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Research Article

A Potent Collocation Approach Based on Shifted Gegenbauer Polynomials for Nonlinear Time Fractional Burgers' Equations

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Abstract: This paper presents a numerical strategy for solving the nonlinear time fractional Burgers' equation (TFBE) to obtain approximate solutions of TFBE. During this procedure, the collocation approach is used. The proposed numerical approximations are supposed to be a double sum of the products of two sets of basis functions. The two sets of polynomials are presented here: a modified set of shifted Gegenbauer polynomials and a shifted Gegenbauer polynomial set. Some specific integers and fractional derivatives are explicitly given as a combination of basis functions to apply the proposed collocation procedure. This method transforms the governing boundary-value problem into a set of nonlinear algebraic equations. Newton's approach can be used to solve the resulting nonlinear system. An analysis of the precision of the proposed method is provided. Various examples are presented and compared to some existing methods in the literature to prove the reliability of the suggested approach.

***Keywords*:** Fractional Burgers' equations, Gegenbaur polynomials, spectral methods, error bound

MSC: 65M60, 11B39, 40A05, 34A08

1. Introduction

There exists a class of partial differential equations known as the nonlinear TFBE that combines nonlinearity and fractional derivatives. These equations are an extension of the traditional Burgers' equation, which is a basic model in fluid dynamics and nonlinear wave phenomena [1]. Because of their importance in a variety of scientific disciplines, the study of the nonlinear TFBE has received a lot of attention. Burgers' equations appear in fluid dynamics; the classical Burgers' equation describes the propagation of shock waves in one-dimensional fluid flow. The TFBE provides a more realistic depiction of complex fluid behavior, such as turbulence and diffusion in porous media, by incorporating fractional derivatives. The Burgers' equation is a simplified form of the Navier-Stokes equations for incompressible fluid flow. Understanding the behavior of solutions to the nonlinear time-fractional Burgers' equation can provide insights into turbulence and shock wave formation in fluid dynamics. This equation can be used to describe heat conduction in materials with nonlinear thermal properties. It helps in studying heat transfer processes, especially in situations where heat conduction is not purely linear. The Burgers' equation can be applied to modeling traffic flow, particularly in

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congested traffic conditions. It helps in understanding the formation and propagation of traffic jams. In plasma physics, Burgers' equation can describe the behavior of plasma waves and turbulence. It plays a role in understanding the dynamics of charged particles in plasmas. The Burgers' equation is a prototypical equation for studying the behavior of nonlinear waves. It can be applied to various physical systems where wave phenomena are present, such as acoustics and optics. Moreover, Burgers' equations and fractional Burgers' equations have been used to model heat conduction in fractal materials. The fractional derivative captures the non-local aspect of heat conduction, making it possible to investigate heat transport in fractal media or materials with memory effects [2-4].

A number of fields, including physics, engineering, and computer science, use orthogonal polynomials. Orthogonal polynomials are commonly used in approximation theory, differential equations (DEs), and spectral methods [5, 6]. Due to their special properties, they are valuable tools for solving theoretical and practical problems that might otherwise be difficult to solve. For more information, see [7, 8].

Gegenbauer polynomials [9], also known as ultraspherical polynomials, this class of polynomials is a fundamental category of orthogonal polynomials with wide-ranging practical significance. They arise in the theory of special functions and are used to solve DEs. In addition, they have an important role to play in the study of quantum mechanics in many body systems, as they provide a natural basis for the expansion of certain functions, see [10, 11].

There are some reasons to use the Gegenbauer polynomials:

- These polynomials enjoy various interesting and useful features.
- High-accurate solutions are obtained if Gegenbauer polynomials are used as basis functions.
- The contributions using the Gegenbauer polynomials are few if compared to the contributions of the other polynomials.

Fractional differential operators are a type of nonlocal differential operator that involves fractional derivatives. For the history of the progress of fractional differential operators, one can consult [12, 13]. Fractional derivatives are natural extensions of classical derivatives and integrals see [14], and they become increasingly important in engineering and science disciplines. Today, fractional calculus has several potential uses outside of the realm of physics and engineering, including economics and biology, see [15, 16]. Also, fractional calculus has gained importance in recent years due to its ability to model and analyze complex phenomena that cannot be described by traditional integer order calculus see [17]. Research into this subject is ongoing, leading to the discovery of novel uses and advancements. Fore more studies, see [18-22].

Spectral methods are numerical techniques utilized for solving DEs, particularly partial DEs. The method involves representing the solution to the partial differential equation in terms of certain special functions, often trigonometric or polynomial functions, and using a set of equations to determine the coefficients of these basis functions. With the spectral method, nonlinear partial DEs can be handled accurately and efficiently. The method is also well suited for problems with periodic boundary conditions, as the basis functions can be chosen to satisfy these conditions exactly. The spectral method has been widely applied in many fields; see [23-26].

The three main techniques for implementing spectral methods are collocation, Galerkin, and tau methods. The collocation method involves evaluating the differential equation at a set of discrete collocation points and requiring that the solution satisfy the differential equation at these points. It is common to choose the collocation points as the roots of a certain set of orthogonal polynomials see, for example, [27-31]. Many mathematical techniques can be used to solve the resulting system of equations, such as Newton's method or Gaussian elimination, see [32, 33]. The Galerkin method involves multiplying the differential equation by a test function and integrating over the solution domain. The test function is chosen to satisfy certain properties, such as being orthogonal to the basis functions used to represent the solution. The resulting system of equations can be solved using techniques such as matrix inversion or iterative solvers, see [34-37]. As opposed to the Galerkin technique, which places constraints on which basis functions can be used, the tau method removes these limitations [38-41]. For more studies, see [42-49].

The main aims of the current article can be summarized in the following three-fold:

- Presenting a new technique for solving the TFBE via basis functions based on the shifted Gegenbauer polynomials by applying the spectral collocation method.
- Reducing the solution of the equation with its homogeneous initial and boundary conditions into a system of algebraic equations, then solving it using a suitable solver.
- Discussion of the error bound of the proposed method.

In accordance with the aforementioned aspects, the advantage of the proposed method is:

By choosing the modified set of shifted Gegenbauer polynomials and a shifted Gegenbauer polynomials set as basis functions, and taking a few terms of the retained modes, it is possible to produce approximations with excellent precision. Less calculation is required. In addition, the resulting errors are small.

In this study, we use the shifted Gegenbauer polynomials to numerically treat the TFBE. We summarize the Caputo fractional derivative and provide some properties and relations of the shifted Gegenbauer polynomials in Section 2 of our research. We solve the nonlinear TFBE using the collocation method in Section 3; we investigate the error bound in Section 4; we provide some illustrative examples in Section 5; and finally, we compare our approach to that of related works for the sake of illustrating the accuracy. The final results are presented in Section 6.

