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Research Article

Pure-Cubic Optical Solitons and Stability Analysis with Kerr Law Nonlinearity

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Abstract: In this research paper, we investigate the effects of third-order dispersion and nonlinear dispersion terms on soliton behavior for pure-cubic solitons in the absence of chromatic dispersion. The research proceeds in several stages. First, we derive the nonlinear ordinary differential equation form by utilizing the complex wave transform. In the second stage, we employ a simplified version of the new extended auxiliary equation method to derive both bright and singular optical solitons. Subsequently, we examine the influence of model parameters on these bright and singular solitons in the third stage. To support our findings, we present solution functions accompanied by effective graphical simulations. We report observations regarding the effects of parameters in the relevant sections. The validity of our results is confirmed through their satisfaction of the model equation. Furthermore, we apply the Vakhitov-Kolokolov stability criterion to ensure the stability of the obtained bright soliton solution. Notably, the novelty of this paper lies in its application of a simplified version of the extended auxiliary equation approach to recover optical solitons. This study stands apart from previously published works that utilized various expansion approaches, yielding a distinct spectrum of results.

***Keywords*:** pure-cubic soliton, impact of the dispersion, auxiliary equation method, optical soliton, Vakhitov-Kolokolov slope condition

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Acronyms

NLEE: Nonlinear Evolution Equation
NLPDE: Nonlinear Partial Differential Equation
NLSE: Nonlinear Schrödinger Equation
NODE: Nonlinear Ordinary Differential Equation
SAEM26: Sub-equation of auxiliary equation expansion method
CD: Chromatic Dispersion
GVD: Group Velocity Dispersion
SPM: Self-Phase Modulation
ENIAC: Electronic Numerical Integrator And Computer
maser: Microwave Amplification by Stimulated Emission of Radiation
laser: Light Amplification by Stimulated Emission of Radiation
IST: Inverse Scattering Transform
NLRI: Nonlinear Refractive Index
TOD: Third Order Dispersion
FOD: Fourth Order Dispersion
WDM: Wavelength Division Multiplexing
VKSC: Vakhitov-Kolokolov Stability Criterion

1. Introduction

Nonlinear partial differential equations and nonlinear evolution equations are mathematical structures that emerge as a result of efforts to understand, model, and solve physical phenomena in our universe. Certain physical phenomena have become associated with their respective equations, forming the foundation for numerous research endeavors in subsequent periods. For instance, the van der Waals equation characterizes gas behavior [1], Maxwell's equations describe electromagnetic radiation [2], Einstein's iconic mass-energy equivalence underpins the theory of relativity [3], the Laplacian equation governs harmonic functions in physics [4], the Helmholtz equation finds application in thermodynamics [5], the Keller-Segel equation models diffusion [6], Newton's law explains gravity [7], and a plethora of other examples exist [8, 9]. Undoubtedly, a seminal advancement that catalyzed extensive research in the realm of nonlinear optics and across various engineering branches was E. Schrödinger's presentation of quantization as an eigenvalue problem in 1927, later recognized as the nonlinear Schrödinger equation [10]. This precursor development has laid the groundwork for diverse investigations and explorations.

The dimensionless form of the nonlinear Schrödinger equation, derived from Maxwell's formulas, is expressed as follows [11, 12]:

$$i\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+|u|^2 u=0,\ \ i=\sqrt{-1}, \tag{1}$$

where the complex wave amplitude is denoted as $u = u(x, t)$. The first term corresponds to linear temporal evolution, the second term corresponds to CD, also known as group velocity dispersion, and the final term corresponds to the Kerr law nonlinearity, which leads to intensity-dependent phase modulation. The independent variables x and t represent the transverse spatial coordinates, signifying the normalized distance within the fiber, and the retarded time, respectively. It's evident that equation (1) is non-integrable, as it doesn't pass the Painlevé test of integrability. Specifically, although equation (1) can be solved through the IST, it also possesses a Hamiltonian form with an infinite number of conserved quantities [13].

The discovery of the first electronic computer ENIAC in 1943 and the semiconductor (transistor) at Bell Laboratories in 1947 were pivotal breakthroughs that opened the door to numerous developments in the fields of electronics and computers. The addition of the "maser" in 1953 and the "laser" in 1960 to this transformative timeline also marked significant milestones in the realm of nonlinear optics. These discoveries have served as primary foundations for the research efforts of numerous scholars, acting as the driving forces behind studies on laser applications, telecommunications, and the internet. Within this landscape, telecommunications, data transmission, and optical solitons stand as key vertices in the golden triangle of nonlinear optics, forming interconnected topics. Central to these studies is the optical soliton, a stable envelope of light waves, which constitutes a central focus of inquiry.

Hasegawa and Tappert illustrated that, in the case of an anomalous GVD regime, lossless fibers can accommodate the bright soliton structure, which exhibits maximum intensity in the time domain [14, 15]. In 1980, the first experimental observation of an optical soliton in a fiber was reported by Mollenauer et al. [16]. Subsequent research aimed to enhance the comprehension of the mathematical and experimental principles introduced by Hasegawa [13] and Mollenauer [16], leading to clearer explication through analytical approaches. For instance, Zakharov and Shabat employed the IST to solve the nonlinear Schrödinger equation [17]. Of particular significance, the revelation that the NLSE can be rendered integrable as an infinite-dimensional Hamiltonian system, featuring an unlimited number of conserved quantities and Lax pair properties [18-20], paved the way for a multitude of studies within the field of nonlinear optics.

Over the past half-century, telecommunications and, particularly, the internet have gained immense significance. This has driven numerous theoretical and experimental studies within the field of nonlinear optics.

The central focus lies in achieving rapid transmission of vast volumes of data over long distances while maintaining high-quality signals. Extensive research in this domain has introduced concepts such as CD and refractive index to the literature. CD denotes the relationship between group velocity and wavelength. In simpler terms, CD signifies changes in pulse shape that occur when the velocity of the optical signal's power along the optical fiber is influenced by the optical frequency or wavelength. CD, thus, constitutes a fundamental characteristic of the fiber material employed in optical soliton transmission. The magnitude of CD can vary, depending on the fiber material and system design, but it usually manifests as the prominent linear effect. While it's technically possible to manufacture fiber material with zero CD (no negative impact on CD soliton transmission), it is generally more practical to manage the CD effect as controllable rather than attempting to eliminate it entirely. One evident reason for this approach is that even if a fiber with zero CD is produced, it won't be compatible with WDM systems due to the nonlinear effects it introduces. As a result, until recently, the focus has been on managing CD rather than eliminating it altogether. Although studies on CD management and measurement date back to the 1980s, CD was formally introduced into the literature by Yan et al. in 2006 [20], and CD-related investigations have remained among the most prominent topics in the realm of nonlinear optics [21-30].

However, a recent development emerged in 2016 with theoretical and experimental investigations that introduced 'pure-quartic solitons'. These solitons were attained by introducing the Fourth-Order Dispersion (FOD) term in the absence of CD [31-34]. This advancement subsequently triggered explorations into various models achieved by excluding the CD [35-37]. Consequently, the study of pure-cubic optical solitons encompasses numerous contemporary and significant issues that warrant investigation and interpretation.

