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Research Article

Benefits of Preservation, Green and Quality Improvement Investment for Waste Management in Sustainable Supply Chain under Fuzzy Learning and Inflation

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Abstract: In the modern world, waste management, incorporating the quality of the products, energy consumption, and environmental concern have become significant challenges for supply chain managers. Also, smart devices are essential for daily life in the current socioeconomic environment, and customers primarily contemplate a smart product's price and energy usage before purchasing that. In this situation, to maintain a balance between the selling price, energy consumption, and carbon emission from supply chain operations becomes necessary. So this study develops a two-echelon sustainable inventory model for deteriorating items with an imperfect production process under energy consumption and selling price dependent demand. The producer makes a rework process and quality improvement investment to mitigate defective products and enhance the quality of the products. The present model develops under the influence of inflation. Also, preservation and green technologies are used to mitigate the rate of deterioration and carbon emission, respectively. Firstly, the model is created in a crisp sense, and then expanded into a fuzzy learning model to examine the impact of the learning effect in an imprecise environment. A numerical analysis is performed to validate the proposed model, and the cost function's convexity is shown graphically using mathematica software. The result of the proposed model provides significant insights to decision-makers on how to efficiently reduce waste while still minimizing the total cost of the system by investing in high efficiency preservation, quality improvement and green techniques. Also, due to learning in fuzziness, the fuzzy learning model gives the lowest total cost, followed by the fuzzy and crisp model. Finally, for various parameters, a sensitivity analysis is performed to gather valuable observations and management insights.

***Keywords*:** imperfect production, rework process, preservation technology, carbon tax policy, inflation, price and energy dependent demand, green technology

MSC: 90B05, 90B06, 90B30

1. Introduction

In the last few decades, supply chain managers have emphasized minimizing the environmental effect of their production and logistics systems, including energy use and greenhouse gas (GHG) emissions. This interest resulted

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from societal pressure and consumer awareness of the value of sustainability to their communities, which encouraged governments to pass legislation with this perspective to lessen the environmental impact of product manufacture, use, and disposal and improve the preservation of natural resources. A supply chain's efficiency may improve with the efficient use of energy. Supply chain management is distinguished by adding environmental concerns to the usual economic focus but without incorporating the idea of carbon emissions. By using energy more efficiently, supply chain managers can reduce their environmental impact by reducing the emissions produced. It could lead to lower costs in the long run and improved environmental performance. Also, utilizing renewable energy sources and energy storage technologies can reduce emissions and increase efficiency. Reducing carbon emissions and energy use in traditional manufacturing is difficult, but it is possible by managing the production rate.

Traditional production inventory models assumed that items were manufactured to perfection; however, this assumption is rarely achieved in reality. Due to human error, long machine run times, and comprehensive process control, imperfect production is unavoidable. Today's consumers are highly quality-conscious and refuse to accept low-quality products, which made it necessary for manufacturers to proactively monitor their manufacturing processes and filter the products before they are delivered. Companies must constantly improve their quality standards to remain viable in an increasingly competitive marketplace. This is why quality assurance is an integral part of any manufacturing process. Quality control measures must be implemented and monitored to ensure the products meet the required standards and satisfy customer needs. Companies must also focus on continuous improvement and innovation to remain competitive. Quality enhancement technologies help producers avoid out-of-control circumstances that result in producing high-quality products.

In the absence of supply chain coordination, the impact of deterioration increases from one stage to another. It can lead to decreased customer satisfaction, increased operating costs, and disruption of the entire supply chain process. Poor supply chain coordination can lead to product delivery delays and reduced quality. To meet the market's demands, looking into the techniques to stop the deteriorating process is essential. Technologies for preservation are used to prevent products from deteriorating and being converted into waste. It has been established that various products deteriorate at different rates when storage environment conditions and temperatures vary. Therefore, maintaining these parameters constant is the primary purpose of investing in preservation technologies to reduce the rate of deterioration.

1.1 *Novelty of the study*

Achieving environmental and economic sustainability poses significant challenges for manufacturers. They explore various approaches to meet these goals. A review of the work done so far by various researchers shows that significant efforts have been made to reduce carbon emissions from imperfect production inventory systems. Researchers have independently developed several models to improve the value of investments as well as to be compatible with customers, including preservation investment for reducing deteriorating products, green technology for environmental protection, etc. For example Sepehri et al. [1] brought attention to an EPQ(Economic Production Quantity) model involving imperfect production, concerning quality of products, and environmental aspects. The main objective of this research was to discuss the impact of preservation and carbon reduction technologies on total cost/profit.

However, the simultaneous focus on reducing waste and carbon emissions during supply chain operations (such as production, rework, transportation, storage, and deterioration) while prioritizing environmental preservation, especially through the strategic implementation of carbon tax policies, emerged as a pivotal aspect. Using preservation technology to protect products and energy flexibility by improving inventory quality and weighing the impact of inflation under-investment, as well as using learning concepts, is still pending. Therefore, it has not happened, and this is what makes this research paper most unique. Consequently, it is essential to examine how decision maker achieve environmental and economic sustainability by integrating reworking, waste management, quality improvement investment, green and preservation technology into their supply chain inventory system under imprecise environment. Therefore, the current study expands on the previously developed flexible manufacturing inventory model by incorporating supply chain, waste management, imperfect production, reworking, learning effect, preservation, and green technology. The study also considers the impact of inflation and a fuzzy environment under investments to reduce the fraction of imperfect production processes.

1.2 *Research question*

Thus, considering this key context, we aim to answer the following questions through this model:

(i) Storing inventory in a warehouse or showroom presents a significant challenge, with the highest risk of product spoilage. Researchers are faced with the important question of which technology to employ to mitigate this issue and preserve inventory.

(ii) The application of fuzzy learning methods introduces a novel dimension to the study. Fuzzy logic enables the simulation of uncertainties and vagueness in decision-making, proving essential when addressing sustainability challenges in the supply chain. This approach facilitates a more realistic analysis of the complexities associated with sustainability investments, prompting the question of how learning will impact the total costs of the proposed model in an imprecise environment.

(iii) Inflation leads to fluctuations in the prices of goods and services, affecting individuals with fixed incomes the most. This predicament places a burden on those within fixed income groups. Conversely, the farming class experiences a favourable impact from inflation, as it results in increased prices for their produce. This raises the question of how inflation will influence the costs of the model.

(iv) At present, every production company, in preparing inventory and delivering it to customers, pollutes the environment with various types of toxic gases. This pollution poses a threat to future generations and releases greenhouse gases such as methane and carbon dioxide (CO_2) into the atmosphere. The rise in Earth's temperature due to the increase of chlorofluorocarbons, among other factors, is a matter of concern. The question that arises is how these emissions can be stopped to protect the environment.

(v) Production Inventory Management involves comprehensive energy inventory such as how to analyze the energy consumption distribution of machines and other equipment to reduce energy consumption, reduce costs, optimize customer pricing, and achieve energy conservation as well as carbon reduction. This has been under consideration till date so that an excellent action plan can be made.

In an attempt to provide logical and analytical answers to the above questions, this model explores the impact of innovative techniques (such as preservation, green technology, and quality improvement investment) on reducing waste in a sustainable supply chain with imperfect production and rework processes, considering fuzzy learning and inflation. To apply this model in real life, the complete conclusion of the inventory model has been obtained by examining various parameters. Additionally, the study highlights important parameters, the slight variation of which affects the model, and their sensitivity analysis is presented to illustrate their impact.

1.3 *Structure of this study*

The current study sections are arranged as follows: Section 1 provides an introduction, while Section 2 presents a literature review to get the motivation for the current work. Section 3 contains problem definitions, notations and basic assumptions required for modeling purposes. The mathematical formulation of the current study is presented in Section 4, and the Solution methodology is given in Section 5. A numerical and sensitivity analysis is presented in Section 6 which helps to validate the model. Section 7 represents a discussion on theoretical implications of current study and managerial implications of the present model in given in section 8. Final concluding remark is presented in Section 9. The contribution of the current study and previous research are summarized in Table 1.

2. Literature review

In this section, we have to discussed the literature review related in different direction (1) Inventory models based on imperfect production (2) Inventory models with carbon emissions & energy usage (3) Inventory models with inflation (4) Inventory models with controllable deterioration (5) Inventory models based on fuzzy learning.

2.1 *Inventory models based on imperfect production*

Imperfect production arises from mechanical errors occurring in the manufacturing process. Numerous researchers have explored the concept of imperfect production in the existing literature. Initially, Rosenblatt and Lee [2] investigated the impact of imperfect manufacturing processes on the quality deterioration of items throughout the optimal production cycle. In a similar vein, Cheng [3] put forward an EOQ(Economic Order Quantity) model that incorporated demand-oriented unit manufacturing cost while considering the presence of imperfect manufacturing processes. Hayek and Salameh [4] developed a lot sizing inventory model in the influence of defective quality products, with a consistent rate of rework. Liao et al. [5] investigated an economic production quantity model for deteriorating items with imperfect production under a comprehensive rework and maintenance process. Sarkar et al. [6] studied an imperfect production quantity model that accommodated continuous and stochastic demand for items, while also investigating the percentage decrease in overall cost. Sarkar et al. [7] introduced an imperfect production inventory model with an investment in quality improvement under controllable lead time. Ruidas et al. [8] introduced an EOQ model for imperfect and rework items using Particle Swarm Optimization (PSO) technique. In this model, the demand of customers is fulfilled by perfect and reworked items, and scrapped items are sold in the secondary market at a discounted price. Guchhait et al. [9] investigated an inventory model with fully backlogged shortage under an imperfect production and a free product minimal repair warranty policy. They observed the impact of setup cost reduction and quality improvement on the optimal production cycle time. Gautam et al. [10] introduced a proficient rework strategy for manage defective products in imperfect production inventory system considering energy usage and the cost of carbon emissions during the manufacturing process. They reveal that rework process is helpful for making the products market-ready at its original price. Sepehri et al. [11] developed a sustainable inventory model considering imperfect production process under different cases of shortages. This study examines the effects of shortages on a green inventory system of imperfect products while simultaneously taking quality improvement and inspection procedures into account. Bhatnagar et al. [12] introduced an economic production quantity inventory system for managing waste from imperfect production considering rework process and the backorderd shortages.

2.2 *Inventory models with carbon emissions & energy usage*

The impact of renewable energy is crucial for developing sustainable supply chains. Economic, environmental, and social pillars make up the three fundamental components of sustainability. The impact of renewable energy is linked to all three pillars in the following way: for economic development, companies use traditional energy more expensively. Many developing countries have supported emission reduction policies (such as cap and trade) to reduce emissions and have developed technologically advanced equipment because rising demand always causes maximum carbon dioxide emissions. In this direction, the authors investigated how government regulation can reduce carbon emissions and energy from various operations associated with production systems. Font et al. [13] described a supply chain inventory model adding sustainability to manage business activities by implementing environmental, social, and economic implications. Researchers Ahi & Searcy [14] have widely studied the sustainability concept's incorporation into the supply chain. Sarkar et al. [15] developed a sustainable inventory model for multi-items considering imperfect production process under optimum energy consumption. Mashud et al. [16] created a sustainable inventory approach with controllable carbon emissions. Thomas et al. [17] established a circular economy integrated inventory model intending to reduce waste and control pollution with the help of 3D printing and different emission reduction mechanisms. They observed a significant increment in profit for plastic reforming industries while applying waste and carbon reduction technologies. Ruidas et al. [18] described a sustainable economic production quantity model for green degree products considering green subsidy. They found that higher subsidy intensity increases product greenness, and simultaneous investment in greening innovation and emission reduction technology benefits both the manufacturer and the environment. Jauhari [19] established a supply chain system for imperfect products considering optimum energy consumption under controllable production rate. Ruidas et al. [20] introduced a production inventory model with price and green degree-dependent demand, considering cap-and-trade policy. They observed that both the manufacturing company of the green product and the environment benefit from the joint investment in Green Innovation (GI) and Emission Reduction Technology (ERT). Sarkar et al. [21] developed a sustainable inventory system for substitutable products under a dual channel policy and a fully controlled emission production system. They observed that green investment has a positive impact on the environment.

2.3 *Inventory models with inflation*

Involving inflation in modelling is a realistic method since it has a variety of effects on the economy. The purchasing power of unit money decreases as a result of high inflation. Buzacott [22] initially investigated an inventory system under the influence of inflation. Yang [23] described a two-storage inventory model considering the impact of inflation and shortages. Singh et al. [24] presented an EPQ model for deteriorating items in an inflationary environment that incorporated time dependent demand and shortages. Singh et al. [25] presented an inventory model for decaying items with two-storage facility when demand is time-dependent, taking inflation and partial backlogging shortages into account. Kumar et al. [26] presented a two-warehouse inventory model for smart products considering imprecise and inflationary environment. Padiyar et al. [27] introduced an integrated inventory model for imperfect production process having preservation facilities under the influence of inflation. Singh and Chaudhary [28] established an economic order quantity model considering multivariate demand. They observed that inflation has a positive impact to reduce market disruption. Yadav et al. [29] investigated an inventory model for deteriorating items considering smart production process and controllable carbon emission in an inflationary environment. Padiyar et al. [30] developed three-echelon integrated inventory model for deteriorating products considering the inflationary environment. Singh et al. [31] developed a economic ordered quantity inventory model considering controllable carbon emission under different cases of shortages. This study examines the effects of shortages on a green inventory system under multivariate demand while simultaneously taking carbon tax policy into account.

2.4 *Inventory models with controllable deterioration*

In the existing inventory system, another critical task to manage is deterioration. After a certain time, every product loses its usefulness and freshness. The first inventory model based on the concept of constant deterioration was developed by Ghare and Schrader [32]. Preservation investment is a crucial component in decreasing the deterioration effect. In this regard, several commercial firms and organizations are required to include preservation technologies into their inventory management systems. The concept of preservation technology was first presented by Hsu et al. [33]. With the preservation technology, they established an inventory model with constant demand. Dye and Hsieh [34] examined the impact of preservation investment in an economic production quantity model with time-dependent deterioration rate. Yang et al. [35] developed a preservation inventory model with permissible delay in payment under the effect of deterioration. Under typical resource constraints, Zhang et al. [36] presented an inventory model for a deteriorating object with an investment preservation technique. Mishra et al. [37] proposed a preservation inventory model with shortage considering price-dependent demand. Mahapatra et al. [38] examined the impact of preservation investment on a fuzzy inventory model of deteriorating items under the learning effect. Saha et al. [39] introduced an application of green preservation investment to reduce deterioration, and waste in an inventory system, and also optimize selling prices and dynamic costs with the help of Pontryagin's maximum rule.

2.5 *Inventory models based on fuzzy learning*

A review of the supply chain literature reveals that many static models have been established for the purpose of simplicity, with various inventory characteristics thought to be precisely known. It demonstrates that the rapid increase in the environment's complexity makes it difficult to precisely define different inventory cost components. To address this issue, academics and researchers created fuzzy set theory. Zadeh [40] was the first to present the concept of fuzzy set theory. A fuzzy mixture inventory model based on triangular fuzzy numbers and probabilistic fuzzy sets with time-dependent lead times was created by Chang et al. [41]. By taking into account learning in fuzziness, only a very few of researchers have relaxed the idea of constant fuzziness. Taking learning into consideration, Kazemi et al. [42] looked into the fuzzy inventory model. They found that human learning in fuzziness decreased the total cost of inventory system. The effect of learning on imprecision on the economic order quantity model with imprecise demand and a finite time horizon was examined by Soni and Suthar [43]. They proposed that if parameter imprecision is high, learning in fuzziness leads to better decisions. Mahapatra et al. [38] developed an economic quantity model for deteriorating items in three different environments (crisp, fuzzy, and fuzzy learning). They observed that crisp model leads to lowest total cost than the fuzzy and fuzzy learning environment. Kumar et al. [44] investigated sustainable inventory system considering environmental, economical and social responsibilities under fuzzy learning environment.

Table 1. Research gap table

References	Supply Chain	Imperfect Production	Quality Improvement Invest-mint	Inflation	Preservation Techno-logy	Carbon reduction / Green Investment	Price & energy sensitive demand	Fuzzy Learn-ing	Energy
Lo et al. [45]	-	√	-	√	-	-	-	-	-
Jawla and Singh [46]	-	√	-	√	√	-	-	-	-
Sarkar et al. [7]	-	√	√	-	-	-	-	-	-
Sarkar et al. [15]	√	√	-	-	-	-	-	-	√
Kumar et al. [26]	-		-	√	-	-	-	√	-
Ruidas et al. [47]	-	√	-	-	-	-	-	-	-
Bhuniya et al. [48]	√	√	-	-	-	-	√	-	√
Sepehri et al. [1]	-	√	√	-	√	√	-	-	-
Ruidas et al.[18]	-	√	-	-	-	√	-	-	-
Ruidas et al.[49]	-	√	-	-	-	√	-	-	-
Mahapatra et al. [38]	-		-	-	√	-	-	√	-
Jauhari [19]	√	√	-	-	-	-	-	√	√
Ruidas et al.[20]	-	√	-	-	-	√	-	-	-
Kumar et al. [44]	-	-	-	-	√	-	-	√	-
Present paper	√	√	√	√	√	√	√	√	√

2.6 *Contribution of current research work*

As mentioned earlier, the research illustrates that researchers have focused on carbon emissions (transportation, storage, deterioration, production, etc.), imperfect production, quality improvement investment, price and energy usage dependent demand, controllable deterioration, fuzzy learning and inflation while independently developing an inventory model. From the literature mentioned above, it is evident that no research explored with the impact of all these issues simultaneously. All these issues are related. Therefore, it is essential to examine the effect of all these on the decision-making process for a sustainable supply chain system.