2. Summary on the Caputo fractional derivative

We give here some elementary properties of fractional calculus. In addition, some properties of the Gegenbauer polynomials and their shifted polynomials are accounted for.

Definition 1 [8, 14] In Caputo's sense, the s fractional-derivative of $h(z)$ is defined as:

$$D_z^s h(z) = \frac{1}{\Gamma(\ell - s)} \int_0^z (z-t)^{\ell - s - 1} h^{(\ell)}(t)dt, \; s > 0, \; z > 0, \tag{1}$$

where $\ell - 1 \le s < \ell$, $\ell \in \mathbb{Z}^+$.

D_z^s meets the following properties for $\ell - 1 \le s < \ell$, $\ell \in \mathbb{Z}^+$:

$$D_z^s b = 0, \quad b \text{ is a constant.} \tag{2}$$

$$D_z^s z^\ell = \begin{cases} 0, & \text{if } \ell \in \mathcal{N}_0 \quad \text{and} \quad \ell < \lceil s \rceil, \\ \dfrac{\Gamma(\ell+1)}{\Gamma(\ell - s + 1)} z^{\ell - s}, & \text{if } \ell \in \mathcal{N}_0 \quad \text{and} \quad \ell \ge \lceil s \rceil, \end{cases} \tag{3}$$

where $\mathcal{N}_0 = \{0\} \cup \mathbb{Z}^+$. Also the notations $\lceil \nu \rceil$ and $\Gamma(x)$ represent the ceiling and gamma functions respectively.

2.1 *Some fundamentals of the shifted Gegenbauer polynomials*

The Gegenbaur polynomials $G_m^\nu(z)$ of degree $m \in \mathbb{Z}^+$ are real-valued functions associated with the parameter $\nu > -\frac{1}{2}$ [50].

$$G_m^\nu(z) = \frac{(2\nu)_m}{\left(\nu + \frac{1}{2}\right)_m} P_m^{\left(\nu - \frac{1}{2}, \nu - \frac{1}{2}\right)}(z), \; m \ge 0,$$

where $P_m^{\left(\nu - \frac{1}{2}, \nu - \frac{1}{2}\right)}(z)$ are the classical Jacobi polynomials and $(\nu)_m$ is the Pochhammer symbol.

It is useful to define the shifted polynomials on [0, 1], as:

$$\tilde{G}_m^\nu(z) = G_m^\nu(2z - 1).$$

The orthogonality relation of $\tilde{G}_k^\nu(z)$ is given by

$$\int_0^1 \tilde{G}_n^{\nu}(z)\tilde{G}_m^{\nu}(z)(z-z^2)^{\nu-\frac{1}{2}}dx = \frac{\pi\Gamma(2\nu+n)}{2^{4\nu-1}(\nu+n)(\Gamma(\nu))^2\Gamma(n+1)}\delta_{n,m}, \tag{4}$$

where $\delta_{n,m}$ is the well-known Kronecker delta function.

The following two results are useful hereafter.

Lemma 1 [51] The Gegenbauer polynomials meet the following relation:

$$G_k^{\nu}(z)-\eta_{k,\nu}G_{k+2}^{\nu}(z) = \frac{4\nu(\nu+k+1)}{(2\nu+k)(2\nu+k+1)}(1-z^2)G_k^{\nu+1}(z), \tag{5}$$

where $\eta_{k,\nu}$ is given by

$$\eta_{k,\nu} = \frac{(k+1)(k+2)}{(2\nu+k)(2\nu+k+1)}. \tag{6}$$

Theorem 1 [52] Let $m, p \in \mathbb{Z}^+$ with $m \geq p \geq 1$. The derivative of $\tilde{G}_m^{\nu}(z)$ are given by the formula

$$\frac{d^p\,\tilde{G}_m^{\nu}(z)}{dx^p} = \sum_{\ell=0}^{m-p} d_{\ell,m,p}\,\tilde{G}_{\ell}^{\nu}(z), \tag{7}$$

where

$$d_{\ell,m,p} = \frac{2^{1-p}(\nu+\ell)\Gamma\left(\ell+\nu+\frac{1}{2}\right)\Gamma(\ell+m+p+2\nu)}{(-\ell+m-p)!}\times$$

$${}_3F_2\left(\begin{matrix}\ell-m+p,\nu+\ell+\frac{1}{2},2\nu+\ell+m+p\\ \nu+\ell+p+\frac{1}{2},2\nu+2\ell+1\end{matrix}\middle|1\right), \tag{8}$$

where ${}_rF_s$ represents the celebrated generalized hypergeometric function given as [53]

$${}_rF_s\left(\begin{matrix}p_1,p_2,\cdots,p_r\\ q_1,q_2,\cdots,q_s\end{matrix}\middle|x\right) = \sum_{m=0}^{\infty}\frac{(p_1)_m(p_2)_m\cdots(p_r)_m}{(q_1)_m(q_2)_m\cdots(q_s)_m}\frac{x^m}{m!}. \tag{9}$$

3. Collocation approach for the treatment of nonlinear TFBE

In this section, we will choose the basis functions that will be suitable to propose our shifted Gegenbauer collocation method (SGCM) to solve the nonlinear TFBE.

3.1 *Basis functions choice*

We will consider the following two sets of basis functions $\phi_i(z)$ and $\psi_j(t)$ as

$$\phi_i(z) = \tilde{G}_i^{(\nu)}(z) - \eta_{i,\nu}\,\tilde{G}_{i+2}^{(\nu)}(z), \tag{10}$$

$$\psi_j(t) = \tilde{G}_j^{(\nu)}(t), \tag{11}$$

where $\eta_{i,\nu}$ is given in (6).

Now, we will give two important results regarding the basis functions $\phi_i(z)$ and $\psi_i(z)$. More definitely, the first result expresses the first- and second-order derivatives of $\phi_i(z)$ as combinations of $\psi_i(z)$. The second formula expresses the fractional derivatives of $\psi_j(z)$ in terms of their original ones.

Corollary 1 Let $i \in \mathbb{Z}^+$. The first and second derivatives of $\phi_i(z)$ are given by the following two formulas:

$$\frac{d\phi_i(z)}{dx} = \sum_{k=0}^{i-1} d_{k,i,1}\,\tilde{G}_k^{\nu}(z) - \eta_{i,\nu}\sum_{k=0}^{i+1} d_{k,i+2,1}\,\tilde{G}_k^{\nu}(z), \tag{12}$$

$$\frac{d^2\phi_i(z)}{dx^2} = \sum_{k=0}^{i-2} d_{k,i,2}\,\tilde{G}_k^{\nu}(z) - \eta_{i,\nu}\sum_{k=0}^{i} d_{k,i+2,2}\,\tilde{G}_k^{\nu}(z), \tag{13}$$

where $d_{\ell,m,p}$ are those given in (8).

