Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

A Fixed Point Theorem for Time-Fractional Fornberg-Whitham Equation Arising in Atmospheric Science

Ashok Kumar Badsara, Virendra Singh Chouhan*

Department of Mathematics and Statistics, Manipal University Jaipur, India
Email: virendrasingh.chouhan@jaipur.manipal.edu

Received: 29 June 2023; **Revised:** 22 December 2023; **Accepted:** 4 March 2024

Abstract: In this paper, we apply the Homotopy perturbation transform method of the time-fractional Fornberg-Whitham equation with Caputo-Fabrizio fractional derivative. We establish the existence and uniqueness solutions of the time-fractional Fornberg-Whitham equation using fixed point theory. The physical interpretation of the nonlinear models are also discussed through the semi analytical solutions, which demonstrate the effectiveness of the proposed method. Finally, 2D and 3D graphs of some derived solutions are depicted through suitable parameter values.

Keywords: the time-fractional fornberg-whitham equation, fractional homotopy perturbation transform method (FHPTM), caputo fabrizio derivative (CFD), stability analysis, fixed point, laplace transform

1. Introduction

The development of fractional calculus and the study of Fractional differential equations (FDEs) as a separate field of mathematics didn't happen until the late nineteenth and mid twentieth hundreds of years [1-2]. In the late nineteenth hundred years, the French mathematician Henri Liouville introduced the concept of fractional derivatives, integrals and published several papers on the topic. Around the same time, the German mathematician Felix Klein also made contributions to the development of fractional calculus. In the early 20th century, the Italian mathematician Umberto Grunwald and the Russian mathematician Mikhail Letnikov independently developed the Grunwald-Letnikov definition of fractional derivatives, which paved the way for the study of FDEs [3-4]. The same definition was independently rediscovered and developed further by the German mathematician Gustav Riesz. In the mid-20th century, the Italian mathematician Giuseppe Caputo introduced the Caputo Fabrizio fractional derivative, which became a widely used definition in the study of FDEs. Caputo's work also laid the foundation for the development of numerical methods for solving FDEs [5]. Since then, the study of FDEs has continued to evolve and has been used to solve a wide range of difficulties in various fields, including physics, engineering, finance, and biology. The theory of FDEs is still a functioning area of exploration, with continuous endeavors to foster new techniques for addressing these conditions, to understand the properties of solutions, furthermore, to track down new applications in different fields. The Time fractional differential equations (TFDEs) are mathematical models used to describe phenomena that exhibit memory or hereditary effects, where the current state of the system depends not just on the immediately preceding state yet in addition on the historical backdrop of the framework over some drawn out time period. These equations are characterized by derivatives of non-integer order, called fractional derivatives, which capture the memory effects

DOI: https://doi.org/10.37256/cm.5220243290

[6-7]. The TFDEs have found a wide range of applications in different fields, including physical science, designing, money, and science, among others [8-9]. For instance, they have been utilized to show the engendering of signs in correspondence frameworks, the diffusion of pollutants in porous media, and the pricing of financial derivatives. The hypothesis of time-FDEs is a relatively recent development, and there is still ongoing research in this area to understand the properties and behavior of solutions of these equations, as well as to develop numerical methods for solving them efficiently and accurately [10-12]. The Fornberg-Whitham equation (FWEs) is a numerical model used to describe the propagation of waves in a medium that is not uniform, such as in shallow water waves. It was first introduced by Bengt Fornberg and Gerald Whitham in the early 1970s. Fornberg and Whitham was interested in studying the behavior of waves in shallow water, which can be affected by the varying depth of the water. They developed the FWEs to describe the changes in wave amplitude and speed as the water depth varies [13-15]. Different methods have been devised to solve the fractional differential equations analytically and numerically since they show the memory features of the system. Its non-linearity trait makes it very difficult to find an exact solution, and the discrete point findings of numerical approaches do not provide consistency. To address this problem, a number of researchers have provided techniques, such as the sine Gordon expansion method [16-19], the Homotopy analysis method [20-21], the Bernoulli sub-equation function method [22], the generalized exponential rational function method [23], the He's variational iteration method [24-26], the Homotopy perturbation technique [27-29], the Chebyshev spectral collocation method [30], the rational sine-cosine and rational sinh-cosh methods [31], the conformable derivative [32], the timefractional Caputo derivative [33], the Atangana-Baleanu derivatives [34], the fractional residual power series method [35], the Residual Power Series Method [36], semi-analytic technique to deal with nonlinear fractional differential equations [37]. For more details we can refer to [38].

One of the important models is the time-fractional Fornberg-Whitham equation [13-14], as follows:

$$ {}_{0}^{CF}D_t^{\alpha}u - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_xu_{xx},\ t > 0,\ 0 < \alpha \leq 1. \tag{1}$$

here the nonlinear term uu_x in (1) to u^2u_x, He's et al. introduced in [24] the modified fractional Fornberg-Whitham equation.

$$ {}_{0}^{CF}D_t^{\alpha}u - u_{xxxt} + u_x = uu_{xxx} - u^2u_x + 3u_xu_{xx},\ t > 0,\ 0 < \alpha \leq 1. \tag{2}$$

where ${}_{0}^{CF}D_t^{\alpha}u$ Caputo Fabrizio fractional derivative of non-integer order of the function $u(x, t)$. If we set $\alpha = 1$, the fractional order FWEs is converted to the classical nonlinear FWEs. Homotopy perturbation transform method can solve a large class of linear and nonlinear functional equations analytically, semi-analytically, and numerically. By hybridizing homotopy perturbation transform method with general transform, it can get the same results with faster convergence and improved accuracy. To prove the efficiency of the proposed homotopy perturbation transform method, we demonstrate our technique to solve the the time-fractional Fornberg-Whitham equation as nonlinear FPDE.

The rest of the paper is organized as follows. In Section 2, we present basic results which are needed in the sequel. In Section 3, we introduce a of the homotopy perturbation transform method for solving nonlinear partial differential equations. In Section 4, we obtain existence and uniqueness of the solution to the time-fractional Fornberg-Whitham equation. Numerical examples are obtained in Section 5 to confirm the accuracy and effectiveness of the proposed technique. The last section contains the final thoughts.

2. Preliminaries

In this section, the author provides definitions of important terms and concepts related to the topic being studied [3-4].

Definition 2.1 If Caputo fractional derivative characterize for $\alpha \geq 0$ & $n \in N \cup 0$ is defined as below (see [10]).

$$ {}_{0}^{C}D_t^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\xi)\frac{d^n}{dt^n}u(\xi)\,d\xi. \tag{3}$$

where ${}_{0}^{C}D_{t}^{\alpha}$ Caputo fractional derivative and $t \geq 0$.

Definition 2.2 Caputo Fabrizio fractional derivative [10], where $u \in H^{1}(a_1, b_1)$, $b_1 > 0$, $0 < \alpha < 1$, defined as

$$ {}_{0}^{CF}D_{t}^{\alpha}u(t) = \frac{M(\alpha)}{(1-\alpha)}\int_{0}^{t} exp[-\frac{\alpha(1-\xi)}{1-\alpha}]u'(\xi)d\xi. \tag{4}$$

with a normalize functions $M(\alpha)$ which is depend on $\alpha \in M(0) = M(1) = 1$.

