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Research Article

On Testing of Fuzzy Hypothesis for Mean and Variance Using Centroid-Based New Distance Function Under Symmetric Fuzzy Environment

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Abstract: This paper discusses the problem of testing fuzzy and statistical hypotheses of the observed data are from a symmetric fuzzy environment. In this approach, many fuzzy tests statistics are obtained based on fuzzy data with varied forms of membership functions of fuzzy sets. To accept or reject the hypothesis of interest, a decision rule based on a new distance function to find the distance between symmetric fuzzy numbers with many forms of membership functions is proposed. The proposed method is employed to test hypotheses for mean and variance, as well as the difference between the means of a normal distribution and two normal distributions, with both known and unknown variances.

***Keywords*:** testing of hypothesis, fuzzy data, fuzzy test statistics, distance function

MSC: 03B52, 03E72, 65C60, 94D05

1. Introduction

Hypothesis testing is a significant study in classical inferential statistics. Traditional approaches rely on exact data, certain assumptions, and parameters. However, various forms of uncertainty arise in practice, particularly while dealing with imprecise data and/or vague concepts. In testing the statistical and fuzzy hypotheses under a fuzzy environment, the observed data from real-life situations and their related statistical parameters possess an inherent vagueness. Most often, the quantification of such vague information is accomplished using fuzzy sets. Currently, authors use different methods to quantify this vagueness, employing various forms of membership functions for fuzzy sets. In most cases, authors choose a specific form of membership function for testing hypotheses under a fuzzy environment. However, there is no guarantee that the quantification with some specific shape of fuzzy quantities will give the solution with good accuracy. Many authors have proposed various ways for testing hypotheses in crisp and fuzzy contexts with crisp or fuzzy data.

In 1995 [1], Romer and Kandel the investigated statistical hypotheses testing with fuzzy data. In this, they have