The NLSE with TOD in the absence of CD is expressed as follows [38]

$$i\left(\frac{\partial u}{\partial t}+a\frac{\partial^3 u}{\partial x^3}\right)+b|u|^2 u=i\left(c_1|u|^2\frac{\partial u}{\partial x}+c_2\frac{\partial |u|^2}{\partial x}u\right),\ i=\sqrt{-1}. \tag{2}$$

Let $u = u(x, t)$ represent a complex-valued function. The term iu_t corresponds to temporal evolution, iau_{xxx} represents the TOD term, $b|u|^2u$ signifies the cubic nonlinearity term, and $ic_1|u|^2u_x$ as well as $ic_2u(|u|^2)_x$ stand for the nonlinear dispersion terms arising from perturbation. Here, a, b, c_1, and c_2 denote non-zero real values. Moreover, when $c_1 = c_2 = 0$, equation (2) reduces to the Hirota equation or the complex modified KdV equation [39].

It is a fact that the unperturbed version of equation (2) is indeed non-integrable, as it does not pass the Painlevé test of integrability. As a result, a couple of Hamiltonian perturbation terms located on the right-hand side of equation (2) have been included to render the perturbed version integrable.

Equation (2) reveals that the model has limitations in its application to problems involving the zero-dispersion limit. Nevertheless, the current model holds mathematical interest, representing a modified version of the standard

NLSE that includes CD. Several pertinent studies regarding equation (2) are listed in [40-44].

The fundamental aim of this research paper is to investigate the intricate interplay between third-order dispersion, nonlinear dispersion terms, and soliton behavior in the context of pure-cubic solitons, while notably excluding the influence of chromatic dispersion. The soliton phenomenon, characterized by its ability to maintain its shape while propagating through a medium, has been a subject of extensive research due to its wide-ranging applications in optical communication and nonlinear optics. In this study, we introduce a novel approach that sheds light on the behavior of pure-cubic solitons under the influence of these specific dispersion effects. Our findings challenge existing paradigms by demonstrating the distinctive characteristics exhibited by solitons in the absence of chromatic dispersion. By employing a unique simplified version of the extended auxiliary equation method, we derive both bright and singular optical soliton solutions. This method, adapted to our specific research context, unveils the underlying dynamics of soliton formation and propagation. These soliton solutions hold substantial importance as they exhibit intriguing behaviors that are not only mathematically significant but also possess physical relevance. Furthermore, we rigorously explore the influence of model parameters on the obtained bright and singular soliton solutions. Through an in-depth analysis of the parameter space, we uncover how various factors modulate the soliton behavior. Our investigation presents a comprehensive understanding of the intricate relationship between these parameters and the resulting soliton profiles. To substantiate our findings, we provide solution functions that precisely characterize the soliton behavior. Additionally, we augment these findings with effective graphical simulations, allowing for a visual grasp of the soliton dynamics. This combination of mathematical insights and visual representations ensures that our results are both comprehensible and credible. The validation of our results is underscored by their alignment with the model equation and the application of the Vakhitov-Kolokolov stability criterion. This rigorous validation process bolsters the robustness of our approach and instills confidence in the reliability of our outcomes.

The originality of the paper lies in its application of a simplified version of the extended auxiliary equation approach to recover optical solitons. This sets it apart from previously published works that utilized various expansion approaches, resulting in a distinct spectrum of results. The approach to investigating the effects of third-order dispersion and nonlinear dispersion terms on soliton behavior for pure-cubic solitons in the absence of chromatic dispersion is unique within the context of the existing literature. The research methodology presented in the paper includes multiple stages: The utilization of the complex wave transform to derive the nonlinear ODE form is a common technique, but the context of applying it to the specific investigation of pure-cubic solitons without chromatic dispersion showcases a unique focus. The introduction of a simplified version of the extended auxiliary equation method for deriving bright and singular optical solitons adds novelty to the paper. While the extended auxiliary equation method has been used in other studies, its adaptation to the specific scenario of pure-cubic solitons without chromatic dispersion, combined with the simplification, represents an original contribution. The examination of the influence of model parameters on bright and singular solitons further contributes to the originality. This exploration of the parameter space and its impact on the soliton behavior within the specified context is a unique aspect of the paper. While presenting solution functions and graphical simulations is a standard practice in scientific research, the application of these elements to the specific investigation of pure-cubic solitons in the absence of chromatic dispersion enhances the originality of the paper. The validation of results through the satisfaction of the model equation and the application of the Vakhitov-Kolokolov stability criterion adds credibility to the study. Although these validation techniques are established, their application within this specific context contributes to the originality of the paper. The originality of the paper can be attributed to the combination of the unique focus on pure-cubic solitons without chromatic dispersion, the application of a simplified version of the extended auxiliary equation approach, the exploration of model parameters and their effects on soliton behavior, and the validation and stability analysis within this specific context. These aspects collectively distinguish the paper from previously published works and contribute to its contribution to the field.

The organization of the article is follows. In section 2, an overview of the extended auxiliary equation approach is provided. Relevant theories, concepts, or methods related to this approach are discussed. Also, the key points from previous research are summarized. In section 3, how the extended auxiliary equation approach is applied in the study is explained. Any modifications or adaptations made to the approach are described. Also, graphs are included. In section 4, the Vakhitov-Kolokolov stability analysis method is introduced. Its significance in the context of the study is explained. Also, mathematical aspects of the method are discussed. In section 5, the results obtained from the analysis using the extended auxiliary equation approach and the Vakhitov-Kolokolov stability analysis are presented. Also, the results are analyzed and interpreted, and their implications in relation to the study's objectives are discussed. In section 6, the main findings of the research are summarized. The implications and significance of the results are discussed. Also, the contributions of the study to the field are highlighted.

2. Extended auxiliary equation approach: A recapitulation

Consider the following NLPE and its associated wave transformation, respectively:

$$N(u, u_t, u_x, u_{xx}, u_{xt}, \ldots), \tag{3}$$

and

$$u(x, t) = U(\xi)\mathrm{e}^{i\theta(x,t)}, \ \theta(x, t) = \alpha x + \beta t + \varphi_0, \ \xi = kx + \omega t. \tag{4}$$

Here, the subscripts indicate partial derivatives with respect to x and t. $U(\xi)$ represents a real field soliton profile, θ denotes the phase component associated with x and t, φ_0 is the soliton's center, α corresponds to the angular wave number, β to the angular velocity, k denotes the soliton's wave number, and ω signifies the soliton's velocity. Additionally, ξ is the new wave variable, while x and t are independent spatial and temporal coordinates, respectively.

By inserting equation (4) into equation (3), the following NODE structure arises:

$$P\left(U, U', U'', U''', \ldots\right), \tag{5}$$

where superscripts refer to derivatives with respect to ξ.