Definition 2.3 Let $0 < \alpha < 1$; for the Caputo Fabrizio fractional derivative sense, the fractional Integral of order α is given by [10],

$$ {}_{0}^{CF}D_{t}^{\alpha}u(t) = \frac{2(1-\alpha)}{M(\alpha)(2-\alpha)}u(t) + \frac{2\alpha}{M(\alpha)(2-\alpha)}\int_{0}^{t} u(\xi)d\xi. \tag{5}$$

Definition 2.4 If Laplace transform (LT) for order $0 < \alpha < 1$ and $m \in N$, Caputo Fabrizio fractional derivative given as [10],

$$\begin{aligned} \mathcal{L}\left[{}_{0}^{CF}D_{t}^{(m+\alpha)}u(t)\right](s) &= \frac{1}{1-\alpha}\mathcal{L}[u^{m+1}(t)]\mathcal{L}\left[exp\left(\frac{-\alpha}{(1-\alpha)}t\right)\right] \\ &= \frac{s^{m+1}\mathcal{L}[u(t)] - s^{m}u(0) - s^{m-1}u'(0)\ldots - u^{m}(0)}{s+\alpha(1-s)}. \end{aligned} \tag{6}$$

we get

$$\mathcal{L}\left[{}_{0}^{CF}D_{t}^{(m+\alpha)}u(t)\right](s) = \frac{s\mathcal{L}(u(t))}{s+\alpha(1-s)},\ m = 0,$$

$$\mathcal{L}\left[{}_{0}^{CF}D_{t}^{(m+\alpha)}u(t)\right](s) = \frac{s^{2}\mathcal{L}(u(t)) - su(o) - u'(0)}{s+\alpha(1-s)},\ m = 1.$$

3. Using HPTM to solve a broad time fractional differential equation in the sense of caputo fabrizio fractional derivative

If FHPTM is a numerical technique applied for approximate answer of nonlinear FDEs and other numerical issues. It is an iterative method that starts with an initial approximation and then perturbs it until a solution is found. We look at the following problem, which consists of a differential equation and a Caputo Fabrizio fractional derivative:

$$ {}_{0}^{CF}D_{t}^{(m+\alpha)}u(x, t) + \beta u(x, t) + \varphi u(x, t) = k(x, t),\ n-1 < \alpha + m \leq n, \tag{7}$$

for the initial conditions

$$\frac{\partial^{l}u(x,\ 0)}{\partial t^{l}} = f_{l}(x),\ l = 0,\ 1,\ 2,\ \ldots\ n-1. \tag{8}$$

when we apply the Laplace transform's derivative rule to equations (7-8), we get

$$\mathcal{L}[u(x,\ t)] = \Theta(x,\ s) - \left(\frac{s+\alpha(1-s)}{s^{n+1}}\right)\mathcal{L}[\beta u(x,\ t) + \varphi u(x,\ t)]. \tag{9}$$

here

$$\Theta(x, s) = \frac{1}{s^{m+1}}[s^m f_0(x) + s^{m-1} f_1(x) + \ldots + f_m(x)] + \frac{s + \alpha(1-s)}{s^{n+1}} \tilde{k}(x, s). \tag{10}$$

Utilizing the inverse Laplace transform on Eq. (9), we yield's

$$u(x, t) = \Theta(x, s) - \mathcal{L}^{-1}\left[\left(\frac{s + \alpha(1-s)}{s^{n+1}}\right)\mathcal{L}[\beta u(x, t) + \varphi u(x, t)]\right], \tag{11}$$

as a result of an infinite series

$$u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t). \tag{12}$$

for nonlinear term is decomposable like

$$\varphi u(x, t) = \sum_{n=0}^{\infty} p^n H_n(x, t). \tag{13}$$

H_n are He's polynomials that can be evaluated using the formula below [27].

$$H_m(u_0, u_1, u_2, \ldots u_n) = \frac{1}{n!}\frac{\partial^m}{\partial p^m}\left[\left(\sum_{m=0}^{\infty} p^i u_i\right)\right]_{p=0}, \ m = 0, 1, 2, \ldots; \tag{14}$$

we substitute (12) and (13) into (9), yields

$$\sum_{m=0}^{\infty} u_m(x, t) = \Theta(x, s) - p\mathcal{L}^{-1}\left[\left(\frac{s + \alpha(1-s)}{s^{m+1}}\right)\mathcal{L}\left[\beta \sum_{m=0}^{\infty} p^m u_m(x, t) + \sum_{m=0}^{\infty} p^m H_m\right]\right]. \tag{15}$$

we obtained following approximations by equating the terms with similar powers in p in Eq. (15)

$$\begin{aligned}
p^0 &: u_0(x, t) = \Theta(x, s),\\
p^1 &: u_1(x, t) = -\mathcal{L}^{-1}\left[\left(\frac{s + \alpha(1-s)}{s^{m+1}}\right)\mathcal{L}\left[\beta u_0(x, t) + H_0(u)\right]\right],\\
p^2 &: u_2(x, t) = -\mathcal{L}^{-1}\left[\left(\frac{s + \alpha(1-s)}{s^{m+1}}\right)\mathcal{L}[\beta u_1(x, t) + H_1(u)]\right],\\
&\vdots\\
p^{m+1} &: u_{m+1}(x, t) = -\mathcal{L}^{-1}\left[\left(\frac{s + \alpha(1-s)}{s^{m+1}}\right)\mathcal{L}[\beta u_{m+1}(x, t) + H_{m+1}(u)]\right].
\end{aligned}$$

Finally, we derive the semi-analytic answer as a truncated series of approximations as

$$u(x, t) = \sum_{m=0}^{\infty} P^n u_m(x, t). \tag{16}$$

Equation (16) represents a series solution, which converge very fast.

4. Analysis of existence and uniqueness for the timefractional fornberg-whitham equation

The fixed-point hypothesis is utilize to exhibit the presence and uniqueness of the answer for the given numerical model by means of the CFFD with no solitary part are important in this segment. Consider the time-fractional FWEs as follows:

$$ {}_{0}^{CF}D_t^{\alpha}u = uu_{xxx} - uu_x + 3u_x u_{xx} - u_x + u_{xxt},\ 0 \leq \alpha < 1,\ t > 0. \tag{17}$$

$$u(x, 0) = h(x).$$

eq. (17) is composed as yield's:

$$u(x, t) - u(x, 0) = I^{\alpha}[uu_{xxx} - uu_x + 3u_x u_{xx} - u_x + u_{xxt}]. \tag{18}$$

The following is how eq. (18) is changed into the Volterra integral equation:

$$\begin{aligned} u(x,\ t) - u(x,\ 0) = {} & \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left[uu_{xxx} - uu_x + 3u_x u_{xx} - u_x + u_{xxt}\right] \\ & + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left[uu_{xxx} - uu_x + 3u_x u_{xx} - u_x + u_{xxt}\right] d\tau. \end{aligned} \tag{19}$$

