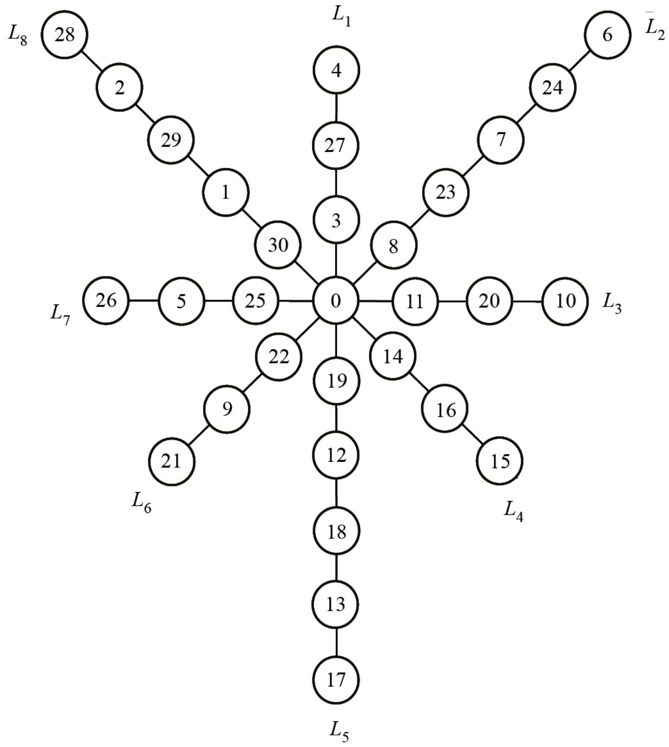

Figure 5. The labeling ζ for $\ell = 3$ and $m = 2$

4. Conclusions

In this paper, the graceful of families of spider graphs with 3ℓ legs, where ℓ and 2ℓ have lengths $2m+1$ and $m+1$ respectively, was demonstrated. Additionally, the graceful of families with $3\ell+2$ legs, where 1, ℓ and $2\ell+1$ legs have lengths $2m+2$, $m+1$ and $2m+1$ respectively. Lastly, graceful was established for the family with $3\ell-1$ legs, where ℓ and $2\ell-1$ have lengths $2m+1$ and $m+1$ respectively. These families represent new results in the literature, contributing to the efforts to prove the conjecture that spider graphs are graceful.

The basic strategy employed for constructing a graceful spider (or construct the graceful labeling function), is to start with a graceful path, from which we remove certain edges along with their labels. Then, we reconstruct these edges within the same graph in a way that results in a graceful spider.

In later works, it is intended to further generalize the families of graceful graphs that appear in Theorems 2, 3 and 4. As well as looking for new families of elegant spiders.

Confilict of interest

The authors declare no competing financial interest.

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