Proof. Formula (25) is a direct consequence of (10) along with Formula (7) for $p = 1$, while Formula (13) is a direct consequence of (10) along with Formula (7) for $p = 2$.

Theorem 2 [11] In Caputo sense, the fractional derivative $D^{(\nu)}\psi_i(t)$ can be expressed as

$$D^{(\nu)}\psi_i(t) = \sum_{j=0}^{\infty} S_{j,i,\nu,v}\,\psi_j(t), \;\; i = \lceil \nu \rceil, \lceil \nu \rceil + 1 \ldots, \tag{14}$$

where

$$S_{j,i,\nu,v} = \sum_{k=\lceil \nu \rceil}^{i} \theta_{k,i,j,\nu,v}, \tag{15}$$

and

$$\theta_{k,i,j,\nu,v} = \frac{(-1)^{i-k}\,2^{4\nu-1}\,\Gamma(j+1)(j+\nu)\Gamma^2(\nu)(2\nu)_{i+k}}{\pi\Gamma(j+2\nu)(i-k)!(\nu+\frac{1}{2})_k\,\Gamma(k-v+1)}$$

$$\times\sum_{l=0}^{j}\frac{(-1)^{j-l}\,L^{-v}\,(2\nu)_{j+l}\,\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(k+l+\nu-v+\frac{1}{2}\right)}{l!(j-l)!\left(\nu+\frac{1}{2}\right)_l\,\Gamma(2\nu+k+l-v+1)}, \;\; j = 0,1,2,\ldots. \tag{16}$$

3.2 *Collocation solution of the nonlinear TFBE*

Our aim in this section is to use the modified shifted Gegenbauer collocation method to solve the following nonlinear TFBE [54]

$$D_t^{\xi} v(z,t) + k_1(z,t)\frac{\partial^2 v(z,t)}{\partial z^2} + k_2(z,t)v(z,t) + k_3(z,t)\frac{\partial v(z,t)}{\partial z}v(z,t) + k_4(z,t)v(z,t)\frac{\partial v(z,t)}{\partial z} = \overline{f}(z,t), \tag{17}$$

directed to the following conditions:

$$v(z,0) = u_0(z),\ z \in [0,1],$$

$$v(0,t) = g_0(t),\ v(1,t) = g_1(t),\ t \in [0,1], \tag{18}$$

where $0 < \xi < 1$ and $k_i(z, t)$, $1 \le i \le 4$ and $\overline{f}(z, t)$ are given continuous functions.

Now, consider the following substitution [55]:

$$v(z,t) := \mathcal{U}(z,t) + \overline{\mathcal{U}}(z,t), \tag{19}$$

where

$$\overline{\mathcal{U}}(z,t) = (1-z)g_0(t) + z g_1(t), \tag{20}$$

that transforms Eq. (17) with (18) into the following modified one:

$$D_t^{\xi} \mathcal{U}(z,t) + k_1(z,t)\frac{\partial^2 \mathcal{U}(z,t)}{\partial z^2} + k_2(z,t)\mathcal{U}(z,t) + k_3(z,t)\frac{\partial \mathcal{U}(z,t)}{\partial z}\mathcal{U}(z,t) + k_4(z,t)\mathcal{U}(z,t)\frac{\partial \mathcal{U}(z,t)}{\partial z} = f(z,t), \tag{21}$$

governed by the conditions:

$$\mathcal{U}(z,0) = u_0(z) - (1-z)g_0(0) - z g_1(0),\ z \in [0,1],$$

$$\mathcal{U}(0,t) = \mathcal{U}(1,t) = 0,\ t \in [0,1], \tag{22}$$

where

$$f(z,t) = \overline{f}(z,t) - D_t^{\xi}\overline{\mathcal{U}}(z,t) - k_1(z,t)\frac{\partial^2 \overline{\mathcal{U}}(z,t)}{\partial z^2} - k_2(z,t)\overline{\mathcal{U}}(z,t) - k_3(z,t)\frac{\partial \overline{\mathcal{U}}(z,t)}{\partial z}\overline{\mathcal{U}}(z,t)$$

$$-k_4(z,t)\overline{\mathcal{U}}(z,t)\frac{\partial \overline{\mathcal{U}}(z,t)}{\partial z}. \tag{23}$$

In this case, rather than solving (17) with (18), we can instead solve the modified equation (21) subject to (22).

Now, we will set

$$S_{\mathcal{M}} = \text{span}\left\{\phi_i(z)\psi_j(t),\ \ i, j = 0, 1, \ldots, \mathcal{M}\right\},$$

$$V_{\mathcal{M}} = \left\{\mathcal{U} \in S_{\mathcal{M}} : \mathcal{U}(0, t) = \mathcal{U}(1, t) = 0,\ \ 0 \le t \le 1\right\}. \tag{24}$$

then any function $\mathcal{U}(z, t) \in V_{\mathcal{M}}$ can be approximated as

$$\mathcal{U}_{\mathcal{M}}(z, t) = \sum_{i=0}^{\mathcal{M}}\sum_{j=0}^{\mathcal{M}} c_{ij}\phi_i(z)\psi_j(t). \tag{25}$$

Now, the residual $R(z, t)$ of Eq. (21) has the following form

$$R(z, t) = D_t^{\xi}\,\mathcal{U}_{\mathcal{M}}(z, t) + k_1(z, t)\frac{\partial^2 \mathcal{U}_{\mathcal{M}}}{\partial z^2}(z, t) + k_2(z, t)\mathcal{U}_{\mathcal{M}}(z, t)$$

$$+k_3(z, t)\frac{\partial \mathcal{U}_{\mathcal{M}}}{\partial z}(z, t)\mathcal{U}_{\mathcal{M}}(z, t) + k_4(z, t)\mathcal{U}_{\mathcal{M}}(z, t)\frac{\partial \mathcal{U}_{\mathcal{M}}}{\partial z}(z, t) - f(z, t). \tag{26}$$

To get the expansion coefficients c_{ij}, we use Theorem 2 and Corollary 1 to obtain the residual $R(z, t)$ given by (26), then we apply the spectral collocation method [56], by forcing the residual $R(z, t)$ to be zero at some collocation points (z_i, t_j), that is, we get

$$R(z_i, t_j) = 0,\ \ i = 1, 2, \ldots, \mathcal{M},\ \ j = 1, 2, \ldots, \mathcal{M} + 1, \tag{27}$$

where $\{(z_i, t_j) : i = 1, 2, ..., \mathcal{M},\ j = 1, 2, ..., \mathcal{M} + 1\}$ are the first distinct roots of $\phi_{\mathcal{M}+1}(z)$ and $\psi_{\mathcal{M}+1}(t)$ respectively.