The objective is to demonstrate that Lipschitz condition (LC) is satisfied by the operator $\phi(x, t, u)$. In other words, we need to choose a constant that is positive L in such a way that

$$\|\phi(x, t, u) - \phi(x, t, v)\| \leq L\|u - v\|.$$

The operator (ϕ) meets the Lipschitz condition requirements. Let u and v are two constrained functions. Then, positive constants S and T exist in such a way $\|u\| \leq S$, $\|v\| \leq T$ now have.

$$\|\phi(x, t, u) - \phi(x, t, v)\| = \|(u_{xxt} - v_{xxt}) - (u_x - v_x) + (uu_{xxx} + 3u_x u_{xx} - vv_{xxx} - 3v_x v_{xx}) - (uu_x - vv_x)\|. \tag{20}$$

Using triangular inequality characteristics, the above becomes

$$\|\phi(x, t, u) - \phi(x, t, v)\| \leq \|(u - v)_{xxt}\| - \|(u - v)_x\| + \|\frac{1}{2}(u^2 - v^2)_{xxx}\| - \|\frac{1}{2}(u^2 - v^2)_x\|. \tag{21}$$

We can get $Q_1$and $Q_2 \in R^+$ using the Lipschitz of the function derivative.

$$\|(u - v)_x\| \leq Q_1\|(u - v)\|. \tag{22}$$

$$\|(u - v)_{xxt}\| \leq Q_1^2 Q_2\|(u - v)\|. \tag{23}$$

and

$$\begin{aligned} \|\frac{1}{2}(u^2 - v^2)_{xxx}\| & \leq \frac{1}{2}Q_1^3\|(u - v)(u + v)\|, \\ & \leq \frac{1}{2}Q_1^3\|(u - v)\|(\|u\|\|v\|), \\ & \leq \frac{1}{2}Q_1^3(S + T)\|(u - v)\|. \end{aligned} \tag{24}$$

by substituting Eqs. (22-23) and (24) into equation (21), we get

$$\| \phi(x, t, u) - \phi(x, t, v) \| \leq \left(Q_1^2 Q_2 + Q_1 + \frac{Q_1^2}{2}(S+T)) + \frac{Q_1}{2}(S+T) \right) \|(u-v)\|.$$

therefore

$$\|\phi(x, t, u) - \phi(x, t, v)\| \leq \mathrm{L}\|(u-v)\|.$$

where

$$L = Q_1^2 Q_2 + Q_1 + \frac{Q_1^2}{2}(S+T)) + \frac{Q_1}{2}(S+T)$$

This demonstrates the Lipschitz condition for ϕ.

Proposition 1 If the following conditions are met,

$$\frac{2L(1-\alpha)}{(2-\alpha)} + \frac{2Lt\alpha}{(2-\alpha)M(\alpha)} < 1. \tag{25}$$

If TFFWEs with beginning condition (1), for example, admits uniqueness and continuous solutions.

Proof. When we consider Eq. (18),

$$u(x, t) - u(x, 0) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left[uu_{xxx} - uu_x + 3u_x u_{xx} - u_x + u_{xxt}\right]$$

$$+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left[uu_{xxx} - uu_x + 3u_x u_{xx} - u_x + u_{xxt}\right] d\tau. \tag{26}$$

This implies the recurrence formula

$$u_0(x, t) = u(x, 0).$$

$$u_n(x, t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\phi(x, t, u_{n-1}) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \phi(x, t, u_{n-1}) d\tau. \tag{27}$$

for

$$\tilde{u}(x, t) = \lim_{n\to\infty} u_n(x, t). \tag{28}$$

$\tilde{u}(x, t) = u(x, t)$ solutions is now shown to be a continuous. Let us begin by establishing

$$U_n(x, t) = u_n(x, t) - u_{n-1}(x, t), \tag{29}$$

it is self-evident

$$u_n(x, t) = \sum_{m=0}^{n} U_m(x, t). \tag{30}$$

In addition, we have gone into much deeper detail.

$$U_n(x,\ t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left[\phi(x,\ t,\ u_{n-1}) - \phi(x,\ t,\ u_{n-2})\right] + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left[\phi(x,\ t,\ u_{n-1}) - \phi(x,\ t,\ u_{n-2})\right] d\tau \tag{31}$$

We obtain by applying the norm to both sides of Eq. (31) as well as the triangular inequality

$$\begin{aligned}
\| U_n(x,\ t)\| &= \| u_n(x,\ t) - u_{n-1}(x,\ t)\| \\
&\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\| \phi(x,\ t,\ u_{n-1}) - \phi(x,\ t,\ u_{n-2})\| + \frac{2\alpha}{(2-\alpha)M(\alpha)}\left\|\int_0^t \left[\phi(x,\ t,\ u_{n-1}) - \phi(x,\ t,\ u_{n-2})\right]\right\| d\tau \\
&\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\| \phi(x,\ t,\ u_{n-1}) - \phi(x,\ t,\ u_{n-2})\| + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \| \phi(x,\ t,\ u_{n-1}) - \phi(x,\ t,\ u_{n-2}) d\tau\|.
\end{aligned} \tag{32}$$

now

$$\| U_n(x,\ t)\| \le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L\| u_{n-1} - u_{n-2}\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L\int_0^t \| u_{n-1} - u_{n-2}\| \, d\tau. \tag{33}$$

which is the same as

$$\| U_n(x,\ t)\| \le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L\| U_{n-1}\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L\int_0^t \| U_{n-1}\| \, d\tau. \tag{34}$$

When the recursive theory is used to Eq.(34), the result is

$$\| U_n(x,\ t)\| \le \left[\left(\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\right)^n + \left(\frac{2\alpha Lt}{(2-\alpha)M(\alpha)}\right)^n\right] u(x,\ 0). \tag{35}$$

illustrated solution exists & is being implemented.

It demonstrates that

$$u(x,\ t) = \lim_{n\to\infty} u_n(x,\ t). \tag{36}$$

Let us consider solutions for Eq.(1).

$$Re_n(x,\ t) = \tilde{u}(x,\ t) - u_n(x,\ t) \text{ for } n \in N. \tag{37}$$

As a result of Eq (37), the differences $Re_n(x, t)$ amid $\tilde{u}(x, t)$ and $u_n(x, t)$ should converge to zero as $n \to \infty$, as shown below.

$$\tilde{u}(x,\ t) - u_n(x,\ t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left[\phi(x,\ t,\ u) - \phi(x,\ t,\ u_n)\right] + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left[\phi(x,\ t,\ u) - \phi(x,\ t,\ u_n)\right] d\tau. \tag{38}$$

now we have

$$\| \tilde{u}(x, t) - u_n(x, t)\| = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\| \phi(x, t, u) - \phi(x, t, u_n)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \| \phi(x, t, u) - \phi(x, t, u_n)\| \, d\tau,$$

$$\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\| u - u_n\| + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \| u - u_n\|,$$

$$\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\| \Re_n\| + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \| \Re_n\|. \tag{39}$$