The SAEM26 method [45] provides the following truncated series solution for equation (5):

$$U(\xi) = \sum_{i=0}^{m} A_i R^i(\xi), \ A_m \neq 0, \tag{6}$$

where the coefficients A_i are real constants that need to be determined. m corresponds to the balancing constant obtained from equation (5) using the balance rule. Additionally, $R(\xi)$ represents the solution of the given ordinary Riccati differential equation as follows:

$$\left(\frac{dR(\xi)}{d\xi}\right)^2 - \delta^2 R^2(\xi)\left(1 - \eta R^4(\xi)\right) = 0. \tag{7}$$

Here, δ and η are non-zero real constants. Equation (7) admits the following well-known solution [45]:

$$R(\xi) = \sqrt{\frac{2}{\mathrm{e}^{2\delta\xi} + \eta\mathrm{e}^{-2\delta\xi}}}. \tag{8}$$

When $\eta = 1$, equation (8) takes the following shape:

$$R(\xi) = \sqrt{\frac{2}{\mathrm{e}^{2\delta\xi} + \mathrm{e}^{-2\delta\xi}}}, \tag{9}$$

or in its equivalent hyperbolic form:

$$R(\xi) = \sqrt{\sec h(2\delta\xi)}, \tag{10}$$

which represents the bright soliton. When $\eta = -1$, equation (8) transforms into:

$$R(\xi) = \sqrt{\frac{2}{e^{2\delta\xi} - e^{-2\delta\xi}}}, \tag{11}$$

or in corresponding hyperbolic form:

$$R(\xi) = \sqrt{\text{csch}(2\delta\xi)}, \tag{12}$$

which depicts the singular soliton.

Here, we are dedicating a few sentences to discuss the method. As is well-known, numerous methods have been developed in the literature for the analytical solution of both NLEE and NLPDE problems. These methods include the tanh expansion, modified extended tanh expansion, sinh-Gordon, extended sinh-Gordon, modified extended function, generalized exponential function, modified simple equation, extended Jacobian elliptic function expansion, G/G' expansion, extended Fan sub-equation, variable-separated ODE, extended hyperbolic function, rational sinh-cosh, improved generalized Riccati equation mapping, unified Riccati equation, Riccati-Bernoulli Sub-ODE, generalized projective Riccati equation, F-expansion, first integral, Backlund transform, Hirota Bilinear, Sardar sub-equation, Cole-Hopf transformation, and various Kudryashov-based methods (Kudryashov R-function, generalized Kudryashov, modified Kudryashov, improved Kudryashov, extended Kudryashov, enhanced Kudryashov, new Kudryashov, etc.), among others. It should be emphasized that most of these methods have modified, extended, and improved versions. A vast number of studies are available on these methods through simple literature searches. While these methods offer unique merits and demerits, their evaluation is not the focus of our current study. As a basic assessment, it can be stated that some methods are more challenging to implement and involve more complex operations. Although they yield numerous solution functions, many of these are repetitive solutions, and some equations are ineffective for solving NLPDEs. When designing an equation to model a physical phenomenon, one crucial consideration is its integrability (passing the Painleve test) and possession of soliton solutions. For soliton solutions, most researchers utilize the solitary ansatz hypothesis to obtain fundamental soliton types such as bright, dark, and singular solitons. In recent years, methods based on extended auxiliary equation expansion (taking the general form $R'(\xi) = \sum_{i=0}^{6} \sqrt{a_i R^i(\xi)}$ [46], and the generalized Riccati equation mapping (in the form of $R'(\xi) = \sum_{i=0}^{4} a_i R^i(\xi)$ [47] have gained increasing popularity, particularly due to their effectiveness in solving higher-order NLSEs. Both the elliptic and auxiliary approaches have various sub-versions.

In this regard, SAEM26 [45] stands as a subversion of the extended auxiliary equation expansion method. This approach requires minimal processing, is easy to implement, and generates bright and singular soliton types using a single solution function that emphasizes a soliton focus. Unlike many other methods, it avoids the proliferation of repetitive solution functions. Particularly in research articles where the objective isn't to obtain various soliton types, it emerges as a preferred choice. The method's selection is chiefly motivated by the article's goal to examine soliton dynamics by considering the influence of model parameters in the absence of the CD term. Furthermore, observations reveal that the method yields effective results not only for higher-order NLPDEs but also for perturbed dispersive higher-order NLPDEs.

3. Utilization of the method

To use the SAEM26 effectively, one must first obtain the NODE. Therefore, let us insert equation (4) into equation

(2), and then derive the following formulas from the imaginary and real components:

$$\left(\omega-\left(c_1+2c_2\right)U(\xi)^2-3a\alpha^2\right)\frac{dU(\xi)}{d\xi}+ak^2\frac{d^3U(\xi)}{d\xi^3}=0, \tag{13}$$

and

$$\left(\beta-a\alpha^3\right)U(\xi)-\left(\alpha c_1+b\right)U(\xi)^3+3a\alpha k^2\frac{d^2U(\xi)}{d\xi^2}=0. \tag{14}$$

If equation (13) is integrated with respect to ξ, considering the integration constant as zero, one can obtain:

$$3\left(\omega-3a\alpha^2k\right)U(\xi)-k\left(c_1+2c_2\right)U(\xi)^3+3ak^3\frac{d^2U(\xi)}{d\xi^2}=0 \tag{15}$$

The homogeneous balance between equation (14) and equation (15) results in the following formula:

$$\frac{\beta-a\alpha^3}{3\left(\omega-3ak\alpha^2\right)}=\frac{\alpha c_1+b}{k\left(c_1+2c_2\right)}=\frac{\alpha}{k}, \tag{16}$$

which leads to the following constraints:

$$\alpha=\frac{b}{2c_2},\ \beta=-\frac{\alpha\left(8ak\alpha^2-3\omega\right)}{k}. \tag{17}$$

One can adopt either the NODE form of equation (14) or equation (15). In our case, we will focus on equation (14). The method employed with equation (6) necessitates determining the number of terms for the truncated series. In fact, it is essential to ascertain the value of m, referred to as the balancing constant [45] or pole order [48] in equation (6).

Let's shift our focus to equations (6), (7), and (14), and apply the homogeneous balance rule [49]. This involves considering both the linear and highest-order nonlinear terms in equation (14). In short, the homogeneous balance of $U''(\xi)$ and $U^3(\xi)$ yields the relation $m + 2 = 3m$, which simplifies to $m = 2$.

Therefore, equation (6) transforms into the following form:

$$U(\xi)=A_0+A_1R(\xi)+A_2R^2(\xi). \tag{18}$$

To determine the unknown parameters, we need to combine equation (18) with equations (7) and (14), resulting in a polynomial expression of $R^p(\xi)$, where $p = 0, ..., 6$. This polynomial form leads to the following system of algebraic equations:

$$R^0(\xi):\ 4kA_0^2c_2^2\left(c_1+2c_2\right)+9ab^2k-12\omega c_2^2=0,$$

$$R^1(\xi):\ \left(4A_0^2c_1c_2^2\left(c_1+2c_2\right)+3ab^2\right)k-4c_2^2\left(ak^3\delta^2+\omega\right)=0,$$

$$R^2(\xi):\ 4A_2c_2^2\left(4ak^3\delta^2+\omega\right)-4kc_2^2A_0\left(A_0A_2+A_1^2\right)\left(2c_2+c_1\right)-3akb^2A_2=0,$$

$$R^3(\xi):\ \left(6A_0A_2+A_1^2\right)\left(c_1+2c_2\right)=0,$$

$$R^4(\xi):\ \left(A_0A_2+A_1^2\right)\left(c_1+2c_2\right)=0,$$

$$R^5(\xi):\ 3ak^2\delta^2\eta+A_2^2\left(c_1+2c_2\right)=0,$$

$$R^6(\xi):\ 24ak^2\delta^2\eta+A_2^2\left(c_1+2c_2\right)=0. \tag{19}$$

The subsequent set, which constitutes the solution to equation (19), reveals the constants in equations (16) to (18):

$$Set^{\mp}=\left\{\omega=\frac{ak\left(3b^2-16\delta^2k^2c_2^2\right)}{4c_2^2},A_0=0,A_1=0,A_2=\mp\frac{2\sqrt{-6\left(c_1+2c_2\right)a\eta}\delta k}{c_1+2c_2}\right\}, \tag{20}$$

while adhering to the necessary constraints $(c_1 + 2c_2)a\eta < 0$, $c_1 \neq 2c_2$, and $c_2 \neq 0$.