3. Notations and assumptions

3.1 *Notations*

The notations are broken down into decision variables, producer's parameters, retailer's parameter and other parameters as follows (Table 2):

Table 2. Representation of notations

Symbol	Description
	Decision variables
T	Cycle time (months)
G	Investment in green technology ($/unit/month)
	Producer's Parameter
t_m	Production time (months)
t_r	Time of reworking process (months)
P	Production rate
R	Rate of rework
σ	Fraction of produced imperfect items after investing in quality improvement
σ_o	Fraction of produced imperfect items before investing in quality improvement
σ_r	Portion of Imperfect items from rework process
C_{sp}	Setup cost for production process ($/cycle)
C_{rp}	Setup cost for rework process ($/cycle)
C_{mp}	Production cost parameter ($/unit/cycle)
C_{mh}	Holding cost for production process ($/unit/cycle)
C_{rh}	Holding cost for rework process ($/unit/cycle)
C_{md}	Deterioration cost for production process ($/unit/cycle)
C_{rd}	Deterioration cost for rework process ($/unit/cycle)
C_{mr}	Rework cost ($/unit/cycle)
C_{mw}	Waste scrapping cost ($/unit/cycle)
C'_{sp}	Carbon emission due to setup of production process (ton CO_2/setup)
C'_{sr}	Carbon emission due to setup of rework process (ton CO_2/setup)
C'_{mp}	Carbon emission caused by manufacturing process (ton CO_2/unit)
C'_{mh}	Carbon emission due to holding inventory from manufacturing process (ton CO_2/unit)
C'_{rh}	Carbon emission caused by holding inventory from rework process (ton CO_2/unit)
C'_{md}	Carbon emission due to deteriorating inventory (ton CO_2/unit)
C'_{mr}	Carbon emission due to reworking process (ton CO_2/unit)
C'_{mw}	Carbon emission due to scrapping waste (ton CO_2/unit)
δ	Fraction of decrease in defectiveness after quality improvement investment
	Retailer's Parameter
t_b	Time when retailer's inventory become vanish (months)
C_{bo}	Ordering cost ($/cycle)
C_{bp}	Purchasing cost for retailer's inventory ($/unit)
C_{bh}	Holding cost for retailer's inventory ($/unit/cycle)
C_{bd}	Retailer's deterioration cost ($/unit/cycle)
C'_{bh}	Carbon emission caused by retailer's holding inventory (ton CO_2/unit)
C'_{bd}	Carbon emission caused by retailer's deteriorating inventory (ton CO_2/unit)
u	Fixed transportation cost ($/delivery)
v	Variable transportation cost ($/delivery)
f_1	Fuel consumption of an empty truck (litre/km)
f_2	Fuel consumption per ton Q (litre/km)
D	Distance travelled from producer to retailer (km)

Table 2. (cont.)

Symbol	Description
f_{e1}	Cost for emission of carbon from vehicles (\$/km)
f_{e2}	Cost for emission of carbon from transporting items (\$/unit/km)
	Other Parameters
θ	Initial deterioration rate ($0 < \theta < 1$)
θ_1	Deterioration rate with investment in preservation ($0 < \theta_1 < 1$)
ζ	Investment in preservation technology (\$/unit/time)
K	Inflation rate
p_{min}	Minimum value of selling price (\$)
P	Selling price (\$)
p_{max}	Maximum value of selling price (\$)
E	Renewable energy consumption (unit/month)
E_{min}	Minimum energy consumption (unit/month)
E_{max}	Maximum energy consumption (unit/month)
Φ_x	Carbon tax (\$/ton CO_2)
λ	Proportion of carbon emission after investment in green ($0 < \lambda < 1$)
μ	Sensitivity parameter for investment in green ($\mu > 0$)

3.2 *Assumptions*

(i) This integrated inventory model is developed for single type of products considering single producer and retailer.

(ii) A quality improvement function is used to reduce the number of defective items, defined as $f(\sigma) = \frac{w}{\delta}\log\left(\frac{\sigma_o}{\sigma}\right)$, $0 \le \sigma \le \sigma_o$, where w represents the total opportunity cost and δ denotes the percentage of deficiency in defectiveness.

(iii) The demand function is considered energy consumption and selling price sensitive for both producer and retailer. It is defined as

$$D(p, E) = a\left(\frac{p_{max} - p}{p - p_{min}}\right) + b\left(\frac{E_{max} - E}{E - E_{min}}\right),$$

where a and b are scaling parameters.

(iv) Deterioration rate is considered constant, and preservation technology is used to mitigate the rate of deterioration. After preservation investment, the reduced deterioration rate becomes $\theta_1 = \theta(1 - m(\zeta))$, where $m(\zeta) = (1 - e^{-\alpha\zeta})$ is a twice differentiable function with respect to ζ.

(v) The model is developed under the influence of inflation to avoid market disruption.

(vi) Carbon emissions occur due to various operations of the supply chain system, such as the setup, production process, rework process, transportation, waste management of scrap, holding inventory, and deterioration etc.

(vii) To reduce rate of carbon emission, a carbon tax policy with an innovative green technology investment is used by decision maker.

4. Mathematical formulation

4.1 *Supply chain model in crisp environment*

The supply chain is critical to supplying the necessities for everyday life under the current unusual circumstances.

Furthermore, smart products have become inevitable in this environment as facilitators and carriers throughout various stages of the conventional process. Customers prefer smart products with minimal energy usage; therefore, variables like pricing and energy efficiency affect the demand for these items. In this model, the producer receives an order for a specific quantity Qm of smart products from the retailer. The producer then produces these products within a predetermined time t_m and delivers the perfect quality products in m shipment to the retailer. A detailed explanation of this model is provided in the subsequent section.

4.1.1 *Producer's inventory model*

The producer's inventory model for deteriorating items is illustrated in Figures 1 and 2 for serviceable and reworkable inventory, respectively. At time $t = 0$ the production process begins with a constant production rate P, reaching its maximum level at $t = t_m$. During the manufacturing period, a portion σP of imperfect items is transferred to the reworking station, where reworking starts at the end of the production process. In the time duration $[t_m, t_r]$ the rework process takes place, considering that the rework rate R is greater than the rate of demand D. A certain proportion σ_r is identified as non-reworkable and treated as scrap, undergoing the waste management process with consideration for the environment. The remaining proportion $(1 - \sigma_r)$ is considered as good items and transferred to the service station, where customer demand is satisfied. Thus, at the service station, during the time period $[t_m, t_r]$, inventory increases due to the reworking process and decreases due to demand and deterioration. At $t = t_r$ reworking inventory becomes zero, and perfect inventory declines due to deterioration and demand. The inventory becomes zero at $t = T$.

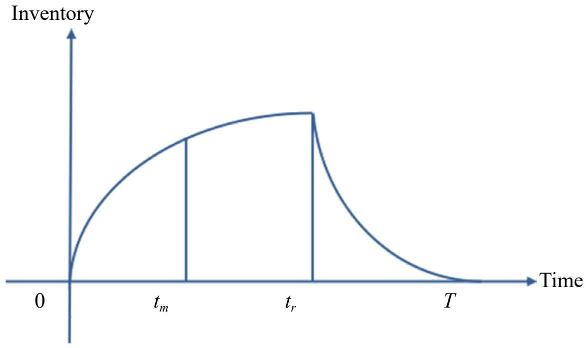

Figure 1. Representation of serviceable inventory of producer

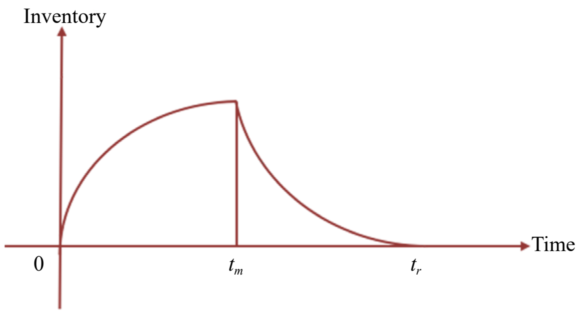

Figure 2. Representation of reworkable inventory of producer

The inventory level for serviceable inventory is represented by following differential equations:

$$I'_{m1}(t) = (1 - \sigma)P - D(p, E) - \theta(1 - m(\zeta))I_{m1}(t), 0 \leq t \leq t_m \tag{1}$$

$$I'_{m2}(t) = (1 - \sigma_r)R - D(p, E) - \theta(1 - m(\zeta))I_{m2}(t),\ t_m \le t \le t_r \tag{2}$$

$$I'_{m3}(t) = - D(p, E) - \theta(1 - m(\zeta))I_{m3}(t),\ t_r \le t \le T \tag{3}$$

With boundary conditions $I_{m1}(0) = 0$, $I_{m1}(t_m) = I_{m2}(t_m)$ and $I_{m3}(T) = 0$.
The inventory level for reworkable inventory is represented by following differential equations

$$I'_{m4}(t) = \sigma P - \theta(1 - m(\zeta))I_{m4}(t),\ 0 \le t \le t_m \tag{4}$$

$$I'_{m5}(t) = - R - \theta(1 - m(\zeta))I_{m5}(t),\ t_m \le t \le t_r \tag{5}$$

With boundary conditions $I_{m4}(0) = 0$ and $I_{m5}(t_r) = 0$.
Solving equations from (1) to (5), we get following inventories

$$I_{m1}(t) = \frac{\left((1-\sigma)P - D\right)}{\theta\left(1-m(\zeta)\right)}\left(1 - e^{-\theta(1-m(\zeta))t}\right) \tag{6}$$

$$I_{m2}(t) = \frac{\left((1-\sigma)P - D\right)}{\theta\left(1-m(\zeta)\right)} e^{-\theta(1-m(\zeta))t}\left(e^{\theta(1-m(\zeta))t_m} - 1\right) + \frac{\left((1-\sigma_r)R - D\right)}{\theta\left(1-m(\zeta)\right)}\left(1 - e^{\theta(1-m(\zeta))(t_m - t)}\right) \tag{7}$$

$$I_{m3}(t) = \frac{D}{\theta\left(1-m(\zeta)\right)}\left(e^{\theta(1-m(\zeta))(T-t)} - 1\right) \tag{8}$$

$$I_{m4}(t) = \frac{\sigma P}{\theta\left(1-m(\zeta)\right)}\left(1 - e^{-\theta(1-m(\zeta))t}\right) \tag{9}$$

$$I_{m5}(t) = \frac{R}{\theta\left(1-m(\zeta)\right)}\left(e^{\theta(1-m(\zeta))(t_r-t)} - 1\right) \tag{10}$$

Now the producer's inventory model consist following sub-costs
(a) Set-up Cost:

$$SC_m = \frac{\left(C_{sp} + C_{sr}\right)}{T}$$

(b) Production Cost:

$$PC_m = \frac{C_{mp}}{T}\int_0^{t_m} Pe^{-kt}\,dt = \frac{PC_{mp}}{kT}\left(1 - e^{-kt_m}\right)$$

(c) Deterioration Cost:

$$DC_m = \frac{C_{md}}{T}\left\{\begin{array}{l}\int_0^{t_m}\theta\left(1-m(\zeta)\right)e^{-kt}I_{m1}(t)\,dt + \\ \int_{t_m}^{t_r}\theta\left(1-m(\zeta)\right)e^{-kt}I_{m2}(t)\,dt + \\ \int_{t_r}^{T}\theta\left(1-m(\zeta)\right)e^{-kt}I_{m3}(t)\,dt +\end{array}\right\} + \frac{C_{rd}}{T}\left\{\begin{array}{l}\int_0^{t_m}\theta\left(1-m(\zeta)\right)e^{-kt}I_{m4}(t)\,dt + \\ \int_{t_m}^{t_r}\theta\left(1-m(\zeta)\right)e^{-kt}I_{m5}(t)\,dt +\end{array}\right\}$$

$$DC_m=\frac{C_{md}\theta\left(1-m(\zeta)\right)}{T}\left\{\begin{array}{l}\frac{\left((1-\sigma)P-D\right)}{\theta\left(1-m(\zeta)\right)}\left(\frac{\left(1-e^{-kt_m}\right)}{k}+\frac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}-1\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\frac{\left((1-\sigma_r)R-D\right)}{\theta\left(1-m(\zeta)\right)}\\ \left(\frac{\left(e^{-kt_m}-e^{-kt_r}\right)}{k}+e^{\theta\left(1-m(\zeta)\right)t_m}\left(\frac{e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_r}-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)\right)+\\ \frac{\left((1-\sigma)P-D\right)}{\theta\left(1-m(\zeta)\right)}\left(e^{\theta\left(1-m(\zeta)\right)t_m}-1\right)\left(\frac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_r}-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\\ \frac{D}{\theta\left(1-m(\zeta)\right)}\left(\frac{\left(e^{-kT}-e^{-kt_r}\right)}{k}+e^{\theta\left(1-m(\zeta)\right)T}\left(\frac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_r}-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)T}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)\right)\end{array}\right\}$$

$$+\frac{C_{rd}\theta\left(1-m(\zeta)\right)}{T}\left\{\begin{array}{l}\frac{\sigma P}{\theta\left(1-m(\zeta)\right)}\left(\frac{\left(1-e^{-kt_m}\right)}{k}+\frac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}-1\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\frac{R}{\theta\left(1-m(\zeta)\right)}\\ e^{\theta\left(1-m(\zeta)\right)t_r}\left(\frac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_r}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\frac{\left(e^{-kt_r}-e^{-kt_m}\right)}{k}\end{array}\right\}$$

(d) Holding Cost:

$$HC_m=\frac{C_{mh}}{T}\left\{\begin{array}{l}\int_0^{t_m}e^{-kt}I_{m1}(t)dt+\int_{t_m}^{t_r}e^{-kt}I_{m2}(t)dt\\ +\int_{t_r}^{T}e^{-kt}I_{m3}(t)dt\end{array}\right\}+\frac{C_{rh}}{T}\left\{\begin{array}{l}\int_0^{t_m}e^{-kt}I_{m4}(t)dt+\\ \int_{t_m}^{t_r}e^{-kt}I_{m5}(t)dt+\end{array}\right\}$$

$$HC_m=\frac{C_{mh}}{T}\left\{\begin{array}{l}\frac{\left((1-\sigma)P-D\right)}{\theta\left(1-m(\zeta)\right)}\left(\frac{\left(1-e^{-kt_m}\right)}{k}+\frac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}-1\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}+\frac{\left((1-\sigma_r)R-D\right)}{\theta\left(1-m(\zeta)\right)}\right)\\ \left(\frac{\left(e^{-kt_m}-e^{-kt_r}\right)}{k}+e^{\theta\left(1-m(\zeta)\right)t_m}\left(\frac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_r}-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)\right)+\\ \frac{\left((1-\sigma)P-D\right)}{\theta\left(1-m(\zeta)\right)}\left(e^{\theta\left(1-m(\zeta)\right)t_m}-1\right)\left(\frac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_r}-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\\ \frac{D}{\theta\left(1-m(\zeta)\right)}\left(\frac{\left(e^{-kT}-e^{-kt_r}\right)}{k}+e^{\theta\left(1-m(\zeta)\right)T}\left(\frac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_r}-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)T}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)\right)\end{array}\right\}$$

$$+\frac{C_{rh}}{T}\begin{Bmatrix}\frac{\sigma P}{\theta(1-m(\zeta))}\left(\frac{\left(1-e^{-kt_m}\right)}{k}+\frac{\left(e^{-(k+\theta(1-m(\zeta)))t_m}-1\right)}{\left(k+\theta(1-m(\zeta))\right)}\right)+\frac{R}{\theta(1-m(\zeta))}\\ e^{\theta(1-m(\zeta))t_r}\left(\frac{\left(e^{-(k+\theta(1-m(\zeta)))t_m}-e^{-(k+\theta(1-m(\zeta)))t_r}\right)}{\left(k+\theta(1-m(\zeta))\right)}\right)+\frac{\left(e^{-kt_r}-e^{-kt_m}\right)}{k}\end{Bmatrix}$$