Consequently, when $n \to \infty$, then $\Re_n \to 0$ while the RHS provides

$$\lim_{n\to\infty} u_n(x, t) = \tilde{u}(x, t). \tag{40}$$

We can proceed with the information shown above $u(x, t) = \tilde{u}(x, t)$, Because the answer to Eq. (1) is sustained,

$$u(x, t) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\phi(x, t, u) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \phi(x, t, u) d\tau$$

$$= \Re_n(x, t) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left[\phi(x, t, u_{n-1}) - \phi(x, t, u)\right] + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left[\phi(x, t, u_{n-1}) - \phi(x, t, u)\right] d\tau. \tag{41}$$

as a result of applying LC to ϕ,

$$\| u(x, t) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\phi(x, t, u) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \phi(x, t, u) d\tau\|$$

$$\le \| \Re_n(x, t)\| + \left[\frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)} + \frac{2\alpha Lt}{(2-\alpha)M(\alpha)}\right]\| \Re_{n-1}(x, t)\|. \tag{42}$$

while taking the limit $n \to \infty$ then the initial condition

$$u(x, t) = u(x, 0) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\phi(x, t, u) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \phi(x, t, u) d\tau \tag{43}$$

therefore, for the sake of essential character, assume u and v to be are two distinct solution to Eq. (1). The LC for ϕ then yields

$$\| u(x, t) - v(x, t)\| \le \frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)}\| u(x, t) - v(x, t)\| + \frac{2\alpha Lt}{(2-\alpha)M(\alpha)}\| u(x, t) - v(x, t)\|. \tag{44}$$

as a result of this,

$$\| u(x, t) - v(x, t)\| \left(1 - \frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)} - \frac{2\alpha Lt}{(2-\alpha)M(\alpha)}\right) \le 0. \tag{45}$$

therefore, $\|u(x, t) - v(x, t)\| = 0$ if

$$\frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)} + \frac{2\alpha Lt}{(2-\alpha)M(\alpha)} < 1. \tag{46}$$

Then the theorem is established.

5. Numerical examples

In this part, we show how the HPTM works by putting it through two starting worth issues connected with the first and adjusted time-fractional FWEs.

Example 1 For the time-fractional Fornberg-Whitham equation with the following initial condition [13].

$$u(x,\ 0) = \frac{4}{3}e^{\frac{1}{2}x}. \tag{47}$$

we get the LT on the two sides of Eqs. (1, 47)

$$\mathcal{L}[u(x,\ t)] = \frac{4}{3s}e^{\frac{1}{2}x} + \left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\left[uu_{xxx} - uu_x + 3u_x u_{xx} - u_x + u_{xxt}\right]. \tag{48}$$

using the inverse of (LT) in Eq. (48),

$$u(x,\ t) = \frac{4}{3}e^{\frac{1}{2}x} + \mathcal{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\left[uu_{xxx} - uu_x + 3u_x u_{xx} - u_x + u_{xxt}\right]\right]. \tag{49}$$

Furthermore, we denote the answer as an unending chain represented

$$u(x,\ t) = \sum_{m=0}^{\infty} u_m(x,\ t). \tag{50}$$

and non-linear period uu_{xxx}, uu_x and $u_x u_{xx}$ can be decomposed by using He's polynomials [27] as

$$\sum_{m=0}^{\infty} H_m(x,\ t) = uu_{xxx}. \tag{51}$$

We can without much of a stretch assess the He's polynomials' parts. The first few components of Hm are written as

$$\begin{aligned}
&H_0(u) = u_0 D_{xxx} u_0,\\
&H_1(u) = u_0 D_{xxx} u_1 + u_1 D_{xxx} u_0,\\
&H_2(u) = u_0 D_{xxx} u_2 + u_1 D_{xxx} u_1 + u_2 D_{xxx} u_0,\\
&\vdots
\end{aligned} \tag{52}$$

for H'_m we find that

$$\sum_{m=0}^{\infty} H'_m(x,\ t) = uu_x.$$

$$H'_0(u) = u_0 D_x u_0,$$

$$H'_1(u) = u_0 D_x u_1 + u_1 D_x u_0,$$

$$H'_2(u) = u_0 D_x u_2 + u_1 D_x u_1 + u_2 D_x u_0,$$

$$\vdots \tag{53}$$

for H''_m we find that

$$\sum_{m=0}^{\infty} H''_m(x,\ t) = uu_{xxx}.$$

$$H''_0(u) = D_x u_0 D_{xx} u_0,$$

$$H''_1(u) = D_x u_0 D_{xx} u_1 + D_x u_1 D_{xxx} u_0, \tag{54}$$

$$H''_2(u) = D_x u_0 D_{xx} u_2 + D_x u_1 D_{xxx} u_1 + D_x u_2 D_{xxx} u_0,$$

$$\vdots$$

using Eqs. (51)-(54) in Eq. (50) gives

$$\sum_{m=0}^{\infty} u_m(x,\ t) = \frac{4}{3} e^{\frac{1}{2}x} + p\mathcal{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\left[\left(\sum_{m=0}^{\infty} p^m u_m(x,\ t)\right)_{xxt} - \left(\sum_{m=0}^{\infty} p^m u_m(x,\ t)\right)_{x}\right]\right]$$

$$\times + \left[\sum_{m=0}^{\infty} p^m H_m(u) - \sum_{m=0}^{\infty} p^m H'_m(u) + 3\sum_{m=0}^{\infty} p^m H''_m(u)\right]. \tag{55}$$

when quotients of the equal powers of p are compared, as follows

$$p^0 : u_0(x,\ t) = \frac{4}{3} e^{\frac{1}{2}x},$$

$$p^1 : u_1(x,\ t) = -\frac{2}{3} e^{x/2}(1-\alpha+t\alpha),$$

$$p^2 : u_2(x,\ t) = -\frac{1}{6} e^{x/2}\left(1+2e^{x/2}\right)t^2\alpha^2 - \frac{1}{6} e^{x/2}(-1+\alpha)\left(-2-4e^{x/2}+\alpha+4e^{x/2}\alpha\right)$$

$$+\frac{1}{6} e^{x/2} t\alpha\left(-4-8e^{x/2}+3\alpha+8e^{x/2}\alpha\right),$$

$$\vdots$$

following the same procedure, the remaining iterates $u(x, t)$ can be calculated easily. Hence, the numerical solution is written as

$$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \ldots$$

Example 2 As the starting value problem, the modified time-fractional FornbergWhitham equation (2) is utilised, for the initial condition [13].

$$u(x,\ 0) = \frac{3}{4}(\sqrt{15}-5)\operatorname{sec}h^2(cx). \tag{56}$$

with a constant c

$$c = \frac{1}{20}\sqrt{10(5-\sqrt{15})}.$$

The LT on both sides Eq. (2), as follows

$$\mathcal{L}[u(x,\ t)] = \frac{3}{4s}(\sqrt{15}-5)\operatorname{sec}h^2(cx) + \left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\left[uu_{xxx} - u^2u_x + 3u_xu_{xx} - u_x + u_{xxt}\right]. \tag{57}$$