The combined effect of equation (18) with equations (4), (8), and the Set^{+} in equation (20) yields the desired optical soliton solution of equation (2) as follows:

$$\mathrm{u(x,t)}=\frac{4\sqrt{-6\left(c_1+2c_2\right)a\eta}\delta k}{\left(c_1+2c_2\right)\left(\mathrm{e}^{2\delta(kx+\omega t)}+\eta\mathrm{e}^{-2\delta(kx+\omega t)}\right)}\mathrm{e}^{i(\alpha x+\beta t+\phi_0)}, \tag{21}$$

where α and β are defined as in equation (17), and ω is provided in equation (20).

For $\eta = \mp1$, equation (21) takes the following forms, respectively:

The bright soliton solution of equation (2) is:

$$\mathrm{u(x,t)}=\frac{2\sqrt{-6\left(c_1+2c_2\right)a}\delta k\,\mathrm{sech}\left(2\delta\left(kx+\omega t\right)\right)}{c_1+2c_2}\mathrm{e}^{i(\alpha x+\beta t+\phi_0)}, \tag{22}$$

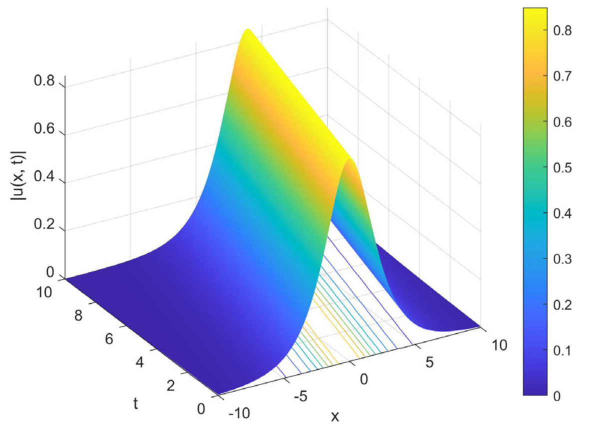
while the singular soliton solution of equation (2) is:

$$\mathrm{u(x,t)}=\frac{2\sqrt{6\left(c_1+2c_2\right)a}\delta k\,\mathrm{csch}\left(2\delta\left(kx+\omega t\right)\right)}{c_1+2c_2}\mathrm{e}^{i(\alpha x+\beta t+\phi_0)}. \tag{23}$$

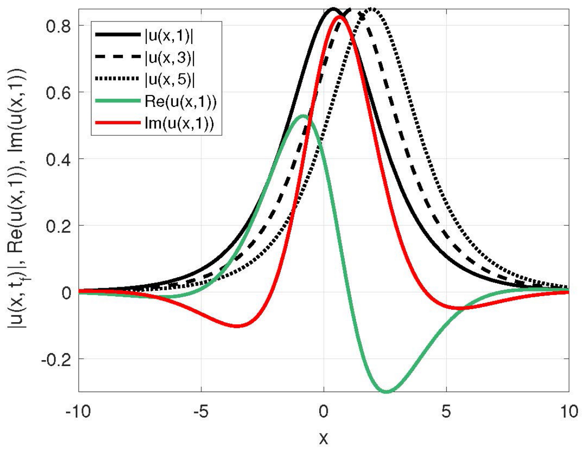
The subsequent section of the article comprises the graphical representation of the bright and singular soliton solutions obtained using equation (22) and equation (23), along with an exploration of how the model parameters influence soliton formation, as depicted in these graphs.

Figure 1 illustrates the bright soliton aspect of $u(x, t)$ in equation (22). To depict equation (22), the parameters are set as follows: $\delta = 1.2$, $a = -1$, $\eta = 1$, $k = 0.25$, $\varphi_0 = 0.65$, $b = 1$, and $c_1 = c_2 = 1$. Figure 1(a) showcases $|\mathrm{u(x, t)}|$ and represents the bright soliton. In Figure 1(b), 2D plots of $|\mathrm{u(x, t}_f)|$, $Re(\mathrm{u(x, t)})$, and $\mathrm{Im(u(x, t))}$ are shown for specific times $t_f = 1, 3, 5$ and $t = 1$, respectively. As depicted in Figure 1(b), the bright soliton moves to the right (from black solid line to dotted line), while composite dark-bright soliton features of the real and imaginary parts are observed at $t = 1$ (green and red lines). The effect of parameter "a", originating from the coefficient of the TOD, is presented in Figure 1(c). The amplitude of the soliton decreases with increasing negative values of "a". A similar examination is conducted for positive values of "a" in Figure 1(d). In this case, with increasing positive values of "a", the soliton's amplitude increases. Therefore, in the absence of the CD term, the amplitude of the bright soliton character tends to increase, depending on the absolute positive value of "a". The impact of the coefficient related to the Kerr nonlinearity term, denoted as "b", is analyzed in Figure 1(e). Similar effects are observed when "b" takes both positive and negative values. In both cases, the soliton shifts to the left. Figure 1(f) demonstrates the influence of c_1, which stems from perturbation. The amplitude decreases as c_1 takes on both increasing negative and positive values. Notably, when c_1 is positive and assumes its maximum value (1.50), the amplitude is minimized (dashed red line). Conversely, when c_1 is negative and assumes its smallest value (-1.50), the amplitude is maximized (solid red line). Moreover, the soliton remains unaffected by horizontal displacement. Figure 1(g) portrays a similar observation for c_2, which represents another coefficient in the perturbation term. In both scenarios, i.e., when c_2 is both negative or positive, the bright soliton shifts to the right as c_2 increases. Depending on positive increases in the value of c_2, a minor amplitude reduction in the bright soliton is observed. On the other hand, substantial amplitude reduction occurs with negative increases in the value of c_2 (solid red to green lines). As a general observation, while model parameters influence the characteristics of the bright soliton in the absence of CD, this influence does not manifest as noise that would distort the nature of the bright soliton. Instead, it mainly manifests as variations in the amplitude and position of the soliton.