Also, the initial condition (22) implies that

$$\mathcal{U}_{\mathcal{M}}(z_i, 0) = u_0(z_i) - (1 - z_i)g_0(0) - z_i\, g_1(0),\ \ i\ :\ 1, \ldots, \mathcal{M} + 1. \tag{28}$$

Therefore, Eqs. (27) and (28) constitute $(\mathcal{M} + 1)^2$ non-linear system of equations. With the help of a numerical solver, this system can be solved.

Algorithm 1 Coding algorithm for the proposed scheme.
Input ξ, $\mathcal{M}$, $k_1(z, t)$, $k_2(z, t)$, $k_3(z, t)$, $k_4(z, t)$ and $f(z, t)$.
Step 1 Assume an approximate solution $\mathcal{U}_{\mathcal{M}}(z, t)$ as in (25).
Step 2 Compute $R(z, t)$ as in (26).
Step 3 Apply the collocation method to obtain the system in (27) and (28).
Step 4 Use *FindRoot* command with initial guess $\{c_{ij} = 10^{-i-j}, i, j : 0, 1, ..., \mathcal{M}\}$, to solve the system in (27) to get c_{ij}.
Output $\mathcal{U}_{\mathcal{M}}(z, t)$.

4. Error bound

Given that the best approximation of $v(z, t)$ is $\mathcal{U}_{\mathcal{M}}(z, t) \in V_{\mathcal{M}}$, we may express the following using the concept of the best approximation:

$$\| v(z, t) - \mathcal{U}_{\mathcal{M}}(z, t) \|_{\infty} \le \| v(z, t) - B_{\mathcal{M}}(z, t) \|_{\infty},\ \ \forall\ B_{\mathcal{M}}(z, t) \in V_{\mathcal{M}}. \tag{29}$$

Inequality (29) also holds if $B_{\mathcal{M}}(z, t)$ denotes the interpolating polynomial for $v(z, t)$ at points (z_i, t_j), where z_i are the zeros of $\phi_i(z)$, while t_j are the roots of $\psi_j(t)$. Procedures analogous to those in [32, 57] lead to

$$v(z,t)-B_{\mathcal{M}}(z,t)=\frac{\partial^{\mathcal{M}+1}v(\eta,t)}{\partial x^{\mathcal{M}+1}(\mathcal{M}+1)!}\prod_{i=0}^{\mathcal{M}}(z-z_i)+\frac{\partial^{\mathcal{M}+1}v(z,\mu)}{\partial t^{\mathcal{M}+1}(\mathcal{M}+1)!}\prod_{j=0}^{\mathcal{M}}(t-t_j)$$

$$-\frac{\partial^{2N+2}v(\hat{\eta},\hat{\mu})}{\partial x^{\mathcal{M}+1}\partial t^{\mathcal{M}+1}((\mathcal{M}+1)!)^2}\prod_{i=0}^{\mathcal{M}}(z-z_i)\prod_{j=0}^{\mathcal{M}}(t-t_j). \tag{30}$$

where η, $\hat{\eta} \in [0, 1]$ and μ, $\hat{\mu} \in [0, 1]$.

Now,

$$\| v(z,t)-B_{\mathcal{M}}(z,t)\|_\infty \le \max_{(z,t)\in\Omega}\left|\frac{\partial^{\mathcal{M}+1}v(\eta,t)}{\partial x^{\mathcal{M}+1}}\right|\frac{\|\prod_{i=0}^{\mathcal{M}}(z-z_i)\|_\infty}{(\mathcal{M}+1)!}+\max_{(z,t)\in\Omega}\left|\frac{\partial^{\mathcal{M}+1}v(z,\mu)}{\partial t^{\mathcal{M}+1}}\right|\frac{\|\prod_{j=0}^{\mathcal{M}}(t-t_j)\|_\infty}{(\mathcal{M}+1)!}$$

$$-\max_{(z,t)\in\Omega}\left|\frac{\partial^{2N+2}v(\hat{\eta},\hat{\mu})}{\partial x^{\mathcal{M}+1}\partial t^{\mathcal{M}+1}}\right|\frac{\|\prod_{i=0}^{\mathcal{M}}(z-z_i)\|_\infty\|\prod_{j=0}^{\mathcal{M}}(t-t_j)\|_\infty}{((\mathcal{M}+1)!)^2}. \tag{31}$$

The smoothness of $v(z, t)$ on $\Omega = (0, 1)^2$ imples the existence of three constants v_1, v_2 and v_3, in the sense that

$$\max_{(z,t)\in\Omega}\left|\frac{\partial^{\mathcal{M}+1}v(z,t)}{\partial x^{\mathcal{M}+1}}\right|\le v_1,\ \max_{(z,t)\in\Omega}\left|\frac{\partial^{\mathcal{M}+1}v(z,\mu)}{\partial t^{\mathcal{M}+1}}\right|\le v_2,\ \max_{(z,t)\in\Omega}\left|\frac{\partial^{2\mathcal{M}+2}v(\hat{\eta},\hat{\mu})}{\partial x^{\mathcal{M}+1}\partial t^{\mathcal{M}+1}}\right|\le v_3. \tag{32}$$

To minimize the factor $\|\prod_{i=0}^{\mathcal{M}}(z-z_i)\|_\infty$, with the aid of Lemma 1 and the one-to-one mapping $z=\frac{1}{2}(x+1)$ between the intervals [-1, 1] and [0, 1], we get

$$\min_{z_i\in[0,1]}\max_{z\in[0,1]}\left|\prod_{i=0}^{\mathcal{M}}(z-z_i)\right| = \min_{x_i\in[-1,1]}\max_{x\in[-1,1]}\left|\prod_{i=0}^{\mathcal{M}}\frac{1}{2}(x-x_i)\right|$$

$$=\left(\frac{1}{2}\right)^{\mathcal{M}+1}\min_{x_i\in[-1,1]}\max_{x\in[-1,1]}\left|\prod_{i=0}^{\mathcal{M}}(x-x_i)\right|$$

$$=\left(\frac{1}{2}\right)^{\mathcal{M}+1}\min_{x_i\in[-1,1]}\max_{x\in[-1,1]}\left|\frac{\phi^{\nu}_{\mathcal{M}-1}(x)}{a^{\nu}_{\mathcal{M}}}\right|, \tag{33}$$

where x_i are the roots of $\phi^{\nu}_{\mathcal{M}-1}(x)$ and $a^{\nu}_{\mathcal{M}}=-\frac{2^{\mathcal{M}+1}\Gamma(\nu+\mathcal{M}+1)}{(2\nu+\mathcal{M}-1)(2\nu+\mathcal{M})\Gamma(\nu)\Gamma(\mathcal{M})}$ is the leading coefficient of $\phi^{\nu}_{\mathcal{M}-1}(x)=\frac{4\nu(\nu+\mathcal{M})}{(2\nu+\mathcal{M}-1)(2\nu+\mathcal{M})}(1-x^2)G^{\nu}_{\mathcal{M}-1}(x).$