Using the inverse of (LT) in Eq. (57),

$$u(x,\ t) = \frac{3}{4}(\sqrt{15}-5)\operatorname{sec}h^2(cx) + \mathcal{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\left[uu_{xxx} - u^2u_x + 3u_xu_{xx} - u_x + u_{xxt}\right]\right]. \tag{58}$$

In addition, we address the arrangement as a perpetual series as follows

$$u(x,\ t) = \sum_{m=0}^{\infty} u_m(x,\ t). \tag{59}$$

and the non-linear term uu_{xxx}, uu_x and u_xu_{xx} can be decomposed by using He's polynomials [27] as

$$\sum_{m=0}^{\infty} H_m(x,\ t) = uu_{xxx}. \tag{60}$$

Evaluation of the elements of the He's polynomials is simple. The first few components of H_m are written as

$$\begin{aligned} &H_0(u) = u_0D_{xxx}u_0,\\ &H_1(u) = u_0D_{xxx}u_1 + u_1D_{xxx}u_0,\\ &H_2(u) = u_0D_{xxx}u_2 + u_1D_{xxx}u_1 + u_2D_{xxx}u_0,\\ &\vdots \end{aligned} \tag{61}$$

for H'_m we find that

$$\sum_{m=0}^{\infty} H'_m(x,\ t) = u^2u_x.$$

$$\begin{aligned} &H'_0(u) = u_0^2D_xu_0,\\ &H'_1(u) = u_0^2D_xu_1 + 2u_0u_1D_xu_0,\\ &\vdots \end{aligned} \tag{62}$$

for H''_m we find that

$$\sum_{m=0}^{\infty} H''_m(x,\ t) = uu_{xxx}.$$

$$\begin{aligned}
&H''_0(u) = D_x u_0 D_{xx} u_0,\\
&H''_1(u) = D_x u_0 D_{xx} u_1 + D_x u_1 D_{xxx} u_0,\\
&H''_2(u) = D_x u_0 D_{xx} u_2 + D_x u_1 D_{xxx} u_1 + D_x u_2 D_{xxx} u_0,\\
&\vdots
\end{aligned} \tag{63}$$

using Eqs. (59)-(63) in Eq. (58) gives

$$\sum_{m=0}^{\infty} u_m(x,\ t) = \frac{4}{3}\left(\sqrt{5}-5\right)\operatorname{sech}(cx)^2 + p\mathcal{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathcal{L}\left[\left(\sum_{m=0}^{\infty} p^m u_m(x,\ t)\right)_{xxt} - \left(\sum_{m=0}^{\infty} p^m u_m(x,\ t)\right)_x\right]\right]$$

$$\times + \left[\sum_{m=0}^{\infty} p^m H_m(u) - \sum_{m=0}^{\infty} p^m H'_m(u) + 3\sum_{m=0}^{\infty} p^m H''_m(u)\right]. \tag{64}$$

when we compare we obtain the coefficients of identical powers of p.

$$p^0 : u_0(x,\ t) = \frac{4}{3}\left(\sqrt{5}-5\right)\operatorname{sech}(cx)^2$$

$$p^1 : u_1(x,\ t) = (1-\alpha+t\alpha)\times\left(-\frac{4}{3}\left(-5+\sqrt{5}\right)\operatorname{sech}(cx)^2 + \frac{128}{27}\left(-5+\sqrt{5}\right)^3 c * \operatorname{sech}(cx)^6 \tanh(cx)\right)$$

$$\times - 8\left(-5+\sqrt{5}\right)c\operatorname{sech}(cx)^2\tanh(cx)\left(-\frac{8}{3}\left(-5+\sqrt{5}\right)c^2 * \operatorname{sech}(cx)^4 + \right)\times\frac{16}{3}\left(-5+\sqrt{5}\right)c^2\operatorname{sech}(cx)^2\tanh(cx)^2$$

$$+\frac{4}{3}\left(-5+\sqrt{5}\right)\operatorname{sech}(cx)^2\left(\frac{64}{3}\left(-5+\sqrt{5}\right)c^3 * \operatorname{sech}(cx)^4\tanh(cx)\right)\times -\frac{32}{3}\left(-5+\sqrt{5}\right)c^3 * \operatorname{sech}(cx)^2\tanh(cx)^3,$$

$$\vdots$$

By proceeding in this fashion, we may obtain the excess part iterations formulas.
Due to this, the best-guess solution are

$$u(x,\ t) = \sum_{m=0}^{\infty} u_m(x,\ t) \tag{65}$$

6. Conclusion and applications

We establish the existence and uniqueness of the time-fractional FWEs using fixed point theory. The FWEs and its time-fractional counter part provide valuable insights into wave propagation in dispersive media with long-range interactions. The equations describe the behavior of waves in a variety of physical systems, including shallow water, fluid flow, and signal processing. The nonlinearity of the medium and the fractional derivative term in the time-fractional FWEs lead to interesting phenomena such as the formation of solitary waves and long-term memory effects, which have been studied extensively by researchers. By selecting appropriate auxiliary parameters α. The Homotopy perturbation transform method (HPTM) gives a straightforward explanation for changing and managing the series solution's convergence. It is worth noting that the proposed method might also be utilised to solve the TFFWEs, which incorporates Caputo-Fabrizio fractional derivative (CFFD) of order $0 < \alpha \leq 1$, as in the HPTM [3-4]. The investigation has shown that the offered methodologies and the problem's exact answers are most closely related. These cuttingedge techniques demand less time and computational effort while producing results that are quantitatively more accurate. According to the research mentioned, the suggested methodologies can easily be modified to address different scientific and engineering issues. Ahead, 2D & 3D figure (1-4) of different arrangements were displayed to show the actual parts of the procured cures.

The time-fractional FWEs have a great many applications in different fields, including fluid dynamics, atmospheric science, and traffic flow. Some of the applications are:

• Water waves: The equation is used to simulate the propagation of long water waves like tsunamis and ocean swells. This can help researchers better understand how these types of waves behave and make predictions about their possible impact.

• Traffic flow: The equation is often used to analyze traffic flow on motorways and roads. It can be used to simulate the spread of congestion and its impact on traffic flow.

• Atmospheric science: In atmospheric science, the equation is utilized to show the advancement of atmospheric waves, such as Rossby waves and gravity waves. These waves fun a crucial preface in the dynamics of the atmosphere and have a significant impact on weather patterns.

• Plasma physics: In space physics, the time-fractional FWEs is also used to analyze plasma waves. It is used to simulate the behavior of plasma waves in the solar wind and the magnetospheres of planets.

These are just a few of the many uses for the time-fractional FWES. It is a powerful tool for studying wave behavior in complicated systems and its application is projected to grow in the future.