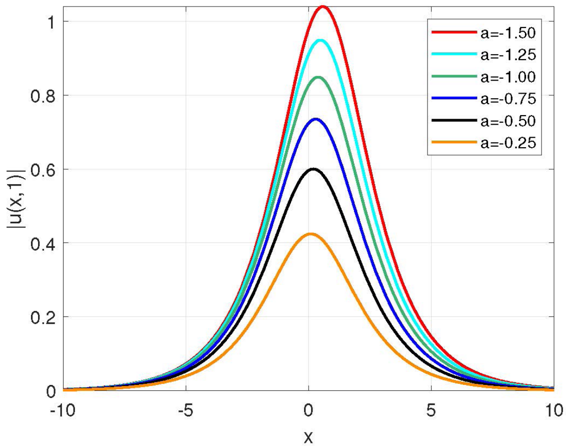
Figure 2 has been generated to observe the soliton behavior of $u(x, t)$ in equation (23). We considered the following parameter values: $\delta = 1.2$, $a = -1$, $\eta = -1$, $k = 0.25$, $\varphi_0 = 0.65$, $b = 1$, and $c_1 = c_2 = -1$. Figure 2(a) represents |u(x, t)| and signifies the singular soliton. In Figure 2(b), 2D representations of $|u(x, t_f)|$, *Re*(u(x, t)), and Im(u(x, t)) are displayed for specific times $t_f = 1, 3, 5$ and $t = 1$, respectively. All soliton profiles in Figure 2(b) exhibit the characteristics of singular solitons, and they travel towards the right. Figure 2(c) illustrates diverse profiles of |u(x, t)| under the influence of "a", which corresponds to the coefficient of the TOD. As "a" takes negative and increasing values, the soliton not only shifts horizontally to the left but also contracts in the skirt regions. A similar trend is observed as "a" increases positively, causing a horizontal shift to the right, as depicted in Figure 2(d) (solid red to orange lines). Figure 2(e) showcases the impact of "b", with increasing values of "b" indicating rightward shifts in the position of the singular soliton along the horizontal axis, regardless of whether "b" is negative or positive. In Figure 2(f), the influence of c_1 is depicted when it is both negative and positive. There is a decrease in the soliton's amplitude at its skirts with the increase in c_1. However, this decrease ranges from the smallest negative value of c_1 to the largest positive value (red solid to dashed lines). Additionally, it's evident that the singular soliton exhibits symmetry with respect to a vertical axis passing through $x = -0.39$. Figure 2(g) presents |u(x, t)| under the influence of c_2. In Figure 2(g), for negative increasing values of c_2, the singular soliton pulsates to the left, while for positive increasing values, it pulsates to the right. Notably, if attention is given to the horizontal distances at the skirts, a maximum forms for the smallest negative values of c_2 and a minimum for the largest positive value of c_2.



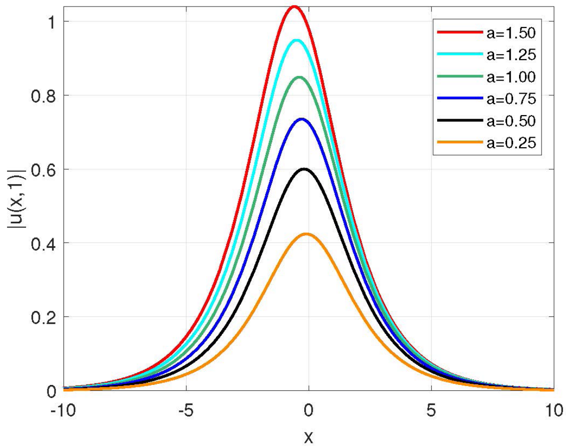
(a) bright soliton of |u(x, t)| in 3D



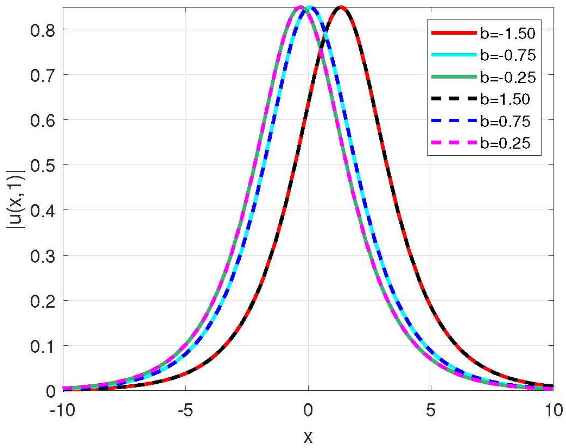
(b) $|u(x, t_f)|$, *Re*(u(x, 1)), *Im*(u(x, 1)) in 2D