In addition, $\|\prod_{j=0}^{\mathcal{M}}(t-t_j)\|_\infty$, can be minimized by means of the one-to-one mapping $t=\frac{1}{2}(\overline{t}+1)$ between [-1, 1] and [0, 1] to get

$$\min_{t_j\in[0,1]}\max_{t\in[0,1]}\left|\prod_{j=0}^{\mathcal{M}}(t-t_j)\right|=\left(\frac{1}{2}\right)^{\mathcal{M}+1}\min_{\bar{t}_j\in[-1,1]}\max_{\bar{t}\in[-1,1]}\left|\frac{\psi^{\nu}_{\mathcal{M}+1}(\bar{t})}{b^{\nu}_{\mathcal{M}}}\right|, \tag{34}$$

where $b^{\nu}_{\mathcal{M}}=\dfrac{2^{\mathcal{M}+1}\Gamma(\nu+\mathcal{M}+1)}{\Gamma(\nu)\Gamma(\mathcal{M}+2)}$ is the leading coefficient of $\psi^{\nu}_{\mathcal{M}+1}(\bar{t})=G^{\nu}_{\mathcal{M}+1}(\bar{t})$ and $\bar{t}_j$ are the zeros of $\psi^{\nu}_{\mathcal{M}+1}(\bar{t})$.

It is known that

$$\eta^{\nu}_{\mathcal{M}}=\max_{x\in[-1,1]}\left|\phi^{\nu}_{\mathcal{M}-1}(x)\right|=\left|\phi^{\nu}_{\mathcal{M}-1}(1)\right|=\frac{4\nu(\nu+\mathcal{M})\Gamma(2\nu+\mathcal{M}-1)}{\Gamma(2\nu+2)\Gamma(\mathcal{M})}. \tag{35}$$

Also,

$$\hat{\eta}^{\nu}_{\mathcal{M}}=\max_{\bar{t}\in[-1,1]}\left|\psi^{\nu}_{\mathcal{M}+1}(\bar{t})\right|=\left|\psi^{\nu}_{\mathcal{M}+1}(1)\right|=\frac{(2\nu)_{\mathcal{M}+1}}{(\mathcal{M}+1)!}. \tag{36}$$

Therefore, inequality (32) along with Equations (33) and (34) enable us to get the following desired result

$$\| v(z,t)-\mathcal{U}_{\mathcal{M}}(z,t)\|_{\infty}\le v_1\frac{\left(\frac{1}{2}\right)^{\mathcal{M}+1}\eta^{\nu}_{\mathcal{M}}}{a^{\nu}_{\mathcal{M}}(\mathcal{M}+1)!}+v_2\frac{\left(\frac{1}{2}\right)^{\mathcal{M}+1}\hat{\eta}^{\nu}_{\mathcal{M}}}{b^{\nu}_{\mathcal{M}}(\mathcal{M}+1)!}+v_3\frac{\left(\frac{1}{4}\right)^{\mathcal{M}+1}\eta^{\nu}_{\mathcal{M}}\hat{\eta}^{\nu}_{\mathcal{M}}}{a^{\nu}_{\mathcal{M}}b^{\nu}_{\mathcal{M}}((\mathcal{M}+1)!)^2}. \tag{37}$$

This gives an upper bound of the absolute error for the approximate and exact solutions.

5. Some test problem

Several test problems are presented in this section to ensure that our algorithm is applicable and efficient.

Test Problem 1 [54] Consider the following TFBE:

$$D^{\xi}_t v(z,t)+k_1(z,t)\frac{\partial^2 v(z,t)}{\partial z^2}+k_2(z,t)v(z,t)+k_3(z,t)\frac{\partial v(z,t)}{\partial z}v(z,t)+k_4(z,t)v(z,t)\frac{\partial v(z,t)}{\partial z}=\bar{f}(z,t), \tag{38}$$

$$v(0,t)=v(1,t)=0,\quad 0\le t\le 1, \tag{39}$$

$$v(z,0)=0,\quad 0\le z\le 1, \tag{40}$$

$$\bar{f}(z,t)=\frac{2t^{2-\alpha}\sin(2\pi z)}{\Gamma(3-\alpha)}+2\pi t^4\sin(2\pi z)\cos(2\pi z)+4\pi^2t^2\sin(2\pi z), \tag{41}$$

and

$$k_1(z,t)=-1,\ k_2(z,t)=k_3(z,t)=0,\ k_4(z,t)=1, \tag{42}$$

whose exact solution is: $v(z,t)=t^2\sin(2\pi z)$.

For $\xi=0.75$ and $\mathcal{M}=16$, both the approximate solution (AS) (left) and the exact solution (ES) (right) are depicted in Figure 1. Figure 2 displays the L_∞ error for $\xi = 0.75$ and $\mathcal{M} = 16$. The absolute error (AE) is displayed in Table 1 for $\xi = 0.75$, $\mathcal{M} = 16$ at different t. Table 2 displays the maximum absolute error (MAE) for various ξ and $\mathcal{M}$. A comparison of MAE between SGCM and the Chebyshev spectral collocation method (CSCM) in [54] and the method in [58] is tabulated in Table 3.