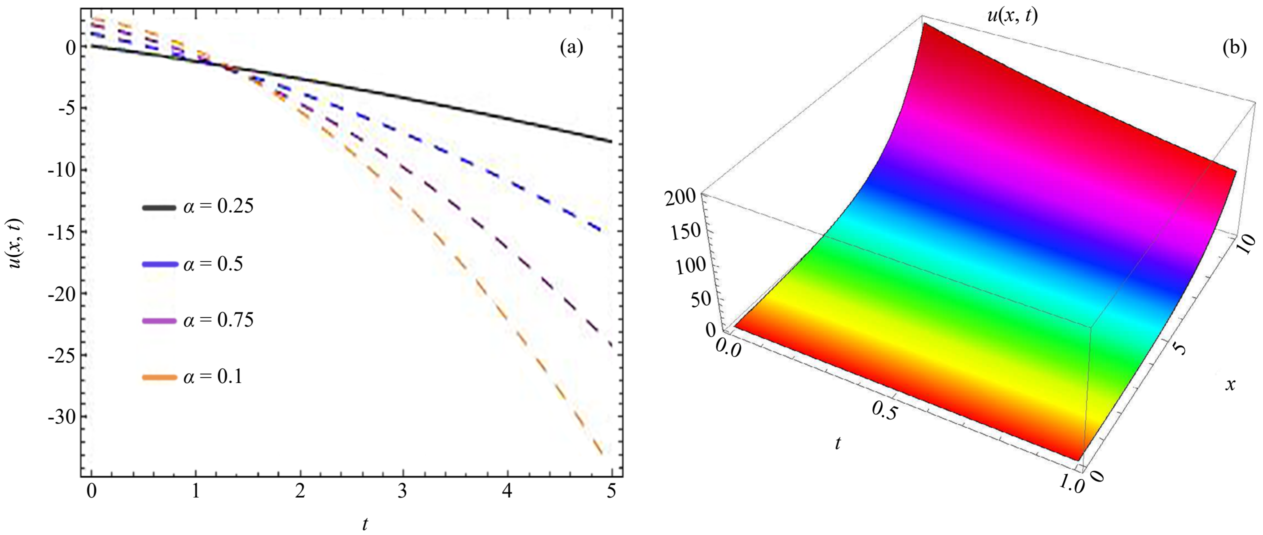

Figure 1. (a) The 2D graph of the HPTM $u(x, t)$ among different values of $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.75$ and $\alpha = 1$. (b) The 3D graph of the exact solution $u(x, t)$

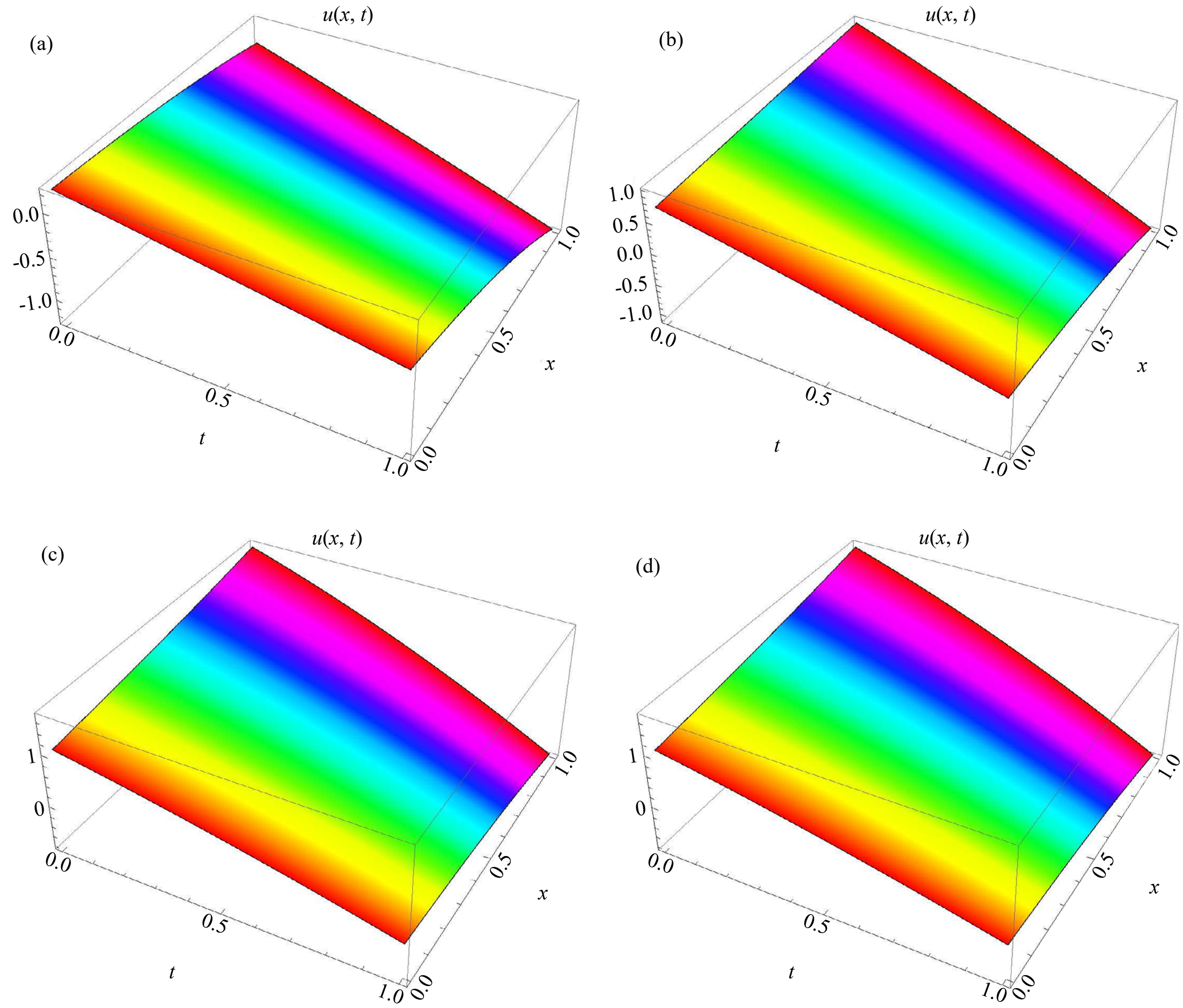

Figure 2. For the time-fractional Fornberg-Whitham equation with the first initial condition (Eq. (47)) of Eq. (1), HPTM result for $\phi(x, t)$ is, respectively (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$ and (d) $\alpha = 1$