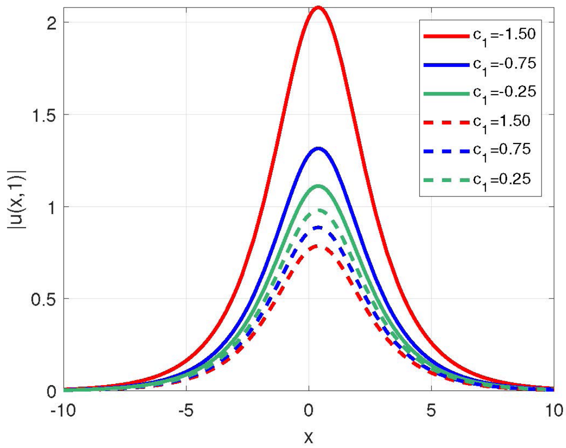
(c) 2D images of |u(x, t)| when $a < 0$



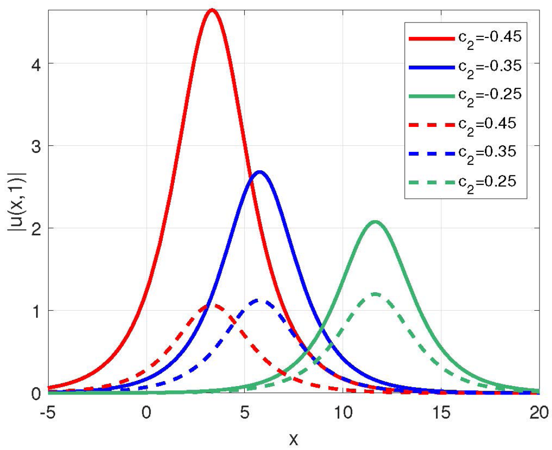
(d) 2D plots of |u(x, t)| when $a > 0$



(e) Impact of b on |u(x, t)|

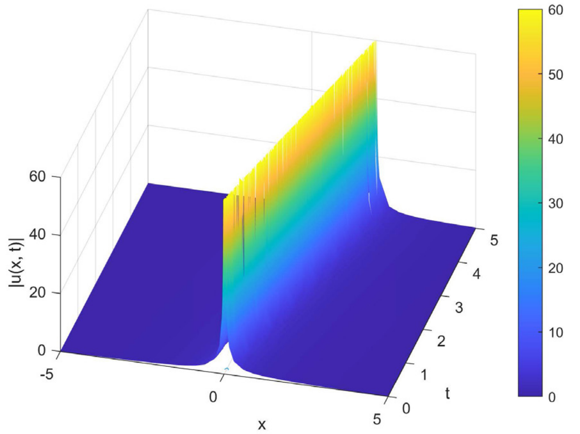


(f) Influence of c_1 on |u(x, t)|

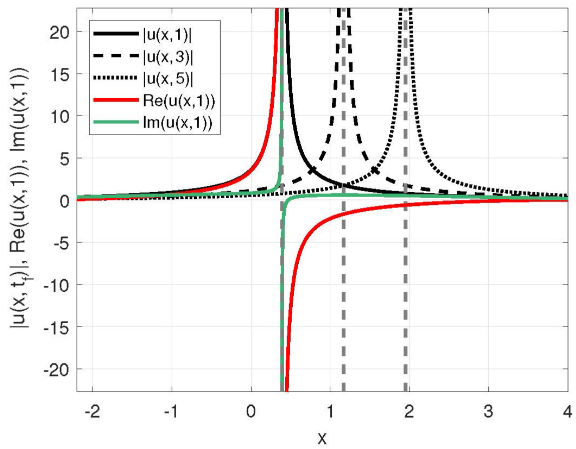


(g) Effect of c_2 on |u(x, t)|

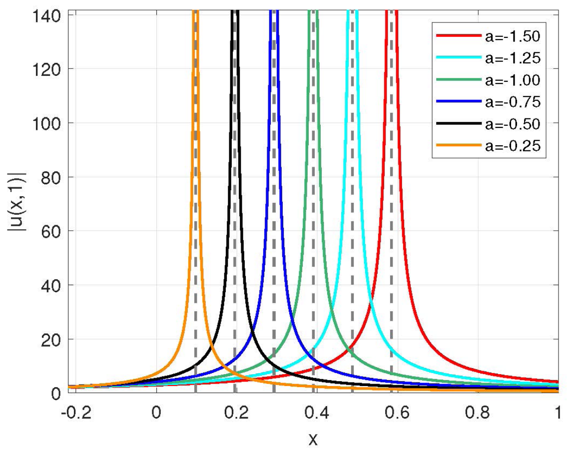
Figure 1. Simulations of the bright soliton for $u(x, t)$ in equation (22) using the obtained parameters: $\delta = 1.2$, $a = -1$, $\eta = 1$, $k = 0.25$, $\varphi_0 = 0.65$, $b = 1$, and $c_1 = c_2 = 1$



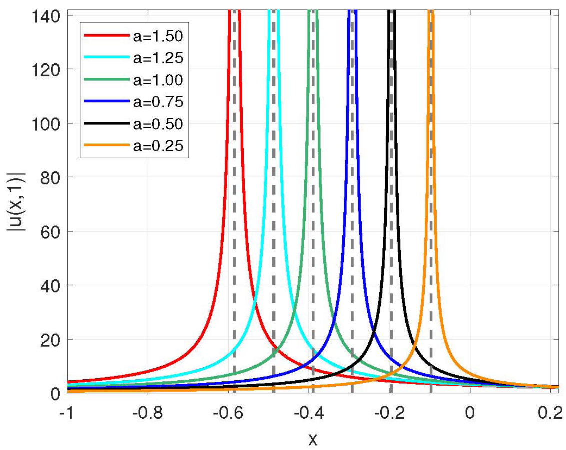
(a) Singular soliton of |u(x, t)| in 3D



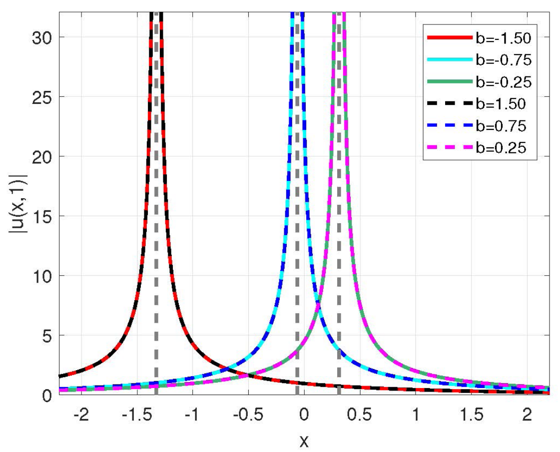
(b) |u(x, t_f)|, *Re*(u(x, 1)), *Im*(u(x, 1)) in 2D



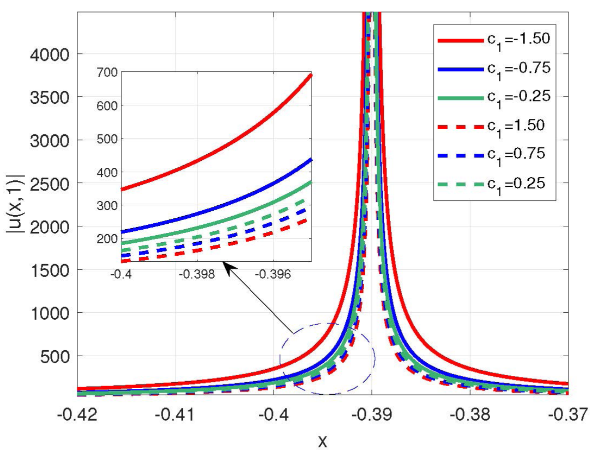
(c) 2D portraits of |u(x, t)| when $a < 0$



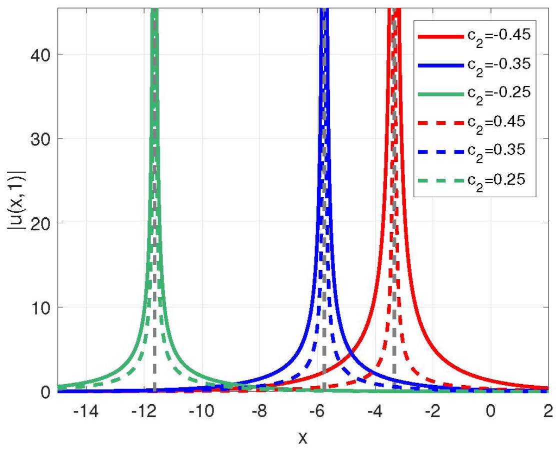
(d) 2D reflections of |u(x, t)| when $a > 0$



(e) Influence of b on |u(x, t)|



(f) Impression of c_1 on |u(x, t)|



(g) Domination of c_2 on |u(x, t)|

Figure 2. Simulations of the singular soliton for $u(x, t)$ in equation (23) using the specified parameters: $\delta = 1.2$, $a = -1$, $\eta = -1$, $k = 0.25$, $\varphi_0 = 0.65$, $b = 1$, and $c_1 = c_2 = -1$

4. Vakhitov-Kolokolov stability analysis

This phase is devoted to the analysis of the stability of the investigated model in equation (29) using the effective Vakhitov-Kolokolov stability criterion. Before moving on to the application of the VKSC, let's list a few sentences about the method and its features. VKSC is an important linear stability criterion used in examining the soliton stability of the nonlinear Schrödinger equations and obtaining effective results. VKSC was introduced to the literature by Vakhitov and Kolokolov in 1973 [50]. The VKSC is also known as the Vakhitov-Kolokolov slope condition [51-54]. According to the VKSC, the necessary conditions for the stability/unstability of the bright soliton are defined as follows: The soliton is stable when

$$\frac{dP(\Omega)}{d\Omega} > 0. \tag{24}$$

equation (24) can also be interpreted as the necessary condition for the stability of the bright soliton: the soliton power curve should have a positive slope as a function of the propagation constant. The soliton is unstable when

$$\frac{dP(\Omega)}{d\Omega} < 0. \tag{25}$$

Different stability properties may arise when the following condition is met:

$$\frac{dP(\Omega)}{d\Omega} = 0. \tag{26}$$

Here, the function $P(\Omega)$ is defined as:

$$P(\Omega) = \int_{-\infty}^{+\infty} |u(x,t)|^2 dx, \tag{27}$$

Here, P stands for soliton power, representing the number of particles contained in the soliton (P is also referred to as the norm), and Ω represents the propagation constant (frequency of the phase). Furthermore, if equations (24)-(26) do not depend on Ω, the Vakhitov-Kolokov stability criterion implies marginal stability. This means that the soliton has a singularity within a finite value of temporal t, and a perturbed soliton cannot maintain its localized form. The marginal stability form that occurs when equations (24)-(26) are not dependent on Ω has not been the subject of our stability analysis in this study.