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aggregated the fuzzy data into a fuzzy sample vector and compared the findings for several possibilities as well as discussed various methods for fuzzifying the fuzzy test results. In 1998 [2], Grzegorzewski introduced a fuzzy estimator and fuzzy confidence interval for the fuzzy median and discussed on the hypothesis testing with vague data. In this, only distribution-free methods are used. In the same year [3], Arnold presented the criteria α and β for generalizing the probabilities of the errors of type I and type II and used one and two-sided Gauß tests. He has also used supplementary graphs to fix the value of β one and two-sided Gauß tests. In 2005 [4], Wu and Ch transacted an h-level set of fuzzy data for statistical hypotheses testing and have proposed decision rules for accepting and rejecting the null hypothesis using degrees of optimism and pessimism. Also, they have provided a computational approach for statistical hypothesis testing. In 2007 [5], Hsien-Chung has employed the conventional Analysis of variance (ANOVA) method with fuzzy data and provided the decision rules for accepting or rejecting the null hypothesis. In addition, a computational procedure was provided by them. In 2009 [6], a bootstrap approach for finding the variance has been developed particularly for hypothesis testing with fuzzy data based on Yao-Wu signed distance by Mohammad Ghasem Akbari and Abdol Hamid Rezaei. In 2009 [7], Przemyslaw describes the many-one problems with fuzzy data and uses an extension of the k-sample median test. In 2009 [8], Torabi and Mirahosseini have employed sequential probability ratio tests to examine the fuzzy hypotheses. In 2009 [9], Wu and Chien-Wei have provided a set of confidence intervals to generate triangular fuzzy numbers for the estimate of the C_{pk} index by modifying Buckley's technique. A three-decision testing rule and step-by-step method are also designed to analyze process performance using fuzzy critical values and fuzzy p-values. In 2009 [10], Torabi and Mirhosseini, have revised some concepts of fuzzy hypotheses testing and provided the Neyman-Pearson lemma for fuzzy hypotheses testing using fuzzy data. In 2009 [11], Elsherif et al. have applied the method for testing fuzzy hypotheses using fuzzy data to the radar detection process. In 2010 [12], Akbari and Rezaei have used Yao-Wu signed distance to develop a bootstrap approach for testing fuzzy hypotheses with fuzzy data. In 2011 [13], Parchami et al. have used fuzzy hypothesis and fuzzy p-value technique to determine whether or not the mean cadmium absorption corresponds to the quantities proposed by Pais and Benton and the degree of acceptance or rejection of the null fuzzy hypothesis estimated for each pollutant treatment in fuzzy hypotheses testing. In this, the findings come from two plants have namely radish and cress, grown on soil contaminated with $CdNO_3$ salt. The findings indicated that employing classical hypotheses testing may result in inconsistent judgments, and that the suggested fuzzy hypotheses testing is a sensible replacement of classical hypotheses testing. In 2011 [14], Mohsen Arefi and Mahmoud Taheri, have developed a method for evaluating the fuzzy hypotheses by introducing a credit level and also applied their method to test fuzzy hypotheses for the mean of a normal distribution and the variance in a normal distribution. In 2011 [15], Ramos and Gonzalez have proposed a technique to analyze the quality of cheese using a categorical scale based on the perception of qualified experts. In 2013 [16], Mohsen Arefi and Mahmoud Taheri have proposed a novel method for testing simple, one-sided and two-sided fuzzy hypotheses and defined two new criteria, namely degree of acceptance (DA) and degree of rejection (DR) to determine the result with the help of fuzzy point estimation. In 2013 [17], Hesamian and Chachi proposed a new approach to develop a two-sample Kolmogorov-Smirnov test for fuzzy hypotheses with fuzzy data. Then new types of fuzzy random variables are presented and transacted α-pessimistic values of the imprecise observations for expanding the two-sample Kolmogorov-Smirnov test. Also, they have introduced a p-value for estimating the fuzzy hypotheses. In 2013 [18], Sevil Bacanli and Duygu Icen employed a sequential probability ratio test for fuzzy hypothesis testing concerning the correlation coefficient for the bivariate normal distribution. In 2014 [19], Pandian and Kalpanapriya have proposed four different types of statistical hypothesis tests with small samples using the testing of significance for the difference of two populations concerning one and two fuzzy sets. They have also provided the rules for the decision of hypotheses and did not employ optimistic and pessimistic approaches, h-level sets, and so on. In 2015 [20], Gajivaradhan and Parthiban have developed a new testing approach for the statistical hypothesis concerning the population means in which the data of the two samples are considered as the real intervals and the decision rules to accept or reject the null hypothesis. They have also extended this approach to trapezoidal fuzzy interval data without employing degrees of optimism and pessimism and h-level set. In 2016 [21], Abbas Parchami et al. have developed two generalized p-values based on Zadeh's probability measure for testing fuzzy hypotheses with crisp data. In 2017 [22], Hesamian and Akbarihave expanded the traditional statistical test using intuitionistic fuzzy hypotheses and also type-I, type-II, power of test, and p-value for intuitionistic fuzzy hypotheses. In 2017 [23], Rabhahas developed a test for hypothesis testing based on the percentage of defectives at a certain electrical part created by a specific company, where the produced unit is crucial and introduced with other units to complete the product. He has also utilized the Neyman-Pearson process as a technique for testing fuzzy hypotheses. In 2017 [24], Mahmoud Taheri and Hesamian have presented an approach for generating a version of linear rank tests for fuzzy data using an index and extended the traditional critical value whenever the level of significance is a fuzzy number. Moreover, the decision-making method for accepting and rejecting the null hypothesis was presented and preference degree was employed for comparing the observed fuzzy test statistic and fuzzy critical value by them. In 2017 [25], Lubiano et al. have used trapezoidal fuzzy numbers for modelling the data using the crisp hypothesis testing about mean values of fuzzy data sets and compared the p-values of tests for trapezoidal assessment vs. LU assessments. In 2017 [26], Parchami et al. have developed two fuzzy notions by examining two complement fuzzy sets and have proposed a novel p-value-based technique for testing the fuzzy hypotheses based on two capability indices C_p and C_{pm}. In 2018 [27], Abbas et al. have used the minimax method for testing fuzzy hypotheses. In 2021 [28], Atalik and Gultekin have provided new intuitionistic fuzzy parametric methods for testing statistical hypotheses for population mean using intuitionistic fuzzy data and intuitionistic fuzzy hypotheses. In this, they have applied the bootstrap method to find intuitionistic fuzzy test statistics. This paper proposes a method for testing statistical and fuzzy hypotheses using fuzzy data under a symmetric fuzzy environment based on a new distance function with a centroid of fuzzy numbers of various shapes.