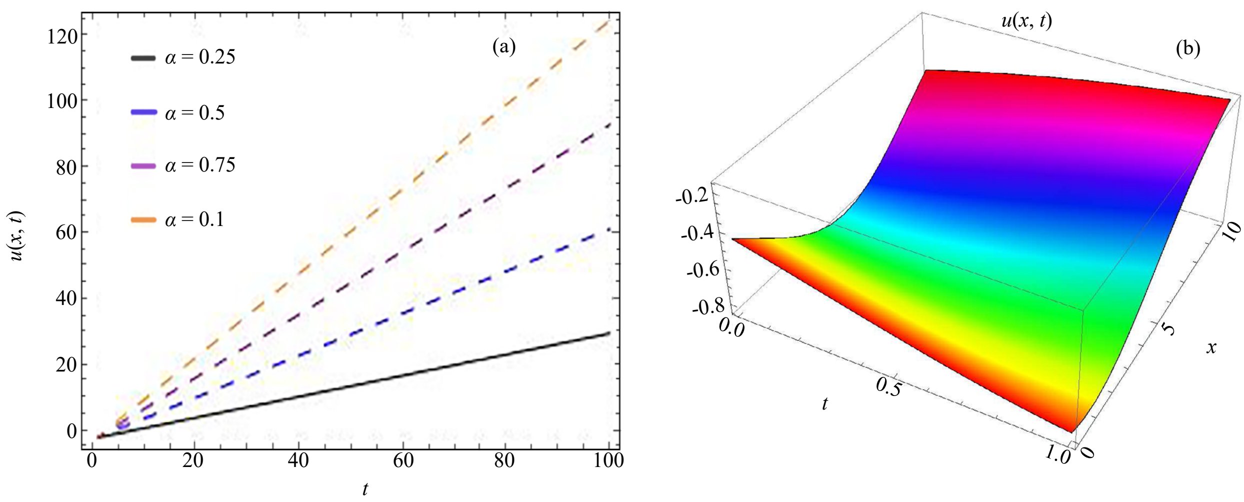

Figure 3. (a) The 2D graph of the HPTM $u(x, t)$ among different values of $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.75$ and $\alpha = 1$. (b) The 3D graph of the exact solution $u(x, t)$

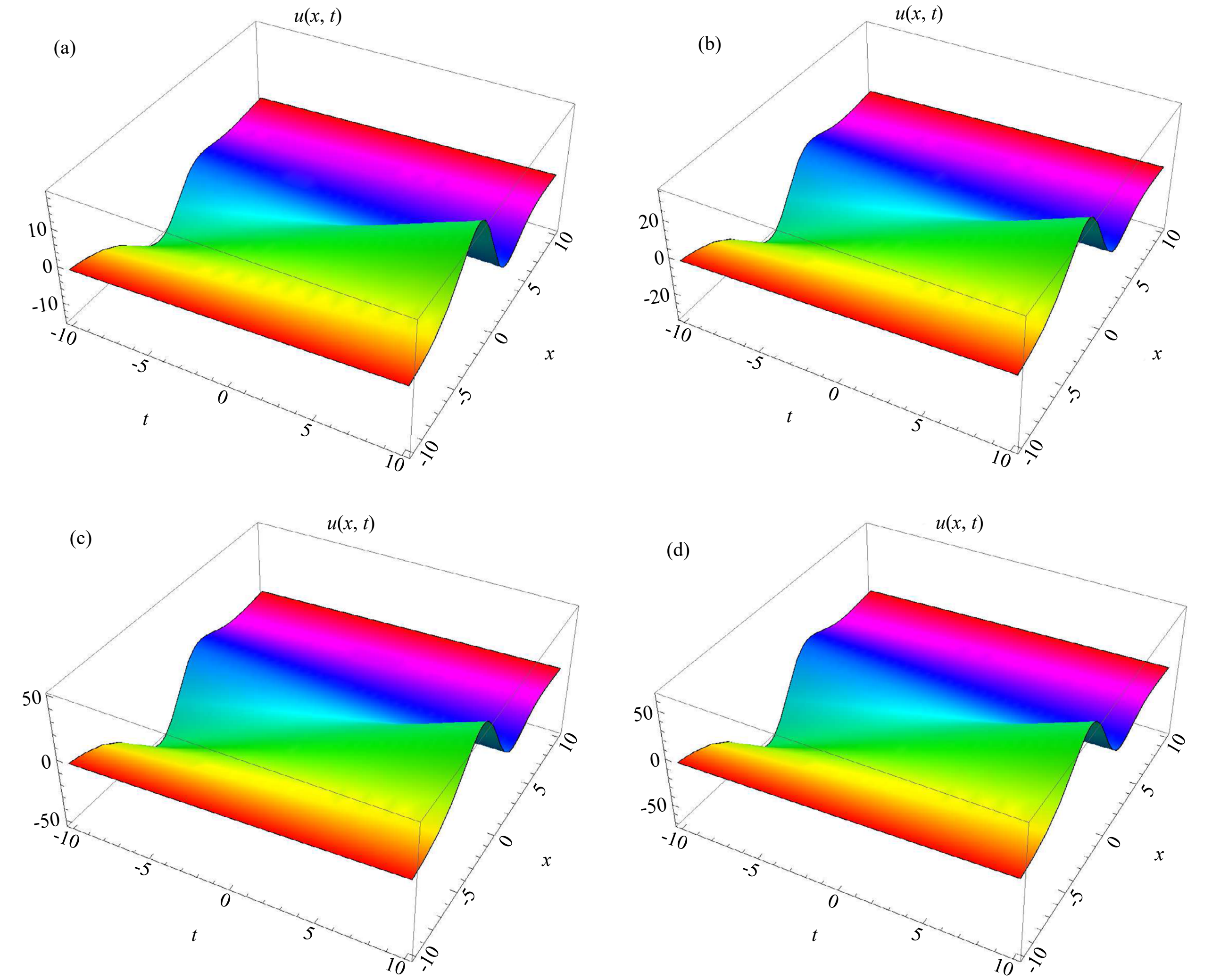

Figure 4. For the time-fractional Fornberg-Whitham equation with the first initial condition (Eq. (56)) of Eq. (1), HPTM result for $\phi(x, t)$ is, respectively (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$ and (d) $\alpha = 1$

Conflict of interest

The authors declare no competing financial interest.

References

[1] Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. Amsterdam: Elsevier; 2006.

[2] Arora S, Mathur T, Agarwal S, Tiwari K, Gupta P. Applications of fractional calculus in computer vision: A survey. *Neurocomputing*. 2022; 489: 407-428.

[3] Kumar A, Prasad RS. On dynamical behavior for approximate solutions sustained by nonlinear fractional damped Burger and Sharma-Tasso-Olver equation. *International Journal of Modern Physics B*. 2023; 37(23): 2350228.

[4] Atangana A, Baleanu D. Nonlinear fractional Jaulent-Miodek and Whitham-Broer-Kaup equations within Sumudu transform. *Abstract and Applied Analysis*. 2013; 2013: 1-8.