Another way to express equation (27) is as follows:

$$P(\Omega) = \int_{-\infty}^{+\infty} \left(U(\xi, \Omega)\right)^2 dx, \tag{28}$$

where

$$u(x,t) = U(\xi, \Omega) e^{i\Omega t}, \ \xi = kx + \omega t. \tag{29}$$

Here, k and ω are real constants. Without requiring further elaboration, similar to equations (24)-(29), it is possible to formulate stability/instability criteria for a dark soliton as well [55].

After this brief overview, let's return to the stability analysis of the bright soliton for the given problem using equation (2) and employing the VKSC.

By inserting equation (29) into equation (2), we obtain:

$$3\Omega^2 \omega V(\xi) - b^2 k \left(c_1 + 2c_2\right) V(\xi)^3 + 3\Omega^2 a k^3 \frac{d^2 V(\xi)}{d\xi^2} = 0, \tag{30}$$

where

$$\mathrm{U}(\xi, \Omega) = \mathrm{V}(\xi, \Omega)^{1/2}. \tag{31}$$

Considering equation (10) and equation (18), we arrive at the bright soliton solution for equation (2) as follows:

$$u(x,t) = \left(\frac{\sqrt{24}\sqrt{-\left(c_1 + 2c_2\right) a\eta}\ \delta\, \Omega\, k \operatorname{sech}\left(2\delta\left(-4a\delta^2 k^3 t + kx\right)\right)}{\left(c_1 + 2c_2\right)} \right)^{1/2} \mathrm{e}^{i\Omega t}. \tag{32}$$

With consideration of equation (27), the following formula is derived:

$$P(\Omega) = \int_{-\infty}^{+\infty} \left|u(x,t)\right|^2 dx = \int_{-\infty}^{+\infty} \left(\frac{\sqrt{24}\sqrt{-\left(c_1 + 2c_2\right) a\eta}\ \delta\, \Omega\, k \operatorname{sech}\left(2\delta\left(-4a\delta^2 k^3 t + kx\right)\right)}{\left(c_1 + 2c_2\right)} \right) dx. \tag{33}$$

The equation (33) can be expressed in the following simplified forms:

$$P(\Omega) = \frac{2\sqrt{6}\sqrt{-\left(c_1 + 2c_2\right) a\eta}\ \delta\, \Omega\, k}{\left(c_1 + 2c_2\right)} \int_{-\infty}^{+\infty} \operatorname{sech}\left(2\delta\left(-4a\delta^2 k^3 t + kx\right)\right) dx, \tag{34}$$

$$P(\Omega)=\frac{2\sqrt{6}\delta k\sqrt{-\left(c_1+2c_2\right)a\eta}\Omega}{\left(c_1+2c_2\right)b}\lim_{L\to\infty}\left(-\frac{\arctan\left(\sinh\left(8a\delta^3k^3t-2\delta kx\right)\right)}{2\delta k}\right)\Bigg|_{-L}^{L}, \tag{35}$$

$$P(\Omega)=\frac{\sqrt{6}\pi\sqrt{-\left(c_1+2c_2\right)a\eta}\Omega}{2\left(c_1+2c_2\right)b}. \tag{36}$$

Considering equation (24), the slope of the soliton power can be expressed as follows:

$$\frac{dP(\Omega)}{d\Omega}=\frac{\sqrt{6}\pi\sqrt{-\left(c_1+2c_2\right)a\eta}}{2\left(c_1+2c_2\right)}. \tag{37}$$

According to the Vakhitov-Kolokolov stability criterion presented in equation (24), the bright soliton is stable if $\frac{dP(\Omega)}{d\Omega}>0$ or unstable if $\frac{dP(\Omega)}{d\Omega}<0$, as shown in equation (37). In essence, the stability of a bright soliton depends on the sign of the fraction on the right side of the equation. Let's explore this situation under different parameter conditions.

Table 1. The sign of $\frac{dP(\Omega)}{d\Omega}$ in equation (37) determines the stability of the bright soliton described by equation (2), following the Vakhitov-Kolokolov stability criterion from equation (24)

c_1+2c_2	a	η	$\frac{dP(\Omega)}{d\Omega}$	Stable/Unstable
>0	<0	1	>0	Stable
<0	>0	1	>0	Stable

When interpreting the table above in conjunction with the method employed in the article, as indicated by equation (9) and equation (10), it becomes evident that the representation of the bright soliton using the method is feasible only when $\eta=1$.

In this scenario, the sign of $\frac{dP(\Omega)}{d\Omega}$ will remain positive ($P'(\Omega)>0$) only when the parameters are chosen as specified in the table. In other words, if the parameter values are selected according to the table, the bright soliton will exhibit stability; otherwise, it will be unstable.

Given that the parameter selections for the bright soliton simulations presented in Figure 1 were based on the initial states provided in Table 1, the solution represented in Figure 1 is stable. Therefore, the answer to the question regarding the stability of the bright soliton obtained from equation (2) within this study is YES.

As a closing point, it is important to emphasize the following. The VKSC serves as a potent algorithm for establishing the stability or instability of the bright soliton, offering an unequivocal yes/no resolution to inquiries regarding its stability. Nevertheless, the VK method does not provide answers to questions such as how to formulate the stability criterion, how to gauge the instability rate based on fluctuations in the soliton power, and whether this assessment will be definitive in cases of exceedingly small instability rates. This absence stems from the observation that in situations where the instability rate is minuscule, experimental evidence has indicated the soliton's stability even when it is theoretically unstable [56]. Considering all these dimensions, when evaluating the merits and limitations of this method, it can be concluded that it remains a potent approach for ascertaining the stability of bright (or dark) solitons, particularly within theoretically grounded mathematical studies. Therefore, it continues to warrant effective utilization [57, 58].

5. Results and discussion

One of the key advantages of our method is its novelty. The application of a simplified version of the extended auxiliary equation approach to recover optical solitons in the context of investigating the effects of third-order dispersion and nonlinear dispersion terms on pure-cubic solitons without chromatic dispersion sets our method apart from previously published works. This uniqueness contributes to the originality and distinctiveness of our research. Our method offers a comprehensive understanding of soliton behavior under specific conditions. By focusing on pure-cubic solitons without chromatic dispersion and accounting for third-order and nonlinear dispersion terms, our method provides insights into a specific scenario that might be overlooked by more generalized approaches. This focused approach can lead to a deeper understanding of the system's behavior. The use of the complex wave transform and the derivation of the nonlinear ordinary differential equation form add mathematical rigor to our method. This formalism helps ensure that our findings are based on solid mathematical foundations and contribute to the credibility of our research. While our method involves a simplified version of the extended auxiliary equation approach, its simplicity is an advantage. The simplified approach retains the core benefits of the extended auxiliary equation method while potentially making it more accessible to researchers who might be new to the technique. This simplicity can aid in disseminating our method and encouraging wider adoption. The application of the Vakhitov-Kolokolov stability criterion ensures the physical relevance of our results. By assessing the stability of the bright soliton solution, our method confirms that the obtained solutions are not only mathematically valid but also physically meaningful. This criterion adds a layer of robustness to our approach. The examination of model parameters and their effects on soliton behavior is a noteworthy advantage. Our method explores the impact of variations in parameters on soliton solutions, contributing to a comprehensive understanding of the system's dynamics. This exploration helps researchers gain insights into how various factors influence soliton behavior in the specific scenario under investigation. The inclusion of effective graphical simulations and solution functions enhances the clarity and communicability of our findings. These visual representations provide an intuitive way for readers to grasp the behavior of the solitons under different conditions, making our results more accessible and understandable. By highlighting the differentiation between our method and previously used expansion approaches, our research positions itself as a superior and innovative alternative. This comparison underscores the advantages of our method in terms of its specific focus and unique contributions.