(e) Reworking Cost:

$$RC_m=\frac{C_{mr}}{T}\int_{t_m}^{t_r}Re^{-kt}dt=\frac{C_{mr}R}{kT}\left(e^{-kt_m}-e^{-kt_r}\right)$$

(f) Waste Scrapping Cost:

$$WS_m=\frac{C_{mw}}{T}\int_{t_m}^{t_r}R\sigma_r e^{-kt}dt=\frac{C_{mw}\sigma_r R}{kT}\left(e^{-kt_m}-e^{-kt_r}\right)$$

(g) Quality Improvement Cost:

$$QI_m=\frac{\omega}{\delta}\log\left(\frac{\sigma_o}{\sigma}\right)\int_0^T e^{-kt}dt=\frac{\omega}{k\delta}\left(1-e^{-kT}\right)\log\left(\frac{\sigma_o}{\sigma}\right)$$

(h) Carbon emission cost for producer's sector: From manufacturer's sector energy and carbon emits because of production process, set-up for manufacturing, deterioration, holding inventory, reworking and scrapping process. Therefore the total amount of carbon emission for manufacturer is

$$TE_m=\begin{bmatrix}\frac{\left(C'_{sp}+C'_{sr}\right)}{T}+\frac{PC'_{mp}}{kT}\left(1-e^{-kt_m}\right)+\frac{\left(C'_{md}\theta(1-m(\zeta))+C'_{mh}\right)}{T}\\ \begin{Bmatrix}\frac{\left((1-\sigma)P-D\right)}{\theta(1-m(\zeta))}\left(\frac{\left(1-e^{-kt_m}\right)}{k}+\frac{\left(e^{-(k+\theta(1-m(\zeta)))t_m}-1\right)}{\left(k+\theta(1-m(\zeta))\right)}\right)+\frac{\left((1-\sigma_r)R-D\right)}{\theta(1-m(\zeta))}\\ \left(\frac{\left(e^{-kt_m}-e^{-kt_r}\right)}{k}+e^{\theta(1-m(\zeta))t_m}\left(\frac{\left(e^{-(k+\theta(1-m(\zeta)))t_r}-e^{-(k+\theta(1-m(\zeta)))t_m}\right)}{\left(k+\theta(1-m(\zeta))\right)}\right)\right)+\\ \frac{\left((1-\sigma)P-D\right)}{\theta(1-m(\zeta))}\left(e^{\theta(1-m(\zeta))t_m}-1\right)\left(\frac{\left(e^{-(k+\theta(1-m(\zeta)))t_r}-e^{-(k+\theta(1-m(\zeta)))t_m}\right)}{\left(k+\theta(1-m(\zeta))\right)}\right)+\\ \frac{D}{\theta(1-m(\zeta))}\left(\frac{\left(e^{-kT}-e^{-kt_r}\right)}{k}+e^{\theta(1-m(\zeta))T}\left(\frac{\left(e^{-(k+\theta(1-m(\zeta)))t_r}-e^{-(k+\theta(1-m(\zeta)))T}\right)}{\left(k+\theta(1-m(\zeta))\right)}\right)\right)\end{Bmatrix}\end{bmatrix}$$

$$\left[\begin{array}{l} +\left\{\begin{array}{l} \dfrac{\sigma P}{\theta\left(1-m(\zeta)\right)}\left(\dfrac{\left(1-e^{-kt_m}\right)}{k}+\dfrac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}-1\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\dfrac{R}{\theta\left(1-m(\zeta)\right)} \\ \left(e^{\theta\left(1-m(\zeta)\right)t_r}\left(\dfrac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_r}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\dfrac{\left(e^{-kt_r}-e^{-kt_m}\right)}{k}\right) \end{array}\right\} \\ \dfrac{\left(C'_{rd}\theta\left(1-m(\zeta)\right)+C'_{rh}\right)}{T}+\left(\dfrac{C'_{mr}R}{kT}+\dfrac{C'_{mw}\sigma_r R}{kT}\right)\left(e^{-kt_m}-e^{-kt_r}\right) \end{array}\right] \tag{11}$$

Now the cost associated with the emission and energy from manufacturing sector under a carbon tax policy is $CE_m = \Phi x(TE_m)$.

After investment in green technology, total carbon emission cost for producer becomes $TCE_m = (1-\pi)(CE_m) = \Phi x(1-\pi)(TE_m)$, where $\pi = \lambda(1-e^{-\mu G})$.

(i) Preservation Investment Cost:

$$PT_m = \zeta\int_0^T e^{-kt}dt = \frac{\zeta}{k}\left(1-e^{-kT}\right)$$

(j) Green Technology Cost:

$$GT_m = G\int_0^T e^{-kt}dt = \frac{G}{k}\left(1-e^{-kT}\right)$$

Thus the producer's total cost is

$$MTC = SC_m + PC_m + DC_m + HC_m + RC_m + WS_m + QI_m + PT_m + GT_m + TEC_m$$

$$MTC = \frac{1}{T}\left[\begin{array}{l} \left(\left(C_{sp}+C_{sr}\right)+\Phi_x\left(1-\pi\right)\left(C'_{sp}+C'_{sr}\right)\right)+\dfrac{P\left(C_{mp}+\Phi_x\left(1-\pi\right)C'_{mp}\right)\left(1-e^{-kt_m}\right)}{k} \\ +\left(\left(C_{mh}+\Phi_x\left(1-\pi\right)C'_{mh}\right)+\theta\left(1-m(\zeta)\right)\left(C_{md}+\Phi_x\left(1-\pi\right)C'_{md}\right)\right) \\ \left\{\begin{array}{l} \dfrac{\left(\left(1-\sigma\right)P-D\right)}{\theta\left(1-m(\zeta)\right)}\left(\dfrac{\left(1-e^{-kt_m}\right)}{k}+\dfrac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}-1\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\dfrac{\left(\left(1-\sigma_r\right)P-D\right)}{\theta\left(1-m(\zeta)\right)} \\ \left(\dfrac{\left(e^{-kt_m}-e^{-kt_r}\right)}{k}+e^{\theta\left(1-m(\zeta)\right)t_m}\left(\dfrac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_r}-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)\right)+ \\ \dfrac{\left(\left(1-\sigma\right)P-D\right)}{\theta\left(1-m(\zeta)\right)}\left(e^{\theta\left(1-m(\zeta)\right)t_m}-1\right)\left(\dfrac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_r}-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+ \\ \dfrac{D}{\theta\left(1-m(\zeta)\right)}\left(\dfrac{\left(e^{-kT}-e^{-kt_r}\right)}{k}+\dfrac{e^{\theta\left(1-m(\zeta)\right)T}\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_r}-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)T}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right) \end{array}\right\} \end{array}\right]$$

$$\left[\begin{array}{l} +\left(\left(C_{rh}+\Phi_x(1-\pi)C'_{mh}\right)+\theta(1-m(\zeta))\left(C_{rd}+\Phi_x(1-\pi)C'_{md}\right)\right) \\ \left\{\begin{array}{l} \left(\dfrac{\sigma P}{\theta(1-m(\zeta))}\left(\dfrac{\left(1-e^{-kt_m}\right)}{k}+\dfrac{\left(e^{-(k+\theta(1-m(\zeta)))t_m}-1\right)}{(k+\theta(1-m(\zeta)))}\right)+\dfrac{R}{\theta(1-m(\zeta))}\right) \\ \left(e^{\theta(1-m(\zeta))t_r}\left(\dfrac{\left(e^{-(k+\theta(1-m(\zeta)))t_m}-e^{-(k+\theta(1-m(\zeta)))t_r}\right)}{(k+\theta(1-m(\zeta)))}\right)+\dfrac{\left(e^{-kt_r}-e^{-kt_m}\right)}{k}\right) \end{array}\right\} \\ +\left(\dfrac{\left(C_{mr}+\Phi_x(1-\pi)C'_{mr}\right)R}{k}+\dfrac{\left(C_{mw}+\Phi_x(1-\pi)C'_{mw}\right)\sigma_r R}{k}\right)\left(e^{-kt_m}-e^{-kt_r}\right) \\ +\dfrac{\omega T}{k\delta}\left(1-e^{-kT}\right)\log\left(\dfrac{\sigma_o}{\sigma}\right)+\dfrac{\zeta T}{k}\left(1-e^{-kT}\right)+\dfrac{GT}{k}\left(1-e^{-kT}\right) \end{array}\right] \tag{12}$$

4.1.2 *Retailer's inventory model*

The behavior of retailer's inventory of deteriorating items with respect to time is shown in Figures 3. At time $t = 0$, the retailer receive an order of quantity Q from the producer which starts decline due to demand and deterioration and becomes zero at $t = t_b$. Here, the manufacturer supplies the inventory to the retailer in m shipments.

The inventory level of retailer's inventory is represented by following differential equation:

$$I'_b(t) = -D(p, E) - \theta(1 - m(\zeta))I_b(t),\ 0 \le t \le t_b \tag{13}$$

under the boundary condition $I_{b1}(0) = Q$ and $I_{b1}(t_b) = 0$.

Using the boundary condition $I_{b1}(t_b) = 0$, the inventory level for retailer at time t is

$$I_b(t) = \frac{D}{\theta(1-m(\zeta))}\left(e^{\theta(1-m(\zeta))(t_b-t)}-1\right) \tag{14}$$

At $t = 0$, the maximum inventory receive from the producer

$$Q = \frac{D}{\theta(1-m(\zeta))}\left(e^{\theta(1-m(\zeta))t_b}-1\right) \tag{15}$$

Now the retailer's inventory model consist following sub-costs

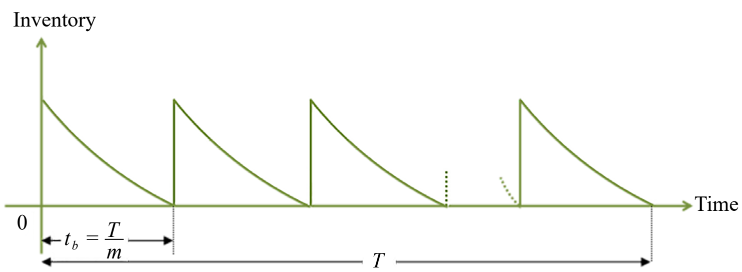

Figure 3. Representation of retailer's inventory level

(a) Ordering Cost:

$$OC_b = \frac{mC_{bo}}{T}$$

(b) Purchasing Cost:

$$PC_b = \frac{mC_{bp}}{T}Q = \frac{D_m C_{bp}}{\theta T\left(1 - m(\zeta)\right)}\left(e^{\theta\left(1-m(\zeta)\right)t_b} - 1\right)$$

(c) Holding Cost:

$$HC_b = \frac{mC_{bh}}{T}\int_0^{t_b} e^{-kt} I_b(t)dt$$

$$HC_b = \frac{mC_{bh}D}{\theta T\left(1 - m(\zeta)\right)}\left(e^{\theta\left(1-m(\zeta)\right)t_b}\left(\frac{\left(1 - e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_b}\right)}{\left(k + \theta\left(1 - m(\zeta)\right)\right)}\right) + \frac{\left(e^{-kt_b} - 1\right)}{k}\right)$$

(d) Deterioration Cost:

$$DC_b = \frac{mC_{bd}}{T}\int_0^{t_b} \theta\left(1 - m(\zeta)\right)e^{-kt} I_b(t)dt$$

$$DC_b = \frac{mC_{bd}D}{T}\left(e^{\theta\left(1-m(\zeta)\right)t_b}\left(\frac{\left(1 - e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_b}\right)}{\left(k + \theta\left(1 - m(\zeta)\right)\right)}\right) + \frac{\left(e^{-kt_b} - 1\right)}{k}\right)$$

(e) Transportation Cost for the retailer:

$$TC_b = \frac{m}{T}\left\{u + v\left(2df_1 + \frac{df_2 D}{\theta\left(1 - m(\zeta)\right)}\left(e^{\theta\left(1-m(\zeta)\right)t_b} - 1\right)\right)\right\}$$

(f) Preservation Investment Cost by retailer:

$$PT_b = m\zeta\int_0^{t_b} e^{-kt}dt = \frac{m\zeta}{k}\left(1 - e^{-kt_b}\right)$$

(g) Green Technology Cost by retailer:

$$GT_b = mG\int_0^{T} e^{-kt}dt = \frac{mG}{k}\left(1 - e^{-kt_b}\right)$$

(h) Carbon emission cost from retailer's sector: From retailer's sector energy and carbon emits because of transportation, carrying inventory, and deterioration. Therefore the total amount of carbon emission for retailer is

$$TE_b = \begin{bmatrix} \frac{m}{T}\left(2de_1 + \frac{de_2 D}{\theta\left(1-m(\zeta)\right)}\left(e^{\theta\left(1-m(\zeta)\right)t_b}-1\right)\right) \\ \frac{mC'_{bh}D}{\theta T\left(1-m(\zeta)\right)}\left(e^{\theta\left(1-m(\zeta)\right)t_b}\left(\frac{\left(1-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_b}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\frac{\left(e^{-kt_b}-1\right)}{k}\right) \\ \frac{mC'_{bd}D}{T}\left(e^{\theta\left(1-m(\zeta)\right)t_b}\left(\frac{\left(1-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_b}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\frac{\left(e^{-kt_b}-1\right)}{k}\right) \end{bmatrix} \tag{16}$$

Now the cost associated with the emission and energy from retailer's sector under a carbon tax policy is $CE_b = \Phi x(TE_b)$.

After investment in green technology, total carbon emission cost for retailer becomes

$$TCE_b = (1 - \lambda(1 - e^{-\mu G}))(CE_b) = \Phi x(1 - \lambda(1 - e^{-\mu G}))(TE_b)$$

$$= \Phi_x\left(1-\lambda\left(1-e^{-\mu G}\right)\right)\begin{bmatrix} \frac{m}{T}\left(2df_{e1} + \frac{df_{e2} D}{\theta\left(1-m(\zeta)\right)}\left(e^{\theta\left(1-m(\zeta)\right)t_b}-1\right)\right) \\ \frac{mC'_{bh}D}{\theta T\left(1-m(\zeta)\right)}\left(e^{\theta\left(1-m(\zeta)\right)t_b}\left(\frac{\left(1-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_b}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\frac{\left(e^{-kt_b}-1\right)}{k}\right) \\ \frac{mC'_{bd}D}{T}\left(e^{\theta\left(1-m(\zeta)\right)t_b}\left(\frac{\left(1-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_b}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\frac{\left(e^{-kt_b}-1\right)}{k}\right) \end{bmatrix}$$

Thus the retailer's total cost is

$$BTC = OC_b + PC_b + DC_b + HC_b + TC_b + PT_m + GT_m + TEC_b$$

$$BTC = \frac{m}{T}\begin{bmatrix} C_{bo} + \frac{DC_{bp}}{\theta\left(1-m(\zeta)\right)}\left(e^{\theta\left(1-m(\zeta)\right)t_b}-1\right) + \\ \frac{D\left(\left(C_{bh}+\Phi_x\left(1-\pi\right)C'_{bh}\right)\right)}{\theta\left(1-m(\zeta)\right)}\left(e^{\theta\left(1-m(\zeta)\right)t_b}\left(\frac{\left(1-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_b}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\frac{\left(e^{-kt_b}-1\right)}{k}\right) + \\ D\left(C_{bd}+\Phi_x\left(1-\pi\right)C'_{bd}\right)\left(e^{\theta\left(1-m(\zeta)\right)t_b}\left(\frac{\left(1-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_b}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\frac{\left(e^{-kt_b}-1\right)}{k}\right) \\ +\left\{u+v\left(2df_1+\frac{df_2 D}{\theta\left(1-m(\zeta)\right)}\left(e^{\theta\left(1-m(\zeta)\right)t_b}-1\right)\right)\right\}+\frac{m\left(\zeta+G\right)T}{k}\left(1-e^{-kt_b}\right) \end{bmatrix}$$

$$\left[+\Phi_x\left(1-\pi\right)\left(2df_{e1}+\frac{df_{e2}D}{\theta\left(1-m(\zeta)\right)}\left(e^{\theta\left(1-m(\zeta)\right)t_b}-1\right)\right)\right] \tag{17}$$

Now the total cost of the supply chain is $STC = MTC + BTC$.