It is organized as follows. Section 2 reviews the essential definitions of a fuzzy set, fuzzy numbers and the quantification of fuzzy numbers with different shapes of membership functions. In Section 3, a new distance function for ordering fuzzy numbers with many forms of membership functions based on its centroid points is proposed to introduce a new decision for testing the fuzzy hypothesis. A method for testing fuzzy hypotheses of some forms investigated by Arefi and Taheri [14] is introduced based on new decision rules using the decision function presented in Section 3. In section 4, the proposed method is applied for testing mean with known variance, variance of normal distribution and difference of means with known variances of two normal distributions. Finally, section 5 provides a conclusion of this work.

2. Preliminary concepts

This section recalls some prominent definitions of fuzzy sets, fuzzy numbers of some specific shapes of membership functions and the idea for quantification of fuzzy numbers with many such shapes.

Definition 2.1 [20] If the set defined by the function $\mu_{\tilde{A}}: A \to [0, 1]$ gives the grade for membership of the element x of the universal set X for its belongingness, then it is called a fuzzy set. It is defined as

$$\tilde{A} = \left\{\left(x,\ \mu_{\tilde{A}}(x)\right);\ x \in X\right\}$$

Definition 2.2 [29] The fuzzy set $\tilde{A}$ of X is said to be a fuzzy number if

(i) $\tilde{A}(x) = 1$ for some x.

(ii) $\tilde{A}[\alpha] = \{x : \mu_{\tilde{A}}(x) \geq \alpha\}$ is a closed bounded interval for $0 < \alpha \leq 1$.

Definition 2.3 [29] A fuzzy number $\tilde{A}$ is called a LR fuzzy number if its membership function is

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\dfrac{m-x}{s_1}\right) & x \leq m \\ 0 & \text{otherwise} \\ R\left(\dfrac{x-m}{s_2}\right) & m < x \end{cases}$$

Where the real number m is called the mean value and s_1 and s_2 are called left and right spreads respectively. Symbolically, the LR fuzzy number $\tilde{A}$ is denoted by $\tilde{A} = (m - s_1, m, m + s_2)$.

Example 2.1 Let $L(x) = \max\{0, 1 - x\}$, $R(x) = e^{-x}$. Then $\tilde{A} = (2, 4, 6)$ with $s_1 = 2$, $s_2 = 2$ and $m = 4$ and $\tilde{B} = (4, 8, 16)$ with $s_1 = 4$, $s_2 = 8$ and $m = 8$, denote an LR fuzzy numbers with membership functions respectively.

$$\mu_{\tilde{A}}(x)=\begin{cases} L\left(\frac{4-x}{2}\right)=\frac{x}{2}-1 & x\le 4 \\ 0 & \text{otherwise} \\ R\left(\frac{x-4}{2}\right)=e^{(4-x)/2} & 4<x \end{cases} \qquad \mu_{\tilde{A}}(x)=\begin{cases} L\left(\frac{8-x}{4}\right)=\frac{x}{4}-1 & x\le 8 \\ 0 & \text{otherwise} \\ R\left(\frac{x-8}{8}\right)=e^{1-\frac{x}{8}} & 8<x \end{cases}$$

Definition 2.4 [20] A fuzzy set defined on the real line $\Re=(-\infty, +\infty)$ is called the generalized triangular, trapezoidal, pentagonal and heptagonal fuzzy number of the form $\tilde{A}=(x_1, x_2, x_3, ..., x_n; \omega)$ where ω is any real number and $0\le\omega\le 1$ for $n = 3, 4, 5, 7$ respectively if its membership functions are given by.

For $n = 3$, $\tilde{A}=(x_1, x_2, x_3; \omega)$ where $x_1 < x_2 < x_3$. It is called a triangular fuzzy number and its membership function is as follows:

$$\mu_{\tilde{A}}(x)=\begin{cases} 0 & x\le x_1 \\ \omega\frac{x-x_1}{x_2-x_1} & x_1\le x\le x_2 \\ \omega\frac{x_3-x}{x_3-x_2} & x_2\le x\le x_3 \\ 0 & x>x_3 \end{cases}$$

If $x_2 - x_1 = x_3 - x_2$ so, then it is called a symmetric triangular fuzzy number.