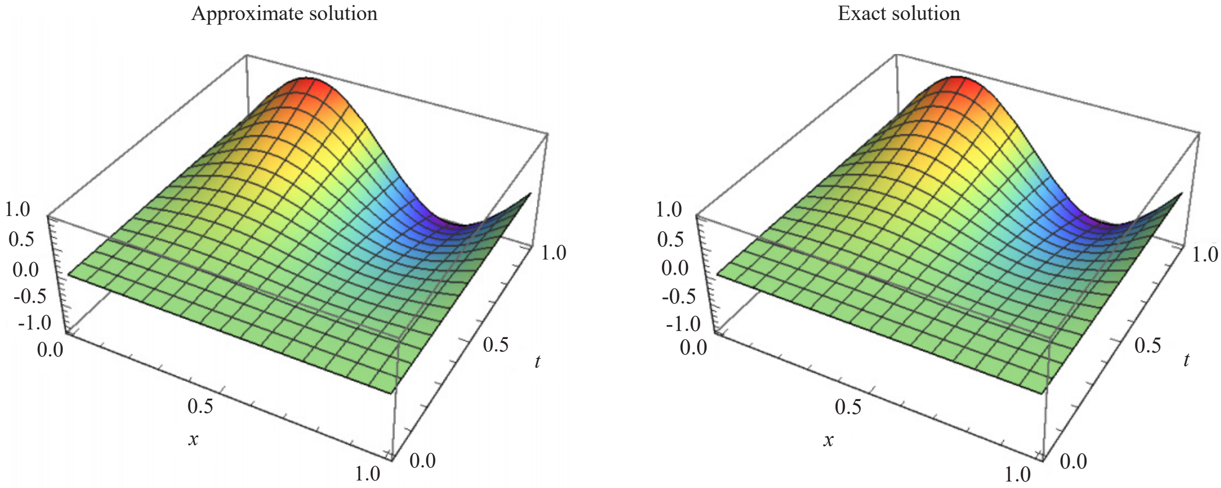

Figure 1. The AS (left) and ES (right) of Problem 1 at $\xi = 0.75$ and $\mathcal{M} = 16$

Table 1. AE of Problem 1 at $\xi = 0.75$ and $\mathcal{M} = 16$

	$\xi = 0.75$			
z	$t = 0.1$	$t = 0.5$	$t = 0.9$	Upper bound of AE
0.1	1.57027×10^{-14}	1.13295×10^{-12}	4.66893×10^{-12}	
0.2	3.36502×10^{-14}	2.33483×10^{-12}	9.84501×10^{-12}	
0.3	5.67862×10^{-14}	3.66857×10^{-12}	1.58273×10^{-11}	
0.4	8.63745×10^{-14}	5.11381×10^{-12}	2.22814×10^{-11}	
0.5	1.27659×10^{-13}	6.70571×10^{-12}	2.86939×10^{-11}	10^{-3}
0.6	1.85665×10^{-13}	8.44963×10^{-12}	3.43515×10^{-11}	
0.7	2.60585×10^{-13}	1.02305×10^{-11}	3.50088×10^{-11}	
0.8	3.70313×10^{-13}	1.24031×10^{-11}	3.82523×10^{-11}	
0.9	5.21337×10^{-13}	1.51795×10^{-11}	2.3182×10^{-11}	

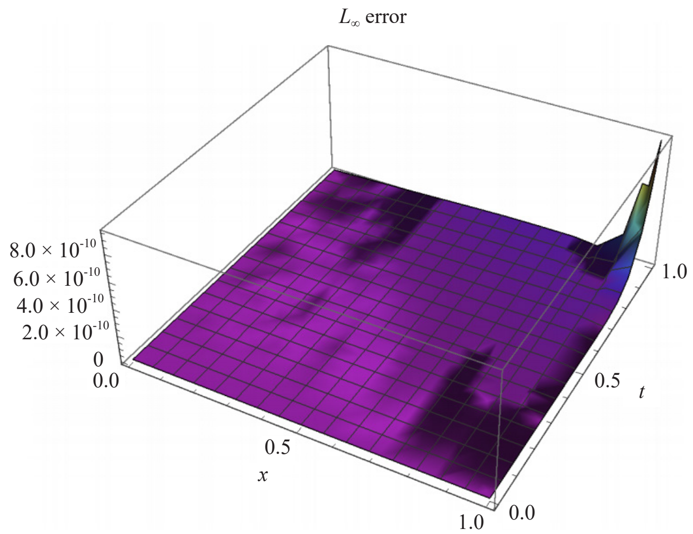

Figure 2. L_∞ error of Problem 1 at $\xi = 0.75$ and $\mathcal{M} = 16$

Table 2. MAE of Problem 1

$\mathcal{M}$	$\xi = 0.5$	$\xi = 0.75$	$\xi = 0.9$
4	5.88639×10^{-1}	6.07029×10^{-1}	6.17303×10^{-1}
6	4.97866×10^{-2}	5.42437×10^{-2}	5.66217×10^{-2}
8	1.5253×10^{-3}	1.76304×10^{-3}	1.90359×10^{-3}
10	3.14771×10^{-5}	3.82333×10^{-5}	4.24249×10^{-5}
12	4.69657×10^{-7}	5.87349×10^{-7}	6.63478×10^{-7}
14	5.41993×10^{-9}	6.91947×10^{-9}	7.91579×10^{-9}
16	5.88383×10^{-10}	5.98725×10^{-10}	2.34618×10^{-9}

Table 3. MAE Comparison of different methods for Problem 1

ξ	$\mathcal{M}$	SGCM	$\mathcal{M}$	CSCM [54]	Method in [58]
0.75	16	$\mathbf{5.98725 \times 10^{-10}}$	15	8.1966×10^{-7}	3.3291×10^{-5}
0.9	16	$\mathbf{2.34618 \times 10^{-9}}$	15	2.9875×10^{-6}	3.735×10^{-5}

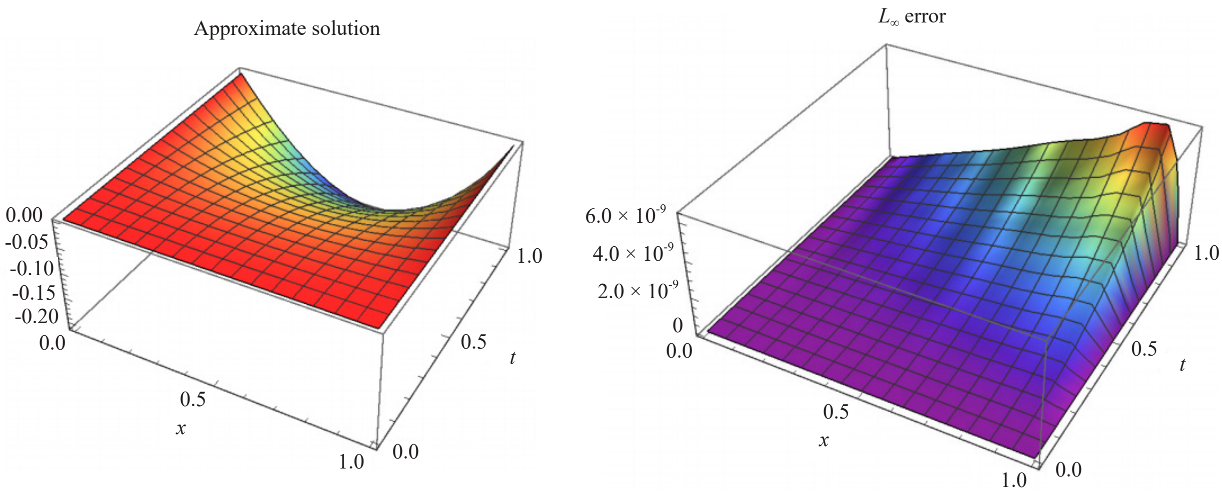

Figure 3. The AS (left) and L_∞ error (right) of Problem 2 at $\xi = 0.5$ and $\mathcal{M} = 6$