$$STC=\frac{1}{T}\left[\begin{array}{l}
\left(\left(C_{sp}+C_{sr}\right)+\Phi_x\left(1-\pi\right)\left(C'_{sp}+C'_{sr}\right)\right)+\dfrac{P\left(C_{mp}+\Phi_x\left(1-\pi\right)C'_{mp}\right)\left(1-e^{-kt_m}\right)}{k}\\
+\left(\left(C_{mh}+\Phi_x\left(1-\pi\right)C'_{mh}\right)+\theta\left(1-m(\zeta)\right)\left(C_{md}+\Phi_x\left(1-\pi\right)C'_{md}\right)\right)\\
\left\{\begin{array}{l}
\dfrac{\left(\left(1-\sigma\right)P-D\right)}{\theta\left(1-m(\zeta)\right)}\left(\dfrac{\left(1-e^{-kt_m}\right)}{k}+\dfrac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}-1\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\dfrac{\left(\left(1-\sigma_r\right)R-D\right)}{\theta\left(1-m(\zeta)\right)}\\
\left(\dfrac{e^{-kt_m}-e^{-kt_r}}{k}+e^{\theta\left(1-m(\zeta)\right)t_m}\left(\dfrac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_r}-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)\right)+\\
\dfrac{\left(\left(1-\sigma\right)P-D\right)}{\theta\left(1-m(\zeta)\right)}\left(e^{\theta\left(1-m(\zeta)\right)t_m}-1\right)\left(\dfrac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_r}-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\\
\dfrac{D}{\theta\left(1-m(\zeta)\right)}\left(\dfrac{\left(e^{-kT}-e^{-kt_r}\right)}{k}+\left(\dfrac{e^{\theta\left(1-m(\zeta)\right)T}\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_r}-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)T}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)\right)
\end{array}\right\}\\
+\left(\left(C_{rh}+\Phi_x\left(1-\pi\right)C'_{mh}\right)+\theta\left(1-m(\zeta)\right)\left(C_{rd}+\Phi_x\left(1-\pi\right)C'_{md}\right)\right)\\
\left\{\begin{array}{l}
\dfrac{\sigma P}{\theta\left(1-m(\zeta)\right)}\left(\dfrac{\left(1-e^{-kt_m}\right)}{k}+\dfrac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}-1\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\dfrac{R}{\theta\left(1-m(\zeta)\right)}\\
\left(e^{\theta\left(1-m(\zeta)\right)t_r}\left(\dfrac{\left(e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_m}-e^{-\left(k+\theta\left(1-m(\zeta)\right)\right)t_r}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}\right)+\dfrac{\left(e^{-kt_r}-e^{-kt_m}\right)}{k}\right)
\end{array}\right\}\\
+R\left(\dfrac{\left(C_{mr}+\Phi_x\left(1-\pi\right)C'_{mr}\right)}{k}+\dfrac{\left(C_{mw}+\Phi_x\left(1-\pi\right)C'_{mw}\right)\sigma_r}{k}\right)\left(e^{-kt_m}-e^{-kt_r}\right)\\
+\left(\dfrac{\omega T}{k\delta}\log\left(\dfrac{\sigma_o}{\sigma}\right)+\dfrac{\left(\zeta+G\right)T}{k}\right)\left(1-e^{-kT}\right)+mC_{bo}+\dfrac{mDC_{bp}\left(e^{\theta\left(1-m(\zeta)\right)t_b}-1\right)}{\theta\left(1-m(\zeta)\right)}+\\
\dfrac{mD\left(\left(C_{bh}+\Phi_x\left(1-\pi\right)C'_{bh}\right)+\left(C_{bd}+\Phi_x\left(1-\pi\right)C'_{bd}\right)\right)}{\theta\left(1-m(\zeta)\right)}\left(\dfrac{\left(e^{\theta\left(1-m(\zeta)\right)t_b}-e^{-kt_b}\right)}{\left(k+\theta\left(1-m(\zeta)\right)\right)}+\dfrac{\left(e^{-kt_b}-1\right)}{k}\right)\\
+\dfrac{m\left(\zeta+G\right)T}{k}\left(1-e^{-kt_b}\right)+m\Phi_x\left(1-\pi\right)\left(2df_{e1}+\dfrac{df_{e2}D\left(e^{\theta\left(1-m(\zeta)\right)t_b}-1\right)}{\theta\left(1-m(\zeta)\right)}\right)
\end{array}\right] \tag{18}$$

4.2 *Supply chain model in fuzzy learning environment*

Since it is difficult to set the parameters precisely, we considered that the inflation rate k could fluctuate up to a certain point under a learning effect τ.

Let k represents a triangular fuzzy number as follows

$$\tilde{k} = (k_1, k_2, k_3) = (k - \Delta_1 j^{-\tau}, k, k + \Delta_2 j^{-\tau})$$

where j denotes the repetitions of task and τ represents the learning component.

Under this situation, total cost of the supply chain is $\widetilde{STC} = \left(\widetilde{STC}_1, \widetilde{STC}_2, \widetilde{STC}_3\right)$.

Here, centroid method is used to defuzzify the total cost of the supply chain. Thus, defuzzify total cost is as follows:

$$\widetilde{STC}_d = \frac{\widetilde{STC}_1 + \widetilde{STC}_2 + \widetilde{STC}_3}{3}$$

where

$$\widetilde{STC}_i = \frac{1}{T}\left[\begin{array}{l}
\left(\left(C_{sp} + C_{sr}\right) + \Phi_x(1-\pi)\left(C'_{sp} + C'_{sr}\right)\right) + \dfrac{P\left(C_{mp} + \Phi_x(1-\pi)C'_{mp}\right)\left(1 - e^{-\tilde{k}_i t_m}\right)}{\tilde{k}_i} \\
\left(\left(C_{mh} + \Phi_x(1-\pi)C'_{mh}\right) + \theta(1-m(\zeta))\left(C_{md} + \Phi_x(1-\pi)C'_{md}\right)\right) \\
\left\{\begin{array}{l}
\dfrac{((1-\sigma)P - D)}{\theta(1-m(\zeta))}\left(\dfrac{\left(1 - e^{-\tilde{k}_i t_m}\right)}{\tilde{k}_i} + \dfrac{\left(e^{-(\tilde{k}_i + \theta(1-m(\zeta)))t_m} - 1\right)}{\left(\tilde{k}_i + \theta(1-m(\zeta))\right)}\right) + \dfrac{((1-\sigma_r)R - D)}{\theta(1-m(\zeta))} \\
\left(\dfrac{\left(e^{-\tilde{k}_i t_m} - e^{-\tilde{k}_i t_r}\right)}{\tilde{k}_i} + \left(\dfrac{e^{\theta(1-m(\zeta))t_m}\left(e^{-(\tilde{k}_i + \theta(1-m(\zeta)))t_r} - e^{-(\tilde{k}_i + \theta(1-m(\zeta)))t_m}\right)}{\left(\tilde{k}_i + \theta(1-m(\zeta))\right)}\right)\right) \\
+\dfrac{((1-\sigma)P - D)\left(e^{\theta(1-m(\zeta))t_m} - 1\right)}{\theta(1-m(\zeta))}\left(\dfrac{\left(e^{-(\tilde{k}_i + \theta(1-m(\zeta)))t_r} - e^{-(\tilde{k}_i + \theta(1-m(\zeta)))t_m}\right)}{\left(\tilde{k}_i + \theta(1-m(\zeta))\right)}\right) \\
+\left(\dfrac{D\left(e^{-\tilde{k}_i T} - e^{-\tilde{k}_i t_r}\right)}{\tilde{k}_i + \theta(1-m(\zeta))} + \dfrac{De^{\theta(1-m(\zeta))T}}{\theta(1-m(\zeta))}\left(\dfrac{\left(e^{-(\tilde{k}_i + \theta(1-m(\zeta)))t_r} - e^{-(\tilde{k}_i + \theta(1-m(\zeta)))T}\right)}{\left(\tilde{k}_i + \theta(1-m(\zeta))\right)}\right)\right)
\end{array}\right\} \\
+\left(\left(C_{rh} + \Phi_x(1-\pi)C'_{mh}\right) + \theta(1-m(\zeta))\left(C_{rd} + \Phi_x(1-\pi)C'_{md}\right)\right) \\
\left\{\begin{array}{l}
\dfrac{\sigma P}{\theta(1-m(\zeta))}\left(\dfrac{\left(1 - e^{-\tilde{k}_i t_m}\right)}{\tilde{k}_i} + \dfrac{\left(e^{-(\tilde{k}_i + \theta(1-m(\zeta)))t_m} - 1\right)}{\left(\tilde{k}_i + \theta(1-m(\zeta))\right)}\right) + \dfrac{R}{\theta(1-m(\zeta))} \\
\left(\left(\dfrac{e^{\theta(1-m(\zeta))t_r}\left(e^{-(\tilde{k}_i + \theta(1-m(\zeta)))t_m} - e^{-(\tilde{k}_i + \theta(1-m(\zeta)))t_r}\right)}{\left(\tilde{k}_i + \theta(1-m(\zeta))\right)}\right) + \dfrac{\left(e^{-\tilde{k}_i t_r} - e^{-\tilde{k}_i t_m}\right)}{\tilde{k}_i}\right)
\end{array}\right\} \\
+R\left(\dfrac{\left(C_{mr} + \Phi_x(1-\pi)C'_{mr}\right)}{\tilde{k}_i} + \dfrac{\left(C_{mw} + \Phi_x(1-\pi)C'_{mw}\right)\sigma_r}{\tilde{k}_i}\right)\left(e^{-\tilde{k}_i t_m} - e^{-\tilde{k}_i t_r}\right)
\end{array}\right]$$

$$\left[\begin{array}{l}+\left(\frac{\omega T}{\tilde{k}_i\delta}\log\left(\frac{\sigma_o}{\sigma}\right)+\frac{(\zeta+G)T}{\tilde{k}_i}\right)\left(1-e^{-\tilde{k}_iT}\right)+mC_{bo}+\frac{mDC_{bp}\left(e^{\theta(1-m(\zeta))t_b}-1\right)}{\theta(1-m(\zeta))}\\ +\frac{mD\left(\left(C_{bh}+\Phi_x(1-\pi)C'_{bh}\right)+\left(C_{bd}+\Phi_x(1-\pi)C'_{bd}\right)\right)}{\theta(1-m(\zeta))}\left(\frac{\left(e^{\theta(1-m(\zeta))t_b}-e^{-\tilde{k}_it_b}\right)}{\left(\tilde{k}_i+\theta(1-m(\zeta))\right)}+\frac{\left(e^{-\tilde{k}_it_b}-1\right)}{\tilde{k}_i}\right)\\ +\frac{m(\zeta+G)T\left(1-e^{-\tilde{k}_it_b}\right)}{\tilde{k}_i}+m\Phi_x(1-\pi)\left(2df_{e1}+\frac{df_{e2}D\left(e^{\theta(1-m(\zeta))t_b}-1\right)}{\theta(1-m(\zeta))}\right)\end{array}\right]$$

Here $i = 1, 2, 3$.

5. Solution methodology and theoretical results

5.1 *Solution methodology*

In this case, the objective function is nonlinear with respect to the decision variables. Consequently, conventional optimization methods are inadequate for deriving an optimal solution. The process for obtaining the optimum solution is outlined in this section.

Following steps have been followed to obtain the optimal solution:

Step 1: Find the first order partial derivatives of the total cost for the supply chain:

$$\frac{\partial STC}{\partial T}, \text{ and } \frac{\partial STC}{\partial G}.$$

Step 2: Satisfy the necessary conditions for optimality:

$$\frac{\partial STC}{\partial G} = 0, \text{ and } \frac{\partial STC}{\partial T} = 0.$$

Step 3: Obtain the solution of above system of equations say (G, T).

Step 4: Determine the nature of the Hessian matrix

$$HM = \begin{bmatrix} \frac{\partial^2 STC}{\partial G^2} & \frac{\partial^2 STC}{\partial G \partial T} \\ \frac{\partial^2 STC}{\partial G \partial T} & \frac{\partial^2 STC}{\partial T^2} \end{bmatrix}.$$

At obtain point (G, T). If $H_{11} > 0$ at $H_{22} > 0$ the point (G, T) Then the objective function is convex in nature and the point (T^*, G^*) gives the optimal solution.

5.2 *Theoretical results*

In this section, we engage in a theoretical exploration of the concavity of the objective function concerning various decision variables. The derivation of these theoretical results is based on the application of certain theorems from Cambini and Martein [50]. According to these theorems, if any function can be expressed as...

$$F(y) = \frac{\varphi(y)}{\psi(y)},$$

where $y \in R^n$.

$F(y)$ is pseudo convex (i.e. strictly convex) function, if $\varphi(y)$ is pseudo convex and differentiable and $\psi(y)$ is positive and affine.

Theorem 1 When the selling price (p) and renewable energy consumption (E) are constant then, the Total cost function of vendor-buyer inventory system $STC(G, T)$ is a pseudo convex function and attained minimum value at (G^*, T^*), if $X \geq 0$ and $XY \geq Ш^2$ where the value of X, Y and $Ш$ are given in Appendix A.

Proof. From equation (18), the total cost function $STC(G, T)$ of the supply chain can be written in the form of a function of T such as

$$STC(G,\ T) = \frac{\psi_1(G,\ T)}{\gamma_1(G,\ T)}, \tag{19}$$

Where

$$\psi_1(G,\ T) = \left(\begin{array}{l}
\left(\left(C_{sp} + C_{sr}\right) + \Phi_x(1-\pi)\left(C'_{sp} + C'_{sr}\right)\right) + \dfrac{P\left(C_{mp} + \Phi_x(1-\pi)C'_{mp}\right)\left(1 - e^{-kt_m}\right)}{k} \\
+\left(\left(C_{mh} + \Phi_x(1-\pi)C'_{mh}\right) + \theta_1\left(C_{md} + \Phi_x(1-\pi)C'_{md}\right)\right) \\
\left\{\begin{array}{l}
\dfrac{\left((1-\sigma)P - D\right)}{\theta_1}\left(\dfrac{\left(1 - e^{-kt_m}\right)}{k} + \dfrac{\left(e^{-(k+\theta_1)t_m} - 1\right)}{(k+\theta_1)}\right) + \dfrac{\left((1-\sigma_r)R - D\right)}{\theta_1} \\
\left(\dfrac{\left(e^{-kt_m} - e^{-kt_r}\right)}{k} + \dfrac{\left(e^{-(k+\theta_1)t_r + \theta_1 t_m} - e^{-kt_m}\right)}{(k+\theta_1)}\right) + \dfrac{\left((1-\sigma)P - D\right)\left(e^{\theta_1 t_m} - 1\right)}{\theta_1} \\
\left(\dfrac{\left(e^{-(k+\theta_1)t_r} - e^{-(k+\theta_1)t_m}\right)}{(k+\theta_1)}\right) + \dfrac{D}{\theta_1}\left(\dfrac{\left(e^{-kT} - e^{-kt_r}\right)}{k} + \left(\dfrac{\left(e^{-(k+\theta_1)t_r + \theta_1 T} - e^{-kT}\right)}{(k+\theta_1)}\right)\right)
\end{array}\right\} + \\
\left(\left(C_{rh} + \Phi_x(1-\pi)C'_{mh}\right) + \theta_1\left(C_{rd} + \Phi_x(1-\pi)C'_{md}\right)\right) \\
\left\{\dfrac{\sigma P}{\theta_1}\left(\dfrac{\left(1 - e^{-kt_m}\right)}{k} + \dfrac{\left(e^{-(k+\theta_1)t_m} - 1\right)}{(k+\theta_1)}\right) + \dfrac{R}{\theta_1}\left(\dfrac{e^{\theta_1 t_r}\left(e^{-(k+\theta_1)t_m} - e^{-(k+\theta_1)t_r}\right)}{(k+\theta_1)} + \dfrac{\left(e^{-kt_r} - e^{-kt_m}\right)}{k}\right)\right\} \\
+R\left(\dfrac{\left(C_{mr} + \Phi_x(1-\pi)C'_{mr}\right)}{k} + \dfrac{\left(C_{mw} + \Phi_x(1-\pi)C'_{mw}\right)\sigma_r}{k}\right)\left(e^{-kt_m} - e^{-kt_r}\right) \\
+\left(\dfrac{\omega T}{k\delta}\log\left(\dfrac{\sigma_o}{\sigma}\right) + \dfrac{(\zeta + G)T}{k}\right)\left(1 - e^{-kT}\right) + mC_{bo} + \dfrac{mD\left(C_{bp}\left(e^{\theta_1 t_b} - 1\right)\right)}{\theta_1} \\
+\dfrac{mD\left(\left(C_{bh} + \Phi_x(1-\pi)C'_{bh}\right) + \theta_1\left(C_{bd} + \Phi_x(1-\pi)C'_{bd}\right)\right)}{\theta_1}\left(\dfrac{\left(e^{\theta_1 t_b} - e^{-kt_b}\right)}{(k+\theta_1)} + \dfrac{\left(e^{-kt_b} - 1\right)}{k}\right) \\
+\dfrac{m(\zeta + G)T\left(1 - e^{-kt_b}\right)}{k} + m\Phi_x(1-\pi)\left(2df_{e1} + \dfrac{df_{e2}D\left(e^{\theta_1 t_b} - 1\right)}{\theta_1}\right)
\end{array}\right) \tag{20}$$

and $\gamma_1(G, T) = T$. (21)