The robustness of our approach in the research paper can be evaluated based on the methodology we've described and its potential to yield reliable and consistent results. The utilization of the complex wave transform and the application of a simplified version of the extended auxiliary equation method demonstrate a structured and systematic approach to addressing the research problem. This methodology provides a clear framework for deriving the nonlinear ordinary differential equation form and subsequently obtaining bright and singular optical soliton solutions. The specific focus on investigating the effects of third-order dispersion and nonlinear dispersion terms on pure-cubic soliton behavior in the absence of chromatic dispersion demonstrates the relevance of our approach to the research question. By concentrating on this specific scenario, our approach is tailored to the problem at hand, which enhances its robustness. The validation of our results through their satisfaction of the model equation contributes to the robustness of our findings. This step helps ensure that the solutions obtained through our approach are consistent with the theoretical framework and mathematical formalism we've presented. The application of the Vakhitov-Kolokolov stability criterion adds another layer of robustness to our approach. By assessing the stability of the obtained bright soliton solution, we provide an important validation of the physical significance of the results. This criterion helps ensure that the soliton solutions are not only mathematically valid but also physically meaningful. The examination of the influence of model parameters on the soliton solutions enhances the robustness of our approach. By analyzing how variations in parameters affect soliton behavior, we demonstrate a deeper understanding of the system's dynamics and contribute to the overall reliability of our findings. The inclusion of effective graphical simulations and solution functions adds robustness to our approach by providing visual representations that aid in understanding the behavior of the solitons. This approach allows readers to gain insights from both quantitative data and qualitative visualizations. Our approach stands apart from previously published works that utilized various expansion approaches adds to the robustness of our approach.

By clearly delineating the differences between our methodology and existing methods, we strengthen the uniqueness and credibility of our research. By providing a clear and detailed description of our methodology, including equations, transformations, and computational procedures, we facilitate the potential for other researchers to validate and build upon our work.

The potential applications of our research findings extend across a spectrum of optical and photonics-related fields [59-62]. The insights gained from our investigation into the effects of third-order dispersion, nonlinear dispersion terms, and soliton behavior for pure-cubic solitons without chromatic dispersion hold promise for several practical applications in the field of optics and nonlinear photonics. These applications capitalize on the fundamental understanding of soliton dynamics that our research has illuminated. The field of optical communication heavily relies on the propagation of light pulses with minimal distortion. Our findings regarding soliton behavior under the influence of various dispersion effects can offer valuable insights for designing ultrafast optical communication systems. By accounting for the distinct characteristics of pure-cubic solitons, engineers can optimize the transmission of optical signals over long distances while minimizing pulse broadening and distortion. Nonlinear optical devices play a pivotal role in signal processing, frequency conversion, and generation of novel optical frequencies. The unique soliton behavior elucidated in our research has the potential to enhance the design and efficiency of such devices. By leveraging the knowledge gained from our study, researchers can tailor the characteristics of nonlinear optical devices to achieve specific outcomes, thereby advancing the capabilities of these devices in various applications. The manipulation and processing of optical signals are integral to various technological domains, including signal encryption, modulation, and data compression. Our research outcomes can be harnessed to develop innovative techniques for optical signal processing. By exploiting the distinctive features of soliton behavior that we have uncovered, researchers can explore novel approaches to achieve advanced signal processing functionalities with improved accuracy and efficiency. Ultrashort optical pulses are essential in applications such as high-precision metrology and laser-based medical procedures. Our findings offer avenues for enhancing optical pulse compression techniques. By understanding how different dispersion effects impact soliton behavior, researchers can tailor dispersion profiles to achieve efficient pulse compression, resulting in shorter and more controlled pulse durations. In the emerging field of quantum optics and quantum information processing, precise control over optical states is paramount. The insights from our research can contribute to the development of techniques for controlling and manipulating soliton states in quantum optical systems. The nuanced understanding of soliton dynamics can be leveraged to enhance the generation and propagation of quantum states of light, with potential applications in quantum communication and computing. The behavior of solitons is influenced by the material properties of the medium through which they propagate. Our research findings can guide materials engineers in designing photonic materials that are tailored to optimize soliton behavior. This could lead to the development of new materials with enhanced nonlinear properties, enabling the creation of more efficient and versatile photonic devices.

6. Conclusion

In this article, we analyze a modified version of the NLSE with TOD in the absence of CD. Since the primary focus of this article is to examine the effects of model parameters in the absence of CD, there is no intention to derive different types of soliton solutions. To achieve this, we apply a subversion of the auxiliary equation method, specifically the SAEM26 method, known for its speed, precision, ease of application, and reliability. Through successful implementation of this method, we attain solutions for both bright and singular solitons.

The originality of our research lies in the application of our distinct methodology to address a gap in the existing literature. While prior studies have explored various expansion approaches, our paper stands apart by introducing a simplified version of the extended auxiliary equation method tailored to the study of pure-cubic solitons without chromatic dispersion. As a result, our work presents a fresh perspective that yields unique insights into soliton behavior under these specific conditions. This paper serves as an avenue for delving into the intricate nuances of soliton dynamics, guided by a method that combines mathematical rigor, parameter exploration, validation, and visual representation. Through our results, we offer a more comprehensive and refined understanding of the behaviors exhibited by pure-cubic solitons in the presence of third-order and nonlinear dispersion terms, all within the absence of chromatic dispersion.

Initially, we verify whether the obtained solution functions satisfy the main equation. Subsequently, to ensure the unconditional stability of the achieved bright soliton solution, we apply the Vakhitov-Kolokolov stability criterion, specifically the Vakhitov-Kolokolov slope condition. Following this, we explore the effects of model parameters on these soliton solutions. For enhanced comprehensibility of these effects, we generate and interpret various 2D graphs corresponding to each parameter's influence.

Additionally, apart from highlighting the significant impact of both the pure-cubic nonlinearity and nonlinear dispersion terms on bright and singular solitons, a similar analysis conducted in this study applies the equation in different SPM forms. These considerations encompass other perturbation terms and encompass studies of stochastic and fractional forms. These topics may serve as potential subjects for future projects. In this context, the study provides insights that will prove valuable for numerous future investigations, thereby extending its relevance beyond its pioneering exploration of the addressed problem.

Conflict of interest

The authors declare no conflict of interest.

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