Table 4. The AE of Problem 2 at $\xi = 0.5$ and $\mathcal{M} = 6$

	$\xi = 0.5$			
z	$t = 0.1$	$t = 0.5$	$t = 0.9$	Upper bound of AE
0.1	3.56876×10^{-12}	1.3226×10^{-10}	4.85373×10^{-10}	
0.2	7.47866×10^{-12}	2.71041×10^{-10}	9.86737×10^{-10}	
0.3	1.14878×10^{-11}	4.08283×10^{-10}	1.47383×10^{-9}	
0.4	1.5952×10^{-11}	5.50699×10^{-10}	1.9652×10^{-9}	
0.5	2.19936×10^{-11}	7.23597×10^{-10}	2.5398×10^{-9}	10^{-5}
0.6	2.89658×10^{-11}	9.07452×10^{-10}	3.1321×10^{-9}	
0.7	3.55229×10^{-11}	1.06493×10^{-9}	3.62127×10^{-9}	
0.8	4.57097×10^{-11}	1.29296×10^{-9}	4.32441×10^{-9}	
0.9	6.08531×10^{-11}	1.61788×10^{-9}	5.32715×10^{-9}	

Test Problem 2 [54] Consider the following TFBE:

$$D_t^{\xi} v(z,t) + k_1(z,t)\frac{\partial^2 v(z,t)}{\partial z^2} + k_2(z,t)v(z,t) + k_3(z,t)\frac{\partial v(z,t)}{\partial z}v(z,t) + k_4(z,t)v(z,t)\frac{\partial v(z,t)}{\partial z} = \overline{f}(z,t), \tag{43}$$

where

$$v(0,t) = t^2, v(1,t) = et^2, \ \ 0 < t \le 1, \tag{44}$$

$$v(z,0) = 0, \ \ 0 < z \le 1, \tag{45}$$

$$\bar{f}(z,t) = \frac{2t^{2-\nu}e^z}{\Gamma(3-\nu)} + t^2e^{2z} - t^2e^z, \tag{46}$$

and

$$k_1(z,t) = -1, \ k_2(z,t) = k_3(z,t) = 0, \ k_4(z,t) = 1, \tag{47}$$

and $v(z, t) = t^2e^z$ is the exact solution.

Table 5. MAE of Problem 2

$\mathcal{M}$	$\xi = 0.5$	$\xi = 0.7$	$\xi = 0.9$
2	1.28593×10^{-3}	1.24513×10^{-3}	1.2074×10^{-3}
4	4.29418×10^{-6}	4.38206×10^{-6}	4.44536×10^{-6}
6	6.58415×10^{-9}	7.03846×10^{-9}	7.45864×10^{-9}

Table 6. MAE Comparison of different methods for Problem 3

ξ	$\mathcal{M}$	SGCM	$\mathcal{M}$	CSCM [54]	Method in [58]
0.75	16	$\mathbf{7.03846 \times 10^{-9}}$	8	4.1575×10^{-5}	2.24523×10^{-4}
0.9	16	$\mathbf{7.45864 \times 10^{-9}}$	10	4.7521×10^{-5}	2.32565×10^{-4}

Figure 3 displays the (AS) (left) and L_∞ error (right) at $\xi = 0.5$ and $\mathcal{M} = 6$. Table 4 dispalys the AE for $\xi = 0.5$ and $\mathcal{M} = 6$ at different t. In addition, MAE for various ξ and $\mathcal{M}$ are displayed in Table 5. MAE comparison between SGCM and CSCM in [54] and [58] is tabulated in Table 6.

Test Problem 3 [54] Consider the following TFBE:

$$D_t^{\xi} v(z,t) + k_1(z,t)\frac{\partial^2 v(z,t)}{\partial z^2} + k_2(z,t)v(z,t) + k_3(z,t)\frac{\partial v(z,t)}{\partial z}v(z,t) + k_4(z,t)v(z,t)\frac{\partial v(z,t)}{\partial z} = \bar{f}(z,t), \tag{48}$$

where

$$v(0,t) = t^2, v(1,t) = -t^2 \ \ 0 \le t < 1, \tag{49}$$

$$v(z,0)=0,\ \ 0\leq z\leq 1, \tag{50}$$

$$\bar{f}(z,t)=\frac{2t^{2-\nu}cos(\pi z)}{\Gamma(3-\nu)}-\pi t^4 cos(\pi z)sin(\pi z)+\pi^2 t^2 cos(\pi z), \tag{51}$$

and

$$k_1(z,t)=-1,\ k_2(z,t)=k_3(z,t)=0,\ k_4(z,t)=1, \tag{52}$$

with the exact solution: $v(z, t) = t^2\cos(\pi z)$. Figure 4 displays the AS (left) and ES (right) for $\xi = 0.5$ and $\mathcal{M} = 14$. L_∞ error at $\xi = 0.5$ and $\mathcal{M} = 14$ are displayed in Figure 5. The AE at $\xi = 0.5$ and $\mathcal{M} = 14$ at different values of t are presented in Table 7. Table 8 presents the MAE at different values of ξ and $\mathcal{M}$. A comparison of MAE between SGCM and method in [54] and [58] is tabulated in Table 9.