Differentiating $\psi_1(G, T)$ partially w.r. to G, we have

$$
\frac{\partial \psi_1}{\partial G} = \left[\begin{array}{l}
\dfrac{\left(1-e^{-kT}\right)T}{k} - \lambda\mu\Phi_x\left(C'_{sp}+C'_{sr}\right)e^{-G\mu} - \dfrac{P\lambda\mu\Phi_x C'_{mp}\left(1-e^{-kt_1}\right)e^{-\mu G}}{k} + \dfrac{m\left(1-e^{-kt_3}\right)T}{k} \\
+\left(\dfrac{\left(C'_{bh}+\theta_1 C'_{bd}\right)e^{\theta_1 t_3}}{\theta_1}\left(\dfrac{\left(1-e^{-(k+\theta_1)t_3}\right)}{\left(k+\theta_1\right)} - \dfrac{1-e^{-kt_3}}{k}\right) + \left(2df_{e1} + \dfrac{dDf_{e2}\left(e^{\theta_1 t_3}-1\right)}{\theta_1}\right)\right) \\
m\lambda\mu\Phi_x e^{-G\mu} - \dfrac{\lambda\mu\Phi_x\left(C'_{rh}+\theta_1 C'_{rh}\right)e^{-G\mu}}{\theta_1} \\
\left(R\left(e^{\theta_1 t_2}\left(\dfrac{e^{-(k+\theta_1)t_2}-e^{-(k+\theta_1)t_1}}{k+\theta_1}\right)+\dfrac{e^{-kt_2}-e^{-kt_1}}{k}\right)+\sigma P\left(\dfrac{e^{-(k+\theta_1)t_1}-1}{k+\theta_1}+\dfrac{1-e^{-kt_1}}{k}\right)\right) \\
-\left(\begin{array}{l}
D\left(e^{\theta_1 T}\left(\dfrac{e^{-(k+\theta_1)t_2}-e^{-(k+\theta_1)t_1}}{\left(k+\theta_1\right)}\right)+\dfrac{e^{-kT}-e^{-kt_2}}{k}\right)+\dfrac{\left(P-\left(1-\sigma\right)-D\right)\left(e^{\theta_1 t_1}-1\right)}{\left(k+\theta_1\right)} \\
\left(e^{-(k+\theta_1)t_2}-e^{-(k+\theta_1)t_1}\right)+\left(P-\left(1-\sigma\right)-D\right)\left(\dfrac{e^{-(k+\theta_1)t_1}-1}{k+\theta_1}+\dfrac{1-e^{-kt_1}}{k}\right) \\
\left(R\left(1-\sigma_r\right)-D\right)\left(e^{\theta_1 t_1}\left(\dfrac{e^{-(k+\theta_1)t_2}-e^{-(k+\theta_1)t_1}}{k+\theta_1}\right)+\dfrac{e^{-kt_1}-e^{-kt_2}}{k}\right)
\end{array}\right) \\
\dfrac{\lambda\mu\Phi_x\left(C'_{mh}+\theta_1 C'_{md}\right)e^{-G\mu}}{\theta_1} - R\lambda\mu\Phi_x\left(C'_{mr}+\sigma_r C'_{mw}\right)e^{-G\mu}\left(\dfrac{e^{-kt_1}-e^{-kt_2}}{k}\right)
\end{array} \right] \tag{22}
$$

Solving the following equations $\frac{\partial STC}{\partial G} = 0$ with the help of (22), we can find the value G.

Again differentiating equation (22) partially w.r. to G and T, we have

$$
\frac{\partial^2 \psi_1}{\partial G^2} = \left(\begin{array}{l}
\lambda\mu^2\Phi_x\left(C'_{sp}+C'_{sr}\right)e^{-G\mu} - \dfrac{P\lambda\mu^2\Phi_x C'_{mp}\left(1-e^{-kt_1}\right)e^{-\mu G}}{k} + m\lambda\mu^2\Phi_x e^{-G\mu} \\
\left(\dfrac{\left(C'_{bh}+\theta_1 C'_{bd}\right)e^{\theta_1 t_3}}{\theta_1}\left(\dfrac{1-e^{-(k+\theta_1)t_3}}{\left(k+\theta_1\right)} - \dfrac{1-e^{-kt_3}}{k}\right) + \left(2df_{e1} + \dfrac{dDf_{e2}\left(e^{\theta_1 t_3}-1\right)}{\theta_1}\right)\right) + \dfrac{\lambda\mu^2\Phi_x\left(C'_{rh}+\theta_1 C'_{rh}\right)e^{-G\mu}}{\theta_1} \\
\left(R\left(e^{\theta_1 t_2}\left(\dfrac{e^{-(k+\theta_1)t_2}-e^{-(k+\theta_1)t_1}}{k+\theta_1}\right)+\dfrac{e^{-kt_2}-e^{-kt_1}}{k}\right)+\sigma P\left(\dfrac{e^{-(k+\theta_1)t_1}-1}{k+\theta_1}+\dfrac{1-e^{-kt_1}}{k}\right)\right) \\
+\left(\begin{array}{l}
D\left(e^{\theta_1 T}\left(\dfrac{e^{-(k+\theta_1)t_2}-e^{-(k+\theta_1)t_1}}{\left(k+\theta_1\right)}\right)+\dfrac{e^{-kT}-e^{-kt_2}}{k}\right)+\dfrac{\left(P-\left(1-\sigma\right)-D\right)\left(e^{\theta_1 t_1}-1\right)}{\left(k+\theta_1\right)} \\
\left(e^{-(k+\theta_1)t_2}-e^{-(k+\theta_1)t_1}\right)+\left(P-\left(1-\sigma\right)-D\right)\left(\dfrac{e^{-(k+\theta_1)t_1}-1}{k+\theta_1}+\dfrac{1-e^{-kt_1}}{k}\right)+ \\
\left(R\left(1-\sigma_r\right)-D\right)\left(e^{\theta_1 t_1}\left(\dfrac{e^{-(k+\theta_1)t_2}-e^{-(k+\theta_1)t_1}}{k+\theta_1}\right)+\dfrac{e^{-kt_1}-e^{-kt_2}}{k}\right)
\end{array}\right) \\
\dfrac{\lambda\mu^2\Phi_x\left(C'_{mh}+\theta_1 C'_{md}\right)e^{-G\mu}}{\theta_1} - R\lambda\mu^2\Phi_x\left(C'_{mr}+\sigma_r C'_{mw}\right)e^{-G\mu}\left(\dfrac{e^{-kt_1}-e^{-kt_2}}{k}\right)
\end{array} \right) \tag{23}
$$

$$\frac{\partial^2\psi_1}{\partial T\partial G}=\begin{pmatrix}\dfrac{(1-m)\left(1-e^{-kT}\right)}{k}+\left(e^{-kT}-e^{-kt_3}\right)T-D\lambda\mu\Phi_x\left(C'_{mh}+\theta_1C'_{md}\right)e^{-G\mu}\\ \left(\dfrac{e^{\theta_1T}\left(e^{-(k+\theta_1)t_2}-e^{-(k+\theta_1)t_1}\right)}{(k+\theta_1)}-\dfrac{e^{-kT}}{\theta_1}\right)-\lambda\mu\Phi_xe^{\theta_1t_3-G\mu}\\ \left(C'_{bh}+\theta_1C'_{bd}\right)\left(\left(\dfrac{1-e^{-(k+\theta_1)t_3}}{(k+\theta_1)}-\dfrac{1-e^{-kt_3}}{k}\right)+\dfrac{\left(e^{-(k+\theta_1)t_3}-e^{-kt_3}\right)}{\theta_1}\right)+dDf_{e2}\end{pmatrix} \tag{24}$$

Differentiating $\gamma_1(G, T)$ partially w.r. to T, we have

$$\frac{\partial\psi_1}{\partial T}=\begin{pmatrix}(G+\eta)T\left(e^{-kT}+e^{-kt_3}\right)+\dfrac{(G+\eta)\left(\left(1-e^{-kT}\right)+m\left(1-e^{-kt_3}\right)\right)}{k}+De^{\theta_1T}\\ \left(\left(C_{mh}+\theta_1C_{md}\right)+(1-\pi)\Phi_x\left(C'_{mh}+\theta_1C'_{md}\right)\right)\left(\dfrac{\left(e^{-(k+\theta_1)t_2}-e^{-(k+\theta_1)t_1}\right)}{(k+\theta_1)}-\dfrac{e^{-kT}}{\theta_1}\right)\\ +e^{\theta_1t_3}\left(\left(C_{bh}+\theta_1C_{bd}\right)+(1-\pi)\Phi_x\left(C'_{bh}+\theta_1C'_{bd}\right)\right)\left(\dfrac{1-e^{-(k+\theta_1)t_3}}{(k+\theta_1)}-\dfrac{1-e^{-kt_3}}{k}\right)\\ +e^{\theta_1t_3}\left(\dfrac{\left(\left(C_{bh}+\theta_1C_{bd}\right)+(1-\pi)\Phi_x\left(C'_{bh}+\theta_1C'_{bd}\right)\right)\left(e^{-(k+\theta_1)t_3}-e^{-kt_3}\right)}{\theta_1}\right)\\ +De^{\theta_1t_3}\left(C_{bp}+d\left(vf_2+\Phi_x(1-\pi)f_{e2}\right)\right)+\dfrac{\left(kTe^{-kT}+\left(1-e^{-kT}\right)\right)w\log\left[\dfrac{\sigma_o}{\sigma}\right]}{k\delta}\end{pmatrix} \tag{25}$$

Solving the equations $\frac{\partial STC}{\partial T}=0$ with the help of equation (25), we can find the value T.
Again differentiating equation (25) partially w.r. to G and T, we have

$$\frac{\partial^2\psi_1}{\partial T^2}=\begin{pmatrix}(G+\eta)\left((2-kT)e^{-kT}+(2-kt_3)e^{-kt_3}\right)+\mathrm{D}\\ \left(\left(C_{mh}+\theta_1C_{md}\right)+(1-\pi)\Phi_x\left(C'_{mh}+\theta_1C'_{md}\right)\right)\left(\dfrac{\theta_1e^{\theta_1T}\left(e^{-(k+\theta_1)t_2}-e^{-(k+\theta_1)t_1}\right)}{(k+\theta_1)}+\dfrac{ke^{-kT}}{\theta_1}\right)+\theta_1\\ e^{\theta_1t_3}\left(\dfrac{DC_{bp}}{m}+\dfrac{\left(\left(C_{bh}+\theta_1C_{bd}\right)+(1-\pi)\Phi_x\left(C'_{bh}+\theta_1C'_{bd}\right)\right)}{m}\left(\dfrac{1-e^{-(k+\theta_1)t_3}}{(k+\theta_1)}-\dfrac{1-e^{-kt_3}}{k}\right)\right)\\ +\dfrac{(2-kT)e^{-kT}w\log\left[\dfrac{\sigma_o}{\sigma}\right]}{\delta}+e^{\theta_1t_3}\left(\dfrac{2\left(e^{-(k+\theta_1)t_3}-e^{-kt_3}\right)}{1}-\dfrac{\left((k+\theta_1)e^{-(k+\theta_1)t_3}-ke^{-kt_3}\right)}{\theta_1}\right)\\ \dfrac{\left(\left(C_{bh}+\theta_1C_{bd}\right)+(1-\pi)\Phi_x\left(C'_{bh}+\theta_1C'_{bd}\right)\right)}{m}+\dfrac{dD\theta_1e^{\theta_1t_3}}{m}\left(vf_2+\Phi_x(1-\pi)f_{e2}\right)\end{pmatrix} \tag{26}$$

$$\frac{\partial^2\psi_1}{\partial G\partial T}=\frac{\partial^2\psi_1}{\partial T\partial G}$$

$$\frac{\partial^2 \psi_1}{\partial G \partial T} = \left(\begin{array}{l} \frac{(1+m)\left(1-e^{-kT}\right)}{k} + \left(e^{-kT} - e^{-kt_3}\right)T - D\lambda\mu\Phi_x\left(C'_{mh} + \theta_1 C'_{md}\right)e^{-G\mu} \\ \left(\frac{e^{\theta_1 T}\left(e^{-(k+\theta_1)t_2} - e^{-(k+\theta_1)t_1}\right)}{(k+\theta_1)} - \frac{e^{-kT}}{\theta_1} \right) - \lambda\mu\Phi_x e^{\theta_1 t_3 - G\mu} \\ \left(\left(C'_{bh} + \theta_1 C'_{bd}\right)\left(\left(\frac{1-e^{-(k+\theta_1)t_3}}{-(k+\theta_1)} - \frac{1-e^{-kt_3}}{k} \right) + \frac{\left(e^{-(k+\theta_1)t_3} - e^{-kt_3}\right)}{\theta_1} \right) + dDf_{e2} \right) \end{array} \right) \tag{27}$$

Now $H_{11} = \left| \frac{\partial^2 \psi_1}{\partial G^2} \right| = X > 0$.

$$H_{22} = \begin{vmatrix} \frac{\partial^2 \psi_1}{\partial G^2} & \frac{\partial^2 \psi_1}{\partial G \partial T} \\ \frac{\partial^2 \psi_1}{\partial G \partial T} & \frac{\partial^2 \psi_1}{\partial T^2} \end{vmatrix} = \left(\frac{\partial^2 \psi_1}{\partial G^2}\right)\left(\frac{\partial^2 \psi_1}{\partial T^2}\right) - \left(\frac{\partial^2 \psi_1}{\partial G \partial T}\right)^2 = XY - Ш^2$$

$H_{22} > 0$ as $XY \geq Ш^2$. Since $H_{11} > 0$ and $H_{22} > 0$ at the point (G, T), so the function $\psi_1(G, T)$ is convex and differentiable. Also the function $\gamma_1(G, T) = T$ is an affine and strictly positive function. Hence the objective function $STC(G, T)$ is convex in nature and the point (G^*, T^*) gives the optimal solution. Thus the total cost function is converges to minimum value at (G^*, T^*), only if $X \geq 0$ and $XY \geq Ш^2$, otherwise cost function is diverges from minimum.

Theorem 2 In Fuzzy learning environment, for the discrete value of the selling price (p) and renewable energy consumption (E), the total cost function of vendor-buyer inventory system $\widetilde{STC}_d(G, T)$ is a pseudo convex function and converges to minimum value at (G^*, T^*), if $U \geq 0$, and $UX \geq Ж^2$ where the value of U, V and $Ж$ are given in Appendix B.

Proof. The total cost function $\widetilde{STC}_d(G, T)$ of the supply chain in fuzzy learning environment can be written in the form of a function of G and T such as

$$\widetilde{STC}_d(G, T) = \frac{\psi_2(G, T)}{\gamma_2(G, T)}, \tag{28}$$

Where

$$\psi_2 = \left(\begin{array}{l} \left(\left(C_{sp} + C_{sr}\right) + \Phi_x(1-\pi)\left(C'_{sp} + C'_{sr}\right)\right) + \frac{P\left(C_{mp} + \Phi_x(1-\pi)C'_{mp}\right)\left(1-e^{-\tilde{k}t_m}\right)}{\tilde{k}} \\ + \left(\left(C_{mh} + \Phi_x(1-\pi)C'_{mh}\right) + \theta_1\left(C_{md} + \Phi_x(1-\pi)C'_{md}\right)\right) \\ \left\{ \begin{array}{l} \frac{((1-\sigma)P - D)}{\theta_1}\left(\frac{\left(1-e^{-\tilde{k}t_m}\right)}{\tilde{k}} + \frac{\left(e^{-(\tilde{k}+\theta_1)t_m} - 1\right)}{(\tilde{k}+\theta_1)} \right) + \frac{((1-\sigma_r)R - D)}{\theta_1} \\ \left(\frac{\left(e^{-\tilde{k}t_m} - e^{-\tilde{k}t_r}\right)}{\tilde{k}} + \frac{\left(e^{-(\tilde{k}+\theta_1)t_r + \theta_1 t_m} - e^{-\tilde{k}t_m}\right)}{(\tilde{k}+\theta_1)} \right) + \frac{((1-\sigma)P - D)\left(e^{\theta_1 t_m} - 1\right)}{\theta_1} \\ \left(\frac{\left(e^{-(\tilde{k}+\theta_1)t_r} - e^{-(\tilde{k}+\theta_1)t_m}\right)}{(\tilde{k}+\theta_1)} \right) + \frac{D}{\theta_1}\left(\frac{\left(e^{-\tilde{k}T} - e^{-\tilde{k}t_r}\right)}{\tilde{k}} + \left(\frac{\left(e^{-(\tilde{k}+\theta_1)t_r + \theta_1 T} - e^{-\tilde{k}T}\right)}{(\tilde{k}+\theta_1)} \right) \right) \end{array} \right\} + \end{array} \right)$$