[5] Kumar A, Prakash A, Mehmet Baskonus H. The epidemic COVID-19 model via Caputo-Fabrizio fractional operator. *Waves in Random and Complex Media*. 2022; 1-15. Available from: https://doi.org/10.1080/17455030.20

22.2075954.
[6] Ciancio A, Yel G, Kumar A, Baskonus HM, Ilhan E. On the complex mixed darkbright wave distributions to some conformable nonlinear integrable models. *Fractals*. 2022; 30(1): 2240018.
[7] Gupta AK, Ray SS. An investigation with Hermite Wavelets for accurate solution of fractional Jaulent-Miodek equation associated with energydependent Schrodinger potential. *Applied Mathematics and Computation*. 2015; 70: 458-471.
[8] Kumar A, Ilhan E, Ciancio A, Yel G, Baskonuş HM. Extractions of some new travelling wave solutions to the conformable Date-Jimbo-Kashiwara-Miwa equation. *AIMS Mathematics*. 2021; 6(5): 4238-4264.
[9] Prakash A, Goyal M, Baskonus HM, Gupta S. A reliable hybrid numerical method for a timedependent vibration model of arbitrary order. *AIMS Mathem*. 2020; 5(2): 979-1000.
[10] Prakash A, Kumar A, Baskonus HM, Kumar A. Numerical analysis of nonlinear fractional Klein-Fock-Gordon equation arising in quantum field theory via Caputo-Fabrizio fractional operator. *Mathematical Sciences*. 2021; 15: 269-281.
[11] Kala BS, Rawat MS, Rawat N, Kumar A. Numerical analysis of nonDarcy MHD flow of a Carreau fluid over an exponentially stretching/shrinking sheet in a porous medium. *International Journal of Scientific Research in Mathematical and Statistical Sciences*. 2019; 6(2): 295-303.
[12] Chen Q, Sabir Z, Raja MAZ, Gao W, Baskonus HM. A fractional study based on the economic and environmental mathematical model. *Alexandria Engineering Journal*. 2023; 65: 761-770.
[13] Kumar D, Singh J, Baleanu D. A new analysis of the Fornberg-Whitham equation pertaining to a fractional derivative with Mittag-Leffler-type kernel. *The European Physical Journal Plus*. 2018; 133(70): 1-10.
[14] Abidi F, Omrani K. The homotopy analysis method for solving the Fornberg-Whitham equation and comparison with Adomian's decomposition method. *Computers & Mathematics with Applications*. 2010; 59(8): 2743-2750.
[15] Ahmad H, Seadawy AR, Ganie AH, Rashid S, Khan TA, Abu-Zinadah H. Approximate numerical solutions for the nonlinear dispersive shallow water waves as the Fornberg-Whitham model equations. *Results in Physics*. 2021; 22: 103907.
[16] Baskonus HM, Kumar A, Kumar A, Gao W. Deeper investigations of the (4+1)-dimensional Fokas and (2+1)-dimensional Breaking soliton equations. *International Journal of Modern Physics B*. 2020; 34(17): 2050152.
[17] García Guirao JL, Baskonus HM, Kumar A, Rawat MS, Yel G. Complex patterns to the (3+1)-dimensional b-type kadomtsev-petviashvili-boussinesq equation. *Symmetry*. 2020; 12(1): 17.
[18] Baskonus HM, Guirao JLG, Kumar A, Causanilles FSV, Bermudez GR. Complex mixed dark bright wave patterns to the modified α and modified Vakhnenko-Parkes equations. *Alexandria Engineering Journal*. 2020; 59(4): 2149-2160.
[19] Garcia Guirao JL, Baskonus HM, Kumar A. Regarding new wave patterns of the newly extended nonlinear (2+1)-dimensional boussinesq equation with fourth order. *Mathematics*. 2020; 8(3): 341.
[20] Saberi Nik H, Buzhabadi R. Solving fractional Fornberg-Whitham equation by homotopy analysis method. *Journal of Applied Mathematics, Statistics and Informatics*. 2011; 7(2): 97-105.
[21] Sakar MG, Ergoren H. Alternative variational iteration method for solving the time-fractional Fornberg-Whitham equation. *Applied Mathematical Modelling*. 2015; 39: 3972-3979.
[22] Baskonus HM, Mahmud AA, Muhamad KA, Tanriverdi T, Gao W. Studying on kudryashov-sinelshchikov dynamical equation arising in mixtures liquid and gas bubbles. *Thermal Science*. 2022; 26: 1229-1244.
[23] Baskonus HW, Gao W. Investigation of optical solitons to the nonlinear complex Kundu-Eckhaus and Zakharov-Kuznetsov-Benjamin-Bona-Mahony equations in conformable. *Optical and Quantum Electronics*. 2022; 54(6): 388.
[24] Junfeng L. An analytical approach to the Fornberg-Whitham type equations by using the variational iteration method. *Computers and Mathematics with Applications*. 2011; 61: 2010-2013.
[25] Sakar MG, Erdogan F. The homotopy analysis method for solving the time-fractional Fornberg-Whitham equation and comparison with Adomian'sdecomposition method. *Applied Mathematical Modelling*. 2013; 37(20-21): 8876-8885.
[26] Sakar MG, Erdogan F, Yildirim A. Variational iteration method for the timefractional Fornberg-Whitham equation. *Computers & Mathematics with Applications*. 2012; 63(9): 1382-1388.
[27] He JH. Homotopy perturbation technique. *Computer Methods in Applied Mechanics and Engineering*. 1999; 178(3-4): 257-262.
[28] Singh J, Kumar D, Kumar S. New treatment of fractional Fornberg-Whitham equation via Laplace transform. *Ain Shams Engineering Journal*. 2013; 4(3): 557-562.

[29] Gupta PK, Singh M. Homotopy perturbation method fractional Fornberg-Whitham equation. *Computers & Mathematics with Applications*. 2011; 61(2): 250-254.
[30] Renu K, Kumar A, Kumar A, Kumar J. Effect of transverse hydromagnetic and media permeability on mixed convective flow in a channel filled by porous medium. *Special Topics & Reviews in Porous Media-An International Journal*. 2021; 12(2): 1-23.
[31] Yan L, Kumar A, Guirao JLG, Baskonus HM, Gao W. Deeper properties of the nonlinear Phi-four and Gross-Pitaevskii equations arising mathematical physics. *Modern Physics Letters B*. 2022; 36(4): 2150567.
[32] Yan L, Yel G, Kumar A, Baskonus HM, Gao W. Newly developed analytical scheme and its applications to the some nonlinear partial differential equations with the conformable derivative. *Fractal and Fractional*. 2021; 5(4): 238.
[33] Chauhan JP, Khirsariyal SR. A semi-analytic method to solve nonlinear differential equations with arbitrary order. *Results in Control and Optimization*. 2023; 12: 100267.
[34] Khirsariya1 SR, Rao SB. Solution of fractional Sawada-Kotera-Ito equation using Caputo and Atangana-Baleanu derivatives. *Mathematical Methods in the Applied Sciences*. 2023; 46(15): 16072-16091.
[35] Khirsariya1 SR, Rao SB, Chauhan J. Semi-analytic solution of time-fractional Korteweg-de Vries equation using fractional residual power series method. *Results in Nonlinear Analysis*. 2022; 5(3): 222-234.
[36] Khirsariya SR, Rao SB, Chauhan JP. A novel hybrid technique to obtain the solution of generalized fractional-order differential equations. *Mathematics and Computers in Simulation*. 2023; 205: 272-290.
[37] Khirsariya1 SR, Rao SB. On the semi-analytic technique to deal with nonlinear fractional differential equations. *Journal of Applied Mathematics and Computational Mechanics*. 2023; 22(1): 17-30.
[38] Podlubny I. *Fractional Differential Equations*. New York: Academic Press; 1999.