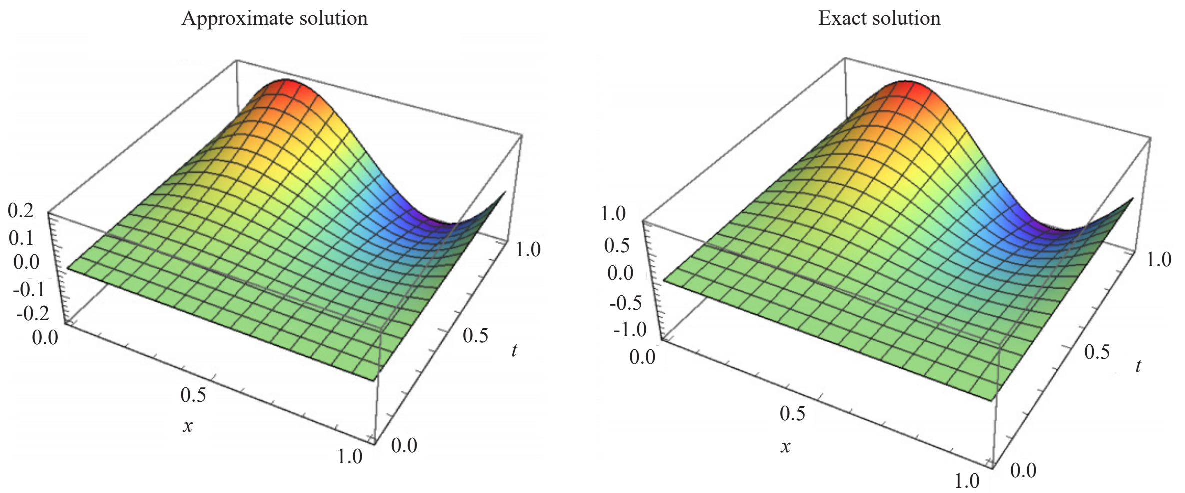

Figure 4. The AS (left) and ES (right) of Problem 3 at $\xi = 0.5$ and $\mathcal{M} = 14$

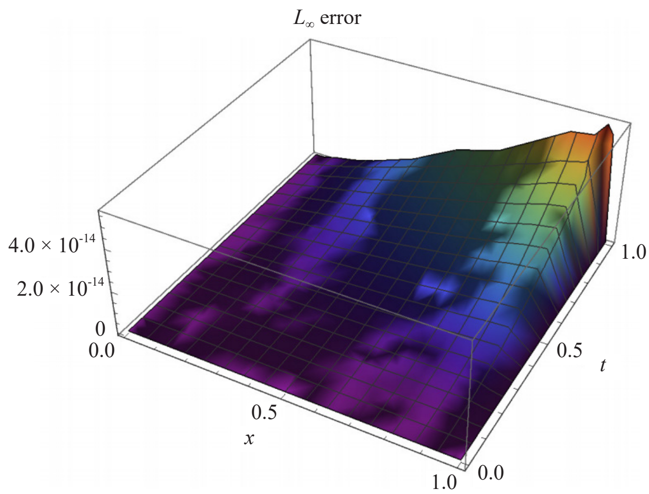

Figure 5. L_∞ error of Problem 3 at $\xi = 0.5$ and $\mathcal{M} = 14$

Table 7. The AE of Problem 3 at $\xi = 0.5$ and $\mathcal{M} = 14$

z	$\xi = 0.5$			Upper bound of AE
	$t = 0.1$	$t = 0.5$	$t = 0.9$	
0.1	2.81893×10^{-17}	9.29812×10^{-16}	3.27516×10^{-15}	
0.2	6.80879×10^{-17}	1.95677×10^{-15}	6.85563×10^{-15}	
0.3	1.11022×10^{-16}	2.98372×10^{-15}	1.06026×10^{-14}	
0.4	1.63064×10^{-16}	4.09395×10^{-15}	1.46411×10^{-14}	
0.5	2.14861×10^{-16}	5.26777×10^{-15}	1.88168×10^{-14}	10^{-14}
0.6	2.7452×10^{-16}	6.56766×10^{-15}	2.33147×10^{-14}	
0.7	3.38271×10^{-16}	8.07687×10^{-15}	2.88936×10^{-14}	
0.8	3.8294×10^{-16}	9.4022×10^{-15}	3.59157×10^{-14}	
0.9	4.48643×10^{-16}	1.13728×10^{-14}	4.90302×10^{-14}	

Table 8. MAE of Problem 3

$\mathcal{M}$	$\xi = 0.1$	$\xi = 0.5$	$\xi = 0.9$
2	2.48361×10^{-1}	2.31915×10^{-1}	2.16388×10^{-1}
4	7.13976×10^{-3}	7.65207×10^{-3}	7.94323×10^{-3}
6	9.81572×10^{-5}	1.16576×10^{-4}	1.3222×10^{-4}
8	6.75632×10^{-7}	9.04135×10^{-7}	1.1282×10^{-6}
10	3.05052×10^{-9}	4.46111×10^{-9}	6.01078×10^{-9}
12	1.06736×10^{-11}	1.6286×10^{-11}	2.30448×10^{-11}
14	2.14525×10^{-12}	1.0002×10^{-13}	1.0002×10^{-13}

Table 9. Comparison of the MAE for Problem 3

ξ	$\mathcal{M}$	SGCM	$\mathcal{M}$	CSCM [54]	Method in [58]
0.5	14	$\mathbf{1.0002 \times 10^{-13}}$	10	9.2905×10^{-7}	8.167×10^{-6}
0.75	14	$\mathbf{1.52464 \times 10^{-13}}$	15	6.7610×10^{-7}	3.443×10^{-6}
0.9	14	$\mathbf{2.27302 \times 10^{-13}}$	25	5.1574×10^{-5}	4.065×10^{-6}

6. Conclusion

Our study introduced a spectral collocation algorithm explicitly designed for solving the time-fractional Burgers' Equation (TFBE). Our approach is grounded in utilizing shifted Gegenbauer polynomials and their modified counterparts, which serve as the basis functions for our numerical method. The central concept behind our method revolves around employing the collocation technique to transform the TFBE into a system of algebraic equations, making it amenable to solution through well-established numerical procedures. Our findings have demonstrated that the presented spectral collocation method exhibits high accuracy and can yield results that are on par with, or even superior to, existing numerical methods for tackling the TFBE. This achievement underscores the potential of our approach as a powerful tool for addressing not only the TFBE but also other important nonlinear differential equations encountered in various scientific and engineering contexts. Our future endeavors are poised to further enhance and deepen our understanding of this method in light of our promising results. Specifically, we intend to comprehensively evaluate and discuss the convergence conditions associated with our spectral collocation algorithm. This analysis will provide valuable insights into the circumstances under which our method excels and helps identify situations where it may require further refinement. Furthermore, we are keenly interested in investigating the possible presence of ghost solutions, a phenomenon studied in prior research [26, 59]. By exploring this aspect, we aim to elucidate any peculiarities or unexpected behavior that might arise when employing our method in specific scenarios. In doing so, we aim to refine our algorithm, ensuring its robustness and applicability across a broader spectrum of nonlinear differential equations, ultimately advancing the field of numerical analysis and computational mathematics. All codes were written and debugged by Mathematica 12 on Dell Inspiron 15, Processor: Intel (R) Core(TM) i5-5200U CPU @ 2.20 GHz 2.20 GHz, 8 GB Ram DDR3, and 1024 GB storage.

Conflict of interest

The authors declare that they have no conflict of interest.

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