$$\left(\begin{array}{l} \left(\left(C_{rh}+\Phi_x(1-\pi)C'_{mh}\right)+\theta_1\left(C_{rd}+\Phi_x(1-\pi)C'_{md}\right)\right) \\ \left\{\dfrac{\sigma P}{\theta_1}\left(\dfrac{\left(1-e^{-\tilde{k}t_m}\right)}{\tilde{k}}+\dfrac{\left(e^{-\left(\tilde{k}+\theta_1\right)t_m}-1\right)}{\left(\tilde{k}+\theta_1\right)}\right)+\dfrac{R}{\theta_1}\left(\dfrac{e^{\theta_1 t_r}\left(e^{-\left(\tilde{k}+\theta_1\right)t_m}-e^{-\left(\tilde{k}+\theta_1\right)t_r}\right)}{\left(\tilde{k}+\theta_1\right)}+\dfrac{\left(e^{-\tilde{k}t_r}-e^{-\tilde{k}t_m}\right)}{\tilde{k}}\right)\right\} \\ +R\left(\dfrac{\left(C_{mr}+\Phi_x(1-\pi)C'_{mr}\right)}{\tilde{k}}+\dfrac{\left(C_{mw}+\Phi_x(1-\pi)C'_{mw}\right)\sigma_r}{\tilde{k}}\right)\left(e^{-\tilde{k}t_m}-e^{-\tilde{k}t_r}\right) \\ +\left(\dfrac{\omega T}{k\delta}\log\left(\dfrac{\sigma_o}{\sigma}\right)+\dfrac{(\zeta+G)T}{\tilde{k}}\right)\left(1-e^{-\tilde{k}T}\right)+mC_{bo}+\dfrac{mD\left(C_{bp}\left(e^{\theta_1 t_b}-1\right)\right)}{\theta_1} \\ +\dfrac{mD\left(\left(C_{bh}+\Phi_x(1-\pi)C'_{bh}\right)+\theta_1\left(C_{bd}+\Phi_x(1-\pi)C'_{bd}\right)\right)}{\theta_1}\left(\dfrac{\left(e^{\theta_1 t_b}-e^{-\tilde{k}t_b}\right)}{\left(\tilde{k}+\theta_1\right)}+\dfrac{\left(e^{-\tilde{k}t_b}-1\right)}{\tilde{k}}\right) \\ +\dfrac{m(\zeta+G)T\left(1-e^{-\tilde{k}t_b}\right)}{\tilde{k}}+m\Phi_x(1-\pi)\left(2df_{e1}+\dfrac{df_{e2}D\left(e^{\theta_1 t_b}-1\right)}{\theta_1}\right) \end{array}\right) \tag{29}$$

where

$$\tilde{k}=\left(k+\frac{(\Delta_2-\Delta_1)}{3}\right)j^{-\tau} \text{ and } \theta_1=\theta\left(1-m(\zeta)\right) \text{ and } \gamma_2(G,\ T)=T. \tag{30}$$

Differentiating $\psi_2(G, T)$ partially w.r. to G, we have

$$\frac{\partial\psi_2}{\partial G}=\left(\begin{array}{l} \dfrac{\left(1-e^{-\tilde{k}T}\right)T}{\tilde{k}}-\lambda\mu\Phi_x\left(C'_{sp}+C'_{sr}\right)e^{-G\mu}-\dfrac{P\lambda\mu\Phi_x C'_{mp}\left(1-e^{-\tilde{k}t_1}\right)e^{-\mu G}}{\tilde{k}}+\dfrac{m\left(1-e^{-\tilde{k}t_3}\right)T}{\tilde{k}} \\ +\left(\dfrac{\left(C'_{bh}+\theta_1 C'_{bd}\right)e^{\theta_1 t_3}}{\theta_1}\left(\dfrac{\left(1-e^{-\left(\tilde{k}+\theta_1\right)t_3}\right)}{\left(\tilde{k}+\theta_1\right)}-\dfrac{1-e^{-\tilde{k}t_3}}{\tilde{k}}\right)+\left(2df_{e1}+\dfrac{dDf_{e2}\left(e^{\theta_1 t_3}-1\right)}{\theta_1}\right)\right) \\ m\lambda\mu\Phi_x e^{-G\mu}-\dfrac{\lambda\mu\Phi_x\left(C'_{rh}+\theta_1 C'_{rh}\right)e^{-G\mu}}{\theta_1} \\ \left(R\left(e^{\theta_1 t_2}\left(\dfrac{e^{-\left(\tilde{k}+\theta_1\right)t_2}-e^{-\left(\tilde{k}+\theta_1\right)t_1}}{\tilde{k}+\theta_1}\right)+\dfrac{e^{-\tilde{k}t_2}-e^{-\tilde{k}t_1}}{\tilde{k}}\right)+\sigma P\left(\dfrac{e^{-\left(\tilde{k}+\theta_1\right)t_1}-1}{\tilde{k}+\theta_1}+\dfrac{1-e^{-\tilde{k}t_1}}{\tilde{k}}\right)\right) \\ \left(\begin{array}{l} D\left(e^{\theta_1 T}\left(\dfrac{e^{-\left(\tilde{k}+\theta_1\right)t_2}-e^{-\left(\tilde{k}+\theta_1\right)t_1}}{\left(\tilde{k}+\theta_1\right)}\right)+\dfrac{e^{-\tilde{k}T}-e^{-\tilde{k}t_2}}{\tilde{k}}\right)+\dfrac{\left(P-(1-\sigma)-D\right)\left(e^{\theta_1 t_1}-1\right)}{\left(\tilde{k}+\theta_1\right)} \\ -\left(e^{-\left(\tilde{k}+\theta_1\right)t_2}-e^{-\left(\tilde{k}+\theta_1\right)t_1}\right)+\left(P-(1-\sigma)-D\right)\left(\dfrac{e^{-\left(\tilde{k}+\theta_1\right)t_1}-1}{\tilde{k}+\theta_1}+\dfrac{1-e^{-\tilde{k}t_1}}{\tilde{k}}\right) \\ \left(R\left(1-\sigma_r\right)-D\right)\left(e^{\theta_1 t_1}\left(\dfrac{e^{-\left(\tilde{k}+\theta_1\right)t_2}-e^{-\left(\tilde{k}+\theta_1\right)t_1}}{\tilde{k}+\theta_1}\right)+\dfrac{e^{-\tilde{k}t_1}-e^{-\tilde{k}t_2}}{\tilde{k}}\right) \end{array}\right) \\ \dfrac{\lambda\mu\Phi_x\left(C'_{mh}+\theta_1 C'_{md}\right)e^{-G\mu}}{\theta_1}-R\lambda\mu\Phi_x\left(C'_{mr}+\sigma_r C'_{mw}\right)e^{-G\mu}\left(\dfrac{e^{-\tilde{k}t_1}-e^{-\tilde{k}t_2}}{\tilde{k}}\right) \end{array}\right) \tag{31}$$

Solving the following equations $\frac{\partial \widetilde{STC}_d}{\partial G} = 0$ with the help of (31), we can find the value G.
Again differentiating equation (31) partially w.r. to G and T, we have

$$\frac{\partial^2 \psi_2}{\partial G^2} = \left(\begin{array}{l} \lambda\mu^2 \Phi_x \left(C'_{sp} + C'_{sr}\right) e^{-G\mu} - \frac{P\lambda\mu^2 \Phi_x C'_{mp}\left(1-e^{-\tilde{k}t_1}\right)e^{-\mu G}}{\tilde{k}} + m\lambda\mu^2\Phi_x e^{-G\mu} \\ \left(\frac{\left(C'_{bh}+\theta_1 C'_{bd}\right)e^{\theta_1 t_3}}{\theta_1}\left(\frac{1-e^{-\left(\tilde{k}+\theta_1\right)t_3}}{\left(\tilde{k}+\theta_1\right)} - \frac{1-e^{-\tilde{k}t_3}}{\tilde{k}} \right) + \left(2df_{e1} + \frac{dDf_{e2}\left(e^{\theta_1 t_3}-1\right)}{\theta_1} \right) \right) + \frac{\lambda\mu^2\Phi_x\left(C'_{rh}+\theta_1 C'_{rh}\right)e^{-G\mu}}{\theta_1} \\ \left(R\left(e^{\theta_1 t_2}\left(\frac{e^{-\left(\tilde{k}+\theta_1\right)t_2} - e^{-\left(\tilde{k}+\theta_1\right)t_1}}{\tilde{k}+\theta_1} \right) + \frac{e^{-\tilde{k}t_2}-e^{-\tilde{k}t_1}}{\tilde{k}} \right) + \sigma P\left(\frac{e^{-\left(\tilde{k}+\theta_1\right)t_1}-1}{\tilde{k}+\theta_1} + \frac{1-e^{-\tilde{k}t_1}}{\tilde{k}} \right) \right) \\ \left(\begin{array}{l} D\left(e^{\theta_1 T}\left(\frac{e^{-\left(\tilde{k}+\theta_1\right)t_2} - e^{-\left(\tilde{k}+\theta_1\right)t_1}}{\left(\tilde{k}+\theta_1\right)} \right) + \frac{e^{-\tilde{k}T}-e^{-\tilde{k}t_2}}{\tilde{k}} \right) + \frac{\left(P-\left(1-\sigma\right)-D\right)\left(e^{\theta_1 t_1}-1\right)}{\left(\tilde{k}+\theta_1\right)} \\ + \left(e^{-\left(\tilde{k}+\theta_1\right)t_2} - e^{-\left(\tilde{k}+\theta_1\right)t_1} \right) + \left(P-\left(1-\sigma\right)-D\right)\left(\frac{e^{-\left(\tilde{k}+\theta_1\right)t_1}-1}{\tilde{k}+\theta_1} + \frac{1-e^{-\tilde{k}t_1}}{\tilde{k}} \right) + \\ \left(R\left(1-\sigma_r\right)-D\right)\left(e^{\theta_1 t_1}\left(\frac{e^{-\left(\tilde{k}+\theta_1\right)t_2} - e^{-\left(\tilde{k}+\theta_1\right)t_1}}{\tilde{k}+\theta_1} \right) + \frac{e^{-\tilde{k}t_1}-e^{-\tilde{k}t_2}}{\tilde{k}} \right) \end{array} \right) \\ \frac{\lambda\mu^2\Phi_x\left(C'_{mh}+\theta_1 C'_{md}\right)e^{-G\mu}}{\theta_1} - R\lambda\mu^2\Phi_x\left(C'_{mr}+\sigma_r C'_{mw}\right)e^{-G\mu}\left(\frac{e^{-\tilde{k}t_1}-e^{-\tilde{k}t_2}}{\tilde{k}} \right) \end{array} \right) \tag{32}$$

$$\frac{\partial^2 \psi_2}{\partial G \partial T} = \left(\begin{array}{l} \frac{\left(1+m\right)\left(1-e^{-\tilde{k}T}\right)}{\tilde{k}} + \left(e^{-\tilde{k}T} - e^{-\tilde{k}t_3}\right)T - D\lambda\mu\Phi_x\left(C'_8 + \theta_1 C'_{10}\right)e^{-G\mu} \\ \left(\frac{e^{\theta_1 T}\left(e^{-\left(\tilde{k}+\theta_1\right)t_2} - e^{-\left(\tilde{k}+\theta_1\right)t_1}\right)}{\left(\tilde{k}+\theta_1\right)} - \frac{e^{-\tilde{k}T}}{\theta_1} \right) - \lambda\mu\Phi_x e^{\theta_1 t_3 - G\mu} \\ \left(\left(C'_{bh}+\theta_1 C'_{bd}\right)\left(\left(\frac{1-e^{-\left(\tilde{k}+\theta_1\right)t_3}}{-\left(\tilde{k}+\theta_1\right)} - \frac{1-e^{-\tilde{k}t_3}}{\tilde{k}} \right) + \frac{\left(e^{-\left(\tilde{k}+\theta_1\right)t_3} - e^{-\tilde{k}t_3}\right)}{\theta_1} \right) + dDf_{e2} \right) \end{array} \right) \tag{33}$$

Differentiating $\psi_2(G, T)$ partially w.r. to T, we have

$$\frac{\partial \psi_2}{\partial T} = \left(\begin{array}{l} \left(G+\eta\right)T\left(e^{-\tilde{k}T} + e^{-\tilde{k}t_3}\right) + \frac{\left(G+\eta\right)\left(\left(1-e^{-\tilde{k}T}\right) + m\left(1-e^{-\tilde{k}t_3}\right)\right)}{\tilde{k}} + De^{\theta_1 T} \\ \left(\left(C_{mh}+\theta_1 C_{md}\right) + \left(1-\pi\right)\Phi_x\left(C'_{mh}+\theta_1 C'_{md}\right)\right)\left(\frac{\left(e^{-\left(\tilde{k}+\theta_1\right)t_2} - e^{-\left(\tilde{k}+\theta_1\right)t_1}\right)}{\left(\tilde{k}+\theta_1\right)} - \frac{e^{-\tilde{k}T}}{\theta_1} \right) \\ + e^{\theta_1 t_3}\left(\left(C_{bh}+\theta_1 C_{bd}\right) + \left(1-\pi\right)\Phi_x\left(C'_{bh}+\theta_1 C'_{bd}\right)\right)\left(\frac{1-e^{-\left(\tilde{k}+\theta_1\right)t_3}}{\left(\tilde{k}+\theta_1\right)} - \frac{1-e^{-\tilde{k}t_3}}{\tilde{k}} \right) \end{array} \right)$$

$$\left(\begin{array}{l} +e^{\theta_1 t_3}\left(\dfrac{\left(\left(C_{bh}+\theta_1 C_{bd}\right)+\left(1-\pi\right)\Phi_x\left(C'_{bh}+\theta_1 C'_{bd}\right)\right)\left(e^{-\left(\tilde{k}+\theta_1\right)t_3}-e^{-\tilde{k}t_3}\right)}{\theta_1} \right) \\ +De^{\theta_1 t_3}\left(C_{bp}+d\left(vf_2+\Phi_x\left(1-\pi\right)f_{e2}\right)\right)+\dfrac{\left(\tilde{k}Te^{-\tilde{k}T}+\left(1-e^{-\tilde{k}T}\right)\right)w\log\left[\dfrac{\sigma_o}{\sigma}\right]}{\tilde{k}\delta} \end{array} \right) \tag{34}$$

Solving the equation $\dfrac{\partial \widetilde{STC}_d}{\partial T}=0$ with the help of equation (34), we can find the value T.
Again differentiating equation (34) partially w.r. to G and T, we have

$$\frac{\partial^2 \psi_2}{\partial T^2}=\left(\begin{array}{l} \left(G+\eta\right)\left(\left(2-\tilde{k}T\right)e^{-\tilde{k}T}+\left(2-\tilde{k}t_3\right)e^{-\tilde{k}t_3}\right)+\mathrm{D} \\ \left(\left(C_{mh}+\theta_1 C_{md}\right)+\left(1-\pi\right)\Phi_x\left(C'_{mh}+\theta_1 C'_{md}\right)\right)\left(\dfrac{\theta_1 e^{\theta_1 T}\left(e^{-\left(\tilde{k}+\theta_1\right)t_2}-e^{-\left(\tilde{k}+\theta_1\right)t_1}\right)}{\left(\tilde{k}+\theta_1\right)}+\dfrac{\tilde{k}e^{-\tilde{k}T}}{\theta_1} \right)+\theta_1 \\ e^{\theta_1 t_3}\left(\dfrac{DC_{bp}}{m}+\dfrac{\left(\left(C_{bh}+\theta_1 C_{bd}\right)+\left(1-\pi\right)\Phi_x\left(C'_{bh}+\theta_1 C'_{bd}\right)\right)}{m}\left(\dfrac{1-e^{-\left(\tilde{k}+\theta_1\right)t_3}}{\left(\tilde{k}+\theta_1\right)}-\dfrac{1-e^{-\tilde{k}t_3}}{\tilde{k}} \right) \right) \\ +\dfrac{\left(2-\tilde{k}T\right)e^{-\tilde{k}T}w\log\left[\dfrac{\sigma_o}{\sigma}\right]}{\delta}+e^{\theta_1 t_3}\left(\dfrac{2\left(e^{-\left(\tilde{k}+\theta_1\right)t_3}-e^{-\tilde{k}t_3}\right)}{1}-\dfrac{\left(\left(\tilde{k}+\theta_1\right)e^{-\left(\tilde{k}+\theta_1\right)t_3}-\tilde{k}e^{-\tilde{k}t_3}\right)}{\theta_1} \right) \\ \dfrac{\left(\left(C_{bh}+\theta_1 C_{bd}\right)+\left(1-\pi\right)\Phi_x\left(C'_{bh}+\theta_1 C'_{bd}\right)\right)}{m}+\dfrac{dD\theta_1 e^{\theta_1 t_3}}{m}\left(vf_2+\Phi_x\left(1-\pi\right)f_{e2}\right) \end{array} \right) \tag{35}$$

$$\frac{\partial^2 \psi_2}{\partial G \partial T}=\frac{\partial^2 \psi_2}{\partial T \partial G}$$

$$\frac{\partial^2 \psi_2}{\partial G \partial T}=\left(\begin{array}{l} \dfrac{\left(1+m\right)\left(1-e^{-\tilde{k}T}\right)}{\tilde{k}}+\left(e^{-\tilde{k}T}-e^{-\tilde{k}t_3}\right)T-D\lambda\mu\Phi_x\left(C'_{mh}+\theta_1 C'_{md}\right)e^{-G\mu} \\ \left(\dfrac{e^{\theta_1 T}\left(e^{-\left(\tilde{k}+\theta_1\right)t_2}-e^{-\left(\tilde{k}+\theta_1\right)t_1}\right)}{\left(\tilde{k}+\theta_1\right)}-\dfrac{e^{-\tilde{k}T}}{\theta_1} \right)-\lambda\mu\Phi_x e^{\theta_1 t_3-G\mu} \\ \left(\left(C'_{bh}+\theta_1 C'_{bd}\right)\left(\left(\dfrac{1-e^{-\left(\tilde{k}+\theta_1\right)t_3}}{-\left(\tilde{k}+\theta_1\right)}-\dfrac{1-e^{-\tilde{k}t_3}}{\tilde{k}} \right)+\dfrac{\left(e^{-\left(\tilde{k}+\theta_1\right)t_3}-e^{-\tilde{k}t_3}\right)}{\theta_1} \right)+dDf_{e2} \right) \end{array} \right) \tag{36}$$

Now $H_{11}=\left|\dfrac{\partial^2 \psi_2}{\partial G^2}\right|=U>0.$

$$H_{22} = \begin{vmatrix} \frac{\partial^2 \psi_2}{\partial G^2} & \frac{\partial^2 \psi_2}{\partial G \partial T} \\ \frac{\partial^2 \psi_2}{\partial G \partial T} & \frac{\partial^2 \psi_2}{\partial T^2} \end{vmatrix} = \left(\frac{\partial^2 \psi_2}{\partial G^2}\right)\left(\frac{\partial^2 \psi_2}{\partial T^2}\right) - \left(\frac{\partial^2 \psi_2}{\partial G \partial T}\right)^2 = UV - Ж^2$$

$H_{22} > 0$ as $UV \geq Ш^2$. Since $H_{11} > 0$ and $H_{22} > 0$ at the point (G, T), so the function $\psi_2(G, T)$ is convex and differentiable. Also the function $\gamma_2(G, T) = T$ is an affine and strictly positive function. Hence the objective function $\widetilde{STC}_d(G, T)$ is convex in nature and the point (G^*, T^*) gives the optimal solution. Thus for fuzzy learning environment, the total cost function is converges to minimum value at (G^*, T^*), only if $U \geq 0$ and $UX \geq Ж^2$, otherwise cost function is diverges from minimum.

6. Numerical and sensitivity analysis

6.1 *Numerical analysis*

The stability and viability of the proposed models are illustrated in this section using a continuous review inventory system. Numerical values for the parameters with appropriate units can be used to illustrate the mathematical model developed as follows:

Example 1 (Crisp Case): $a = 10$, $b = 60$, $p_{max} = 200$, $p = 150$, $p_{min} = 100$, $E_{max} = 220$, $E = 180$, $E_{min} = 150$, $k = 150$, $\theta = 0.4$, $\alpha = 0.04$, $\zeta = 15$, $\lambda = 0.6$, $\mu = 20$, $\Phi = 0.8$, $f_1 = 0.03$, $f_2 = 0.36$, $e_1 = 0.26$, $e_2 = 0.03$, $m = 3$, $u = 10$, $v = 0.01$, $d = 100$, $C_{sp} = 300$, $C_{sr} = 100$, $C'_{sp} = 100$, $C'_{sr} = 50$, $C_{mp} = 300$, $C'_{mp} = 25$, $C_{mh} = 1$, $C'_{mh} = 1$, $C_{md} = 5$, $C'_{md} = 3$, $C_{rh} = 0.5$, $C'_{rh} = 0.3$, $C_{rd} = 0.5$, $C'_{rd} = 0.2$, $C_{mr} = 5$, $C'_{mr} = 20$, $C_{mw} = 4$, $C'_{mw} = 10$, $R = 20$, $C_{bo} = 600$, $C_{bp} = 60$, $C_{bh} = 6$, $C'_{bh} = 0.3$, $C_{bd} = 5$, $C'_{bd} = 0.5$, $\sigma_o = 0.03$, $\sigma = 0.2$, $\sigma_r = 0.3$, $\omega = 200$, $\delta = 2$, $P = 10$, $t_m \rightarrow 2$, $t_r = 3$.

Solution: On applying the said methodology to obtain the optimal solution, we get the optimal solution which is given in Table 3:

Table 3. Optimal solution in crisp sense

G^* (\$/unit/month)	T^* (months)	STC^* (\$)
0.24316	4.6282	5199.02

Here $H_{11} = \frac{\partial^2 STC}{\partial G^2} = 199.37 > 0$ and

$$H_{22} = \begin{vmatrix} \frac{\partial^2 STC}{\partial G^2} & \frac{\partial^2 STC}{\partial G \partial T} \\ \frac{\partial^2 STC}{\partial T \partial G} & \frac{\partial^2 STC}{\partial G^2} \end{vmatrix} = \begin{vmatrix} 199.37 & 11.16 \\ 11.16 & 100.65 \end{vmatrix} = 19942 > 0.$$

Figure 4, Shows the convexity of the total cost function for crisp model.

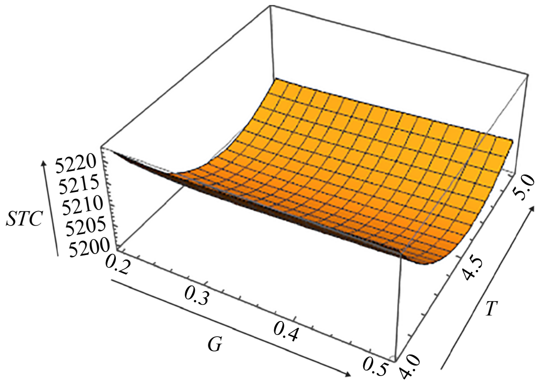

Figure 4. Convexity of the total cost function for crisp environment

Example 2 (Fuzzy Case): Following additional input parameters are used to analyze the production-inventory model in the case of fuzzy:

$$\Delta_2 = 0.04, \Delta_1 = 0.06, j = 1.$$

Solution: On applying the said methodology to obtain the optimal solution, we get the optimal solution for fuzzy case which is represented in Table 4:

Table 4. Optimal solution in fuzzy sense

$\widetilde{G}^*$ (\$/unit/month)	$\widetilde{T}^*$ (months)	$\widetilde{STC}^*$ (\$)
0.237889	4.66555	5188.43

Here $H_{11} = \dfrac{\partial^2 \widetilde{STC}}{\partial G^2} = 185.67 > 0$ and

$$H_{22} = \begin{vmatrix} \dfrac{\partial^2 \widetilde{STC}}{\partial G^2} & \dfrac{\partial^2 \widetilde{STC}}{\partial G \partial T} \\ \dfrac{\partial^2 \widetilde{STC}}{\partial T \partial G} & \dfrac{\partial^2 \widetilde{STC}}{\partial G^2} \end{vmatrix} = \begin{vmatrix} 185.67 & 11.588 \\ 11.588 & 97.23 \end{vmatrix} = 17918.59 > 0.$$

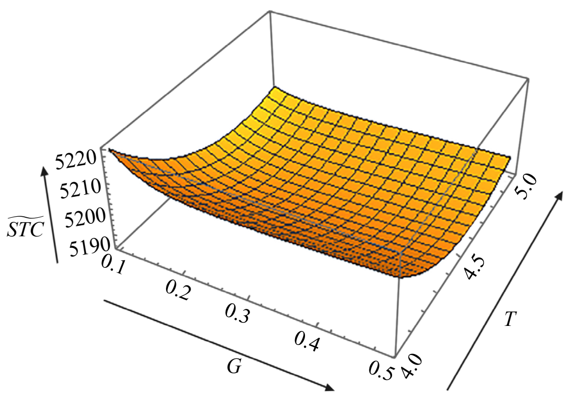

Figure 5. Convexity of the total cost function for fuzzy environment

Figure 5, represents the Convexity of the total cost function for fuzzy model.

Example 3 (Fuzzy Learning Case): Following additional input parameters are used to analyze the production-inventory model in the case of fuzzy learning: In this case number of repetition the task (j) must be greater than 1. Here take $j = 4$ and $\sigma = 0.8$.

Solution: On applying the said methodology to obtain the optimal solution, we get the optimal solution for fuzzy learning case which is represented in Table 5:

Table 5. Optimal solution in fuzzy learning

$\tilde{G}^*$ (\$/unit/month)	$\tilde{T}^*$ (months)	$\widetilde{STC}^*$ (\$)
0.235973	4.67849	5184.79

Here $H_{11} = \frac{\partial^2 \widetilde{STC}}{\partial G^2} = 143.47 > 0$ and

$$H_{22} = \begin{vmatrix} \frac{\partial^2 \widetilde{STC}}{\partial G^2} & \frac{\partial^2 \widetilde{STC}}{\partial G \partial T} \\ \frac{\partial^2 \widetilde{STC}}{\partial T \partial G} & \frac{\partial^2 \widetilde{STC}}{\partial G^2} \end{vmatrix} = \begin{vmatrix} 143.47 & 8.051 \\ 8.051 & 128.72 \end{vmatrix} = 18402.65 > 0.$$

Figure 6, represents the Convexity of the total cost function for fuzzy learning model respectively.

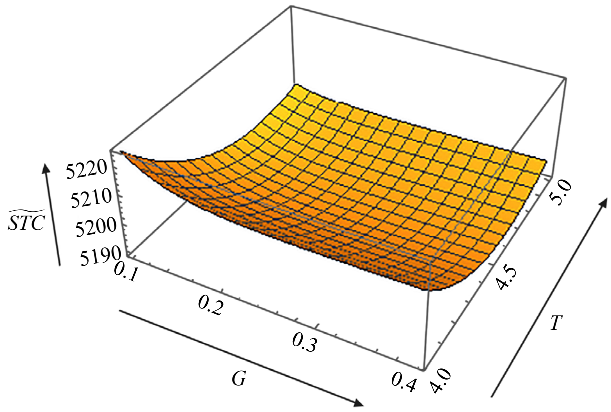

Figure 6. Convexity of the total cost function in fuzzy learning environment

6.2 *Sensitivity analysis*

This section provides a sensitivity analysis for all three cases (crisp, fuzzy, fuzzy learning). To assess the impact of variations in various key parameters related to the supply chain system on total cost changes, the effects of these different parameters visually depicted from Figures 7 to 14 and present a summary in Table 6.

Table 6. Sensitivity analysis

Parameters	% Chan-ges	Crisp Model			Fuzzy Model			Fuzzy Learning Model		
		T	G	STC	T	G	STC	T	G	STC
$\sigma_o = 0.03$	- 20	4.62818	0.2431	5176.70	4.66554	0.23789	5166.11	4.67847	0.23597	5162.47
	- 10	4.62819	0.24316	5188.49	4.66554	0.23789	5177.89	4.67848	0.23597	5174.24
	+ 10	4.62820	0.24316	5208.56	4.66556	0.23789	5197.96	4.67849	0.23597	5194.33
	+ 20	4.62821	0.24316	5217.26	4.66556	0.23789	5206.67	4.67850	0.23597	5203.03
$\sigma = 0.2$	- 20	4.62604	0.24264	5198.09	4.66336	0.23732	5187.41	4.67629	0.23539	5183.77
	- 10	4.62712	0.24290	5198.52	4.66445	0.23761	5187.92	4.67739	0.23568	5184.28
	+ 10	4.62928	0.24341	5199.53	4.66664	0.23817	5188.94	4.67958	0.23626	5185.31
	+ 20	4.63036	0.24366	5200.04	4.66774	0.23845	5189.45	4.68068	0.23655	5185.82
$\sigma_r = 0.03$	- 20	4.62820	0.24316	5221.34	4.66555	0.23789	5210.74	4.67849	0.23597	5207.11
	- 10	4.62820	0.24316	5209.56	4.66555	0.23789	5198.97	4.67849	0.23597	5195.33
	+ 10	4.62820	0.24316	5189.49	4.66555	0.23789	5178.90	4.67849	0.23597	5175.26
	+ 20	4.62820	0.24316	5180.79	4.66555	0.23789	5170.20	4.67849	0.23597	5166.56
$k = 0.01$	- 20	4.63921	0.24164	5195.89	4.67712	0.23618	5185.18	4.68235	0.23539	5183.71
	- 10	4.63368	0.24240	5197.46	4.67131	0.23704	5186.81	4.68041	0.23568	5184.25
	+ 10	4.62275	0.24390	5200.58	4.65983	0.23872	5190.04	4.67656	0.23626	5185.33
	+ 20	4.61734	0.24462	5202.13	4.65415	0.23954	5191.65	4.67464	0.23655	5185.87
$\theta = 0.3$	- 20	5.15138	0.16179	4985.64	5.06963	0.17871	5022.02	5.14727	0.15278	4997.74
	- 10	4.90182	0.21344	5081.94	4.94679	0.20337	5069.81	4.96262	0.19943	5065.64
	+ 10	4.39377	0.26124	5306.25	4.42628	0.25763	5296.89	4.43749	0.25634	5293.68
	+ 20	4.18889	0.27431	5405.85	4.21776	0.27154	5397.47	4.2277	0.27057	5394.61
$p = 0.3$	- 20	3.79270	0.21917	16694.7	3.84401	0.17311	16664.7	3.99016	0.20523	12543.3
	- 10	4.18521	0.24111	8498.62	4.22589	0.23142	8482.06	4.24005	0.227667	8476.36
	+ 10	5.14163	0.24069	3405.17	5.17378	0.23729	3398.35	5.18490	0.23609	3396.00
	+ 20	5.76084	0.23591	2268.61	5.78365	0.233652	2264.70	5.79155	0.23286	2263.36
$E = 180$	- 20	4.24889	0.24194	7828.33	4.28911	0.23330	7812.92	4.30309	0.23000	7807.63
	- 10	4.56811	0.24324	5507.71	4.60596	0.23762	5496.52	4.61907	0.23556	5492.68
	+ 10	4.77670	0.24270	4547.16	4.81274	0.23815	4537.84	4.82521	0.23651	4534.65
	+ 20	4.82376	0.24252	4367.92	4.85935	0.23815	4359.00	4.87167	0.23657	4355.93
$\zeta = 15$	- 20	4.38489	0.26284	5308.54	4.41812	0.25929	5299.03	4.42958	0.25803	5295.77
	- 10	4.50658	0.25373	5253.06	4.54179	0.24948	5243.02	4.55396	0.24796	5239.57
	+ 10	4.74971	0.23036	5146.29	4.78950	0.22347	5135.12	4.80334	0.22090	5131.28
	+ 20	4.87144	0.21380	5094.70	4.91444	0.20389	5082.92	4.92956	0.20003	5078.87
$\Phi_x = 0.8$	- 20	4.55277	0.23628	5190.78	4.58791	0.23149	5180.83	4.60007	0.22976	5177.41
	- 10	4.59062	0.24004	5194.97	4.62686	0.23501	5184.70	4.63941	0.23319	5181.18
	+ 10	4.66550	0.24576	5202.93	4.70398	0.24023	5192.01	4.71732	0.23822	5188.26
	+ 20	4.70255	0.24793	5206.70	4.74217	0.24213	5195.44	4.75591	0.24000	5191.57

Table 6. (cont.)

Parameters	% Chan-ges	Crisp Model			Fuzzy Model			Fuzzy Learning Model		
		T	G	STC	T	G	STC	T	G	STC
$P = 10$	- 20	4.61268	0.23450	5191.48	4.65015	0.22837	5180.90	4.66314	0.22611	5177.26
	- 10	4.62042	0.23904	5195.26	4.65704	0.23289	5184.29	4.67077	0.23130	5181.03
	+ 10	4.63601	0.24695	5202.78	4.67337	0.24200	5192.18	4.68626	0.24021	5188.54
	+ 20	4.64385	0.25044	5206.53	4.68117	0.24577	5195.92	4.69408	0.24409	5192.28
$R = 20$	- 20	4.61609	0.23682	5193.16	4.65337	0.23086	5182.51	4.66629	0.22867	5178.85
	- 10	4.62213	0.24010	5196.10	4.65944	0.23451	5185.47	4.67237	0.23247	5181.82
	+ 10	4.63428	0.24603	5201.97	4.67169	0.24104	5191.38	4.68464	0.23923	5187.75
	+ 20	4.64038	0.24873	5204.87	4.67785	0.24398	5194.33	4.69082	0.24227	5190.71
$t_m = 2$	- 20	4.67898	0.19574	5075.64	4.70861	0.20325	5097.33	4.71737	0.20642	5106.41
	- 10	4.65539	0.22595	5141.05	4.69502	0.21824	5129.55	4.70163	0.22155	5141.10
	+ 10	4.59491	0.25489	5249.73	4.63043	0.25083	5240.01	4.64271	0.24937	5236.67
	+ 20	4.55518	0.26353	5293.24	4.58899	0.26019	5284.36	4.60065	0.25901	5281.31
$C_{sp} = 300$	- 20	4.59961	0.24479	5186.2	4.63689	0.23966	5175.53	4.64979	0.23780	5171.97
	- 10	4.61393	0.24390	5192.53	4.65124	0.23878	5181.99	4.66416	0.23689	5178.37
	+ 10	4.64242	0.24234	5205.50	4.67981	0.23699	5194.85	4.69276	0.23505	5191.20
	+ 20	4.65659	0.24152	5211.95	4.69401	0.23610	5201.25	4.70698	0.23413	5197.58
$C_{mp} = 0.3$	- 20	4.62763	0.24319	5198.77	4.66498	0.23793	5188.17	4.67791	0.23610	5184.54
	- 10	4.62791	0.24317	5198.90	4.66526	0.23791	5188.30	4.67820	0.23599	5184.66
	+ 10	4.62848	0.24314	5199.15	4.66583	0.23787	5188.56	4.67877	0.23595	5184.92
	+ 20	4.62876	0.24313	5199.28	4.66612	0.23785	5188.69	4.67906	0.23593	5185.05

7. Theoretical implications

To understand the model in practical life, its theoretical insights are highlighted in this section.

• The selling price of goods and its quality draw the line of profit and loss in business, hence it is one of the most important parameters. From Table 6, we have observed that when the selling price (p) of the products increases, the total cost of the supply chain (STC) decreases, whereas the cycle time (T) and green investment (G) increase. Moreover, the same type of impact has been seen for fuzzy and fuzzy learning environments that have a minimum cost when it comes to fuzzy learning environments. This demonstrates the positive impact of higher prices on supply chain efficiency and green initiatives. Furthermore, it demonstrates that fuzzy and fuzzy learning environments might be beneficial in terms of cost. This implies that, in order to reduce the total cost of the supply chain, corporations should focus on lowering emissions and investing in green technologies. In order to decrease the costs and improve efficiency, businesses should also invest in fuzzy learning environments. Comparative analysis of the cost incurred in developing the model in all three cases is represented in Figure 7.

• According to Table 6 and the sensitivity graphs, increasing the energy consumption of the products decreases the total cost of the supply chain while increasing the cycle time (T) and green investment (G). Furthermore, the same type of influence has been demonstrated for fuzzy and fuzzy learning settings that have minimal total costs when it comes to fuzzy learning environments. Such a reduction in STC can help organizations reduce costs and enhance productivity. Additionally, increases in T and G can be helpful to the environment. Furthermore, fuzzy learning environments can provide greater insight into the decision-making process, allowing businesses to make smarter decisions. Comparative analysis of the cost incurred in developing the model in all three cases is represented in Figure 8.

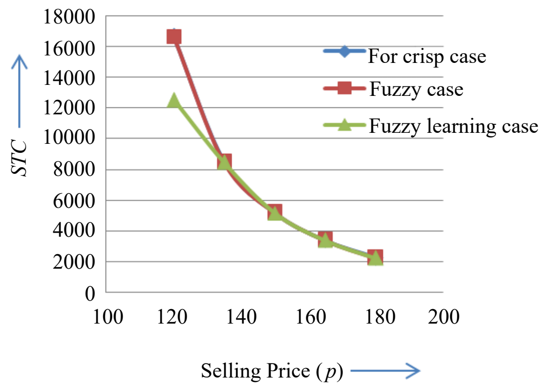

Figure 7. Impact of selling price to the total cost function

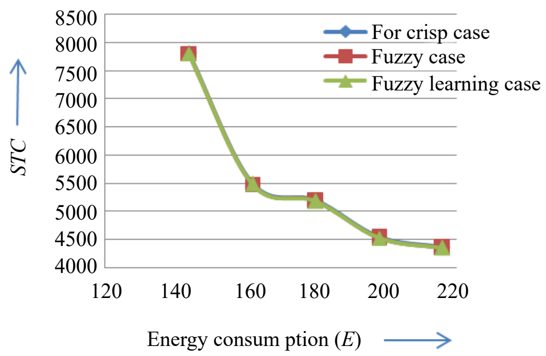

Figure 8. Impact of energy consumption to the total cost function

• When investing in quality improvement, if the quantity of imperfect items produced increases, the inventory cycle time and total cost decrease, but green investment shows less sensitivity to changes in it. A slight change in its value in fuzzy and fuzzy learning results in the same change as in the crisp case. The change in total cost can be understood through Figure 9, where the total cost is increasing at almost the same rate in all three cases, but we achieve the minimum cost when it comes to fuzzy learning environments. From Figure 10, we observe that after investing in quality improvement, as the quantity of produced imperfect items increases, the total cost rises at a reduced rate compared to before investing in quality improvement.

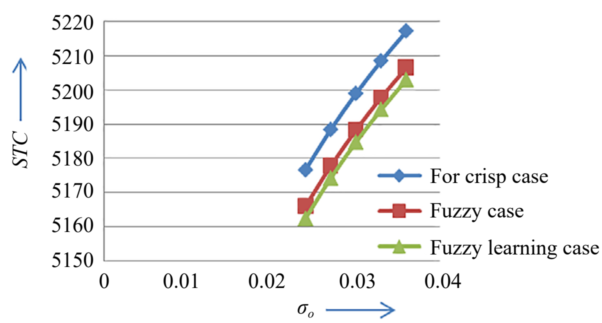

Figure 9. Impact of fraction of produced imperfect items before investing in quality improvement on total cost

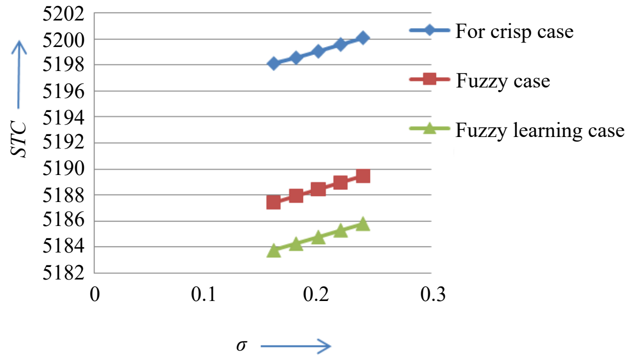

Figure 10. Impact of fraction of produced imperfect items after investing in quality improvement on total cost

• The safety of the inventory is the most important component of inventory management since any mistake in handling it could result in its loss. As a result, the rate at which inventory deteriorates becomes crucial to the model. A reduction in the rate of deterioration leads to a shorter cycle length when the model's behavior is examined by slightly changing its actual values. But this decrease comes with a rise in the overall cost as well as the cost of the green investment. On the other hand, a decline in the green investment rate and the overall cost of inventory is associated with an increase in the deterioration rate. Comparing the total cost with fuzzy and fuzzy learning, Figure 11 illustrates a steady increase in total cost with respect to deterioration in all three cases.

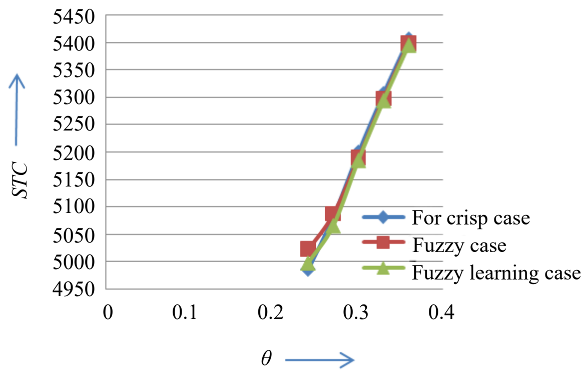

Figure 11. Impact of deterioration rate on total cost of supply chain

• With an increment in preservation investment (ζ), the total cost of the supply chain (STC) & green investment (G) decreases, whereas the cycle time (T) increases. As a result, the supply chain becomes more profitable and efficient, generating less waste. Green investments also contribute to the reduction of emissions into the environment, increasing the sustainability of the supply chain.

• When the manufacturer's production rate (P) increases, the total cost of the supply chain (STC), green investment (G), and cycle time (T) all increase. We have seen the same type of impact for the manufacturer's rework rate (R), carbon tax (Φ_x), and the fraction of imperfectness (σ).

• The portion of incomplete items from the work process is slightly changed from the actual value taken in the model (to −20%), then it is observed that there is no difference in the inventory cycle length and green investment cost. But the total cost has seen an increase. Uncertainty in inflation is analyzed by fuzzy and fuzzy learning to understand the model in real life, in which even if the value of the portion of imperfect goods is reduced, the green investment and inventory cycle length remain constant. But the total cost is increasing here too. The slight change in total cost can be understood from Figure 12, the total cost in all three cases is at a strictly decreasing level.

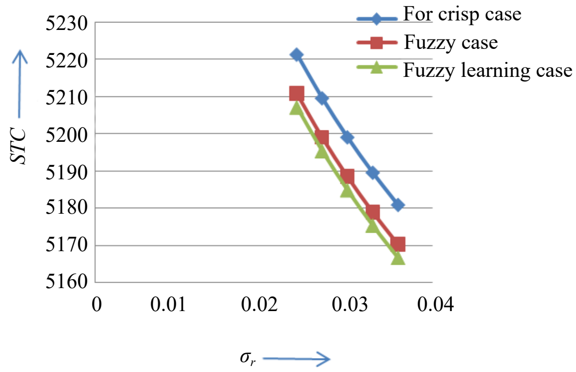

Figure 12. Impact of fraction of imperfect items from rework process on total cost

• The model is designed to reflect the uncertainty of inflation, so it is important to understand how small changes affect the model. By slightly changing the inflation rate from its actual value, it was observed that if its value is reduced to −20% then the value of Inventory Cycle Length is increasing, but the Green Investment and Total Cost are decreasing, which is it is beneficial from business point of view. Analyzed fuzzy and fuzzy learning for inflation rate uncertainty, yet the value of inventory cycle length is increasing, but green investment and total cost are decreasing. How much the total cost is changing in Crisp fuzzy and fuzzy learning can be understood from Figure 13.

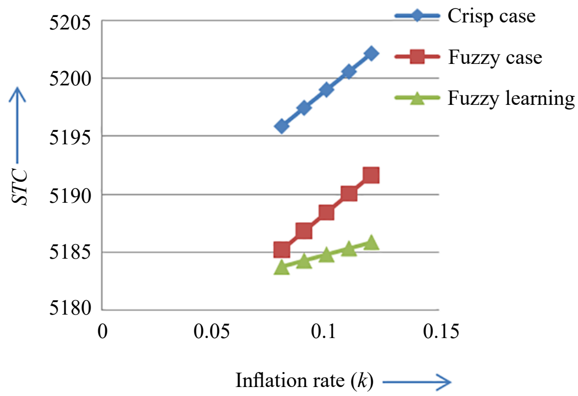

Figure 13. Impact of inflation rate on total cost of supply chain

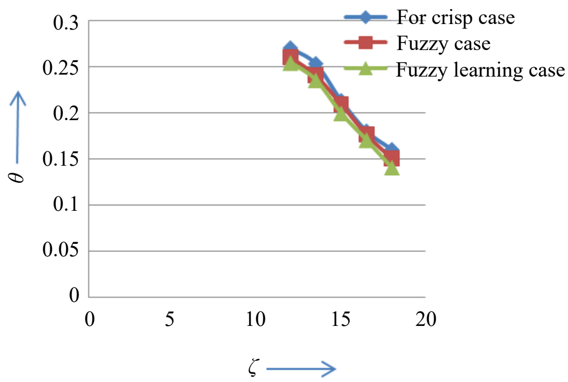

Figure 14. Impact of preservation investment on deterioration rate

8. Managerial implications

The main objective of this paper is to choose profitable strategy to invest in environmental protection and technology conservation with green revolution, production for defective goods, and the aim is to provide a new approach to developing models by giving importance to their conservation and energy consumption as well as inflation uncertainty. Since the models developed in previous research do not help manufacturers and inventory retailers make decisions about quality improvement as well as technology conservation, addressing the uncertainty of inflation. This paper brings together all the above key topics and provides a new direction that is beneficial in helping producers and inventory retailers consider these key issues together. The study also shows that investing in energy consumption and quality improvements using a carbon tax policy reduces total costs by broadly defined numerical values. Furthermore, this study generates several important managerial implications, which can be derived as follows:

(i) According to the current study, the supply chain manager must make decisions in light of the learning effect in a fuzzy environment. The business grows faster when decision-makers adopt a highly proactive learning approach in an imprecise climate from previous activities.

(ii) The current study suggests that investing in quality improvement and preservation technology successfully controls the quality of the products and deterioration process respectively, resulting in a reduction in waste. As a result, this study provides significant insights to decision-makers on how to efficiently reduce waste while still minimizing the total cost of the system.

(iii) The current study suggests that the supply chain manager must use green technologies in his supply chain operations as an investment in green successfully controls the excess of carbon emission and results in the development of a cleaner system.

(iv) According to the current study, discarding defective products leads to financial losses, while implementing a rework process plays an essential role in minimizing total cost. Therefore a flawless rework process must be carried out as the total cost of the proposed supply chain system is highly sensitive with respect to the proportion of defective units that are be reworked.

(v) The environmental repercussions and potential risks to future generations arise from the waste generated in the production process. This research offers guidance to decision-makers on how to fulfill their societal responsibility by implementing mechanisms that improve production system quality.

(vi) Furthermore, smart devices are essential for daily life, and people care about a smart product's price and energy usage before making a purchase. This study also offers insights to decision-makers on how to determine pricing strategies in price-sensitive economies such as India, Indonesia, Iran, Iraq, and other similar nations.

9. Conclusion and future scope

9.1 *Conclusion*

In today's era, companies are making efforts to address the problems related to carbon emissions, waste, quality of products, and energy consumption due to growing environmental concerns. This study aimed to figure out how to solve these issues simultaneously. A smart production integrated inventory system for decaying products with quality improvement, preservation, and green investment is discussed in this study when a customer's demand is price and energy-consumption-sensitive. Firstly, the model is created in a crisp sense, it is expanded into a fuzzy model to account for the imprecise inflationary environment, and further, it is extended to examine the impact of the learning in a imprecise environment. This paper aims to minimize the supply chain's total cost with optimum cycle time and green investment. Algorithms were developed to obtain the optimal solution to the proposed problem for both crisp and fuzzy environments. The results of the quantitative analysis show that investments in quality, carbon reduction efforts, and preservative technology are all paying off. It would be helpful to select more efficient policy technologies as a significant part of the total cost comes from imperfect production, where every precaution is necessary. Carbon reduction, green investment preservation technologies and inflation uncertainty often have a distinct impact on investment in technologies. And ignoring these will reduce the profit received, hence every member of the supply chain must take a decision on this. The findings of the current study reveal that due to learning in fuzziness, the fuzzy learning model results the lowest total cost than the fuzzy and crisp model. Also, it was discovered that smart items might fast generate a sizable profit. This model holds applicability for any company dealing with the development of smart products. This study substantiated that discarding defective products leads to financial losses, while implementing a rework process plays an essential role in minimizing overall cost. Further, analysis is carried out under the influence of inflation to reduce market disruption. Results indicate that preservation, green, and quality improvement investments help minimize waste and emissions for obtaining the objective of a cleaner production system.

9.2 *Future scope*

This study has certain limitations, such as the model's inability to accommodate more than two members of the supply chain and its non-utilization when considering the possibility of inventory holding cost uncertainty. Also, if the rates of imperfect production and rework are stochastic at that time, then for that also this model will not be considered in real life. However, considering them in the future, a new direction can be given to the research. Furthermore, there are many other interesting areas to expand on the current study. Some of these promising areas are to consider various policies of carbon reduction (such as hybrid carbon policy carbon cap and trade, carbon caps, etc.), potential role of digital transition, shortage, muti-items, smart production with maintenance policy etc.

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Conflict of interest

The authors declare no competing financial interest.

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Appendix A

$$X = \frac{\partial^2 \psi_1(G,\ T)}{\partial G^2}$$

$$Y = \frac{\partial^2 \psi_1(G,\ T)}{\partial T^2}$$

$$Ш = \frac{\partial^2 \psi_1(G,\ T)}{\partial G \partial T}$$

Appendix B

$$U = \frac{\partial^2 \psi_2(G,\ T)}{\partial G^2}$$

$$V = \frac{\partial^2 \psi_2(G,\ T)}{\partial T^2}$$

$$Ж = \frac{\partial^2 \psi_2(G,\ T)}{\partial G \partial T}$$