For $n = 4$, $\tilde{A}=(x_1, x_2, x_3, x_4; \omega)$ where $x_1 < x_2 < x_3 < x_4$. It is called a trapezoidal fuzzy number and its membership function is as follows:

$$\mu_{\tilde{A}}(x)=\begin{cases} \omega\frac{x-x_1}{x_2-x_1} & x_1\le x\le x_2 \\ \omega & x_2\le x\le x_3 \\ \omega\frac{x-x_4}{x_3-x_4} & x_3\le x\le x_4 \\ 0 & \text{otherwise} \end{cases}$$

If $x_2 - x_1 = x_4 - x_3$ so, then it is called a symmetric trapezoidal fuzzy number.

For $n = 5$, $\tilde{A}=(x_1, x_2, x_3, x_4, x_5; \omega)$ where $x_1 < x_2 < x_3 < x_4 < x_5$. It is called a pentagonal fuzzy number and its membership function is as follows:

$$\mu_{\tilde{A}}(x)=\begin{cases} \omega\frac{x-x_1}{x_2-x_1} & x_1\le x\le x_2 \\ \omega\frac{x-x_2}{x_3-x_1} & x_2\le x\le x_3 \\ \omega & x=x_3 \\ \omega\frac{x_4-x}{x_4-x_3} & x_3\le x\le x_4 \\ \omega\frac{x_5-x}{x_5-x_4} & x_4\le x\le x_5 \\ 0 & x<x_1,\ x>x_5 \end{cases}$$

If $x_2 - x_1 = x_5 - x_4$ and $x_3 - x_2 = x_4 - x_3$, then it is called a symmetric pentagonal fuzzy number.

For $n = 7$, $\tilde{A} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7; \omega)$ where $x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < x_7$. It is called a heptagonal fuzzy number and its membership function is as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} k\dfrac{x - x_1}{x_2 - x_1} & x_1 \le x \le x_2 \\ k & x_2 \le x \le x_3 \\ k + (\omega - k)\dfrac{x - x_3}{x_4 - x_3} & x_3 \le x \le x_4 \\ k + (\omega - k)\dfrac{x_5 - x}{x_5 - x_4} & x_4 \le x \le x_5 \\ k & x_5 \le x \le x_6 \\ k\dfrac{x_7 - x}{x_7 - x_6} & x_6 \le x \le x_7 \\ 0 & x \le x_1,\ x \ge x_7 \end{cases}$$

where ω is any real number and $0 < k < 1$, $k \le \omega \le 1$.

If $x_2 - x_1 = x_7 - x_6$, $x_3 - x_2 = x_6 - x_5$ and $x_4 - x_3 = x_5 - x_4$, then it is called a symmetric heptagonal fuzzy number.

Definition 2.5 Let us consider the fuzzy number $\tilde{a}$. The quantified values of $\tilde{a}$ using fuzzy sets with triangular, pentagonal and heptagonal shapes of membership functions are defined as follows:

$$(a - \eta, a, a + \delta)$$

$$(a - 2\eta, a - \eta, a, a + \delta, a + 2\delta)$$

$$(a - 3\eta, a - 2\eta, a - \eta, a, a + \delta, a + 2\delta, a + 3\delta)$$

where $\eta, \delta \ge 0$.

The above form of fuzzy numbers will be symmetric if $\eta = \delta$. The same may be quantified with higher shapes of membership functions of fuzzy sets.

3. A new distance function of fuzzy numbers with many forms of membership functions

In this section, a new distance function is proposed for ordering symmetric fuzzy numbers with many forms of membership functions such as triangular, trapezoidal, pentagonal, heptagonal, etc. To demonstrate the proposed ordering technique based on the new distance function, the fuzzy numbers of triangular, pentagonal and heptagonal shapes of membership functions are considered in this section.

3.1 *Proposed centroid point for ordering fuzzy numbers with triangular, pentagonal and heptagonal shape of membership functions*

Three various centroids of generalized trapezoidal fuzzy numbers are found by splitting the trapezoid into three planar figures in three different ways to find the new centroids of trapezoid, pentagon and heptagon. The trapezoid is divided into three triangles in three different ways (Figure 1). Finally, the centroids of centroids of three planar figures are obtained in the following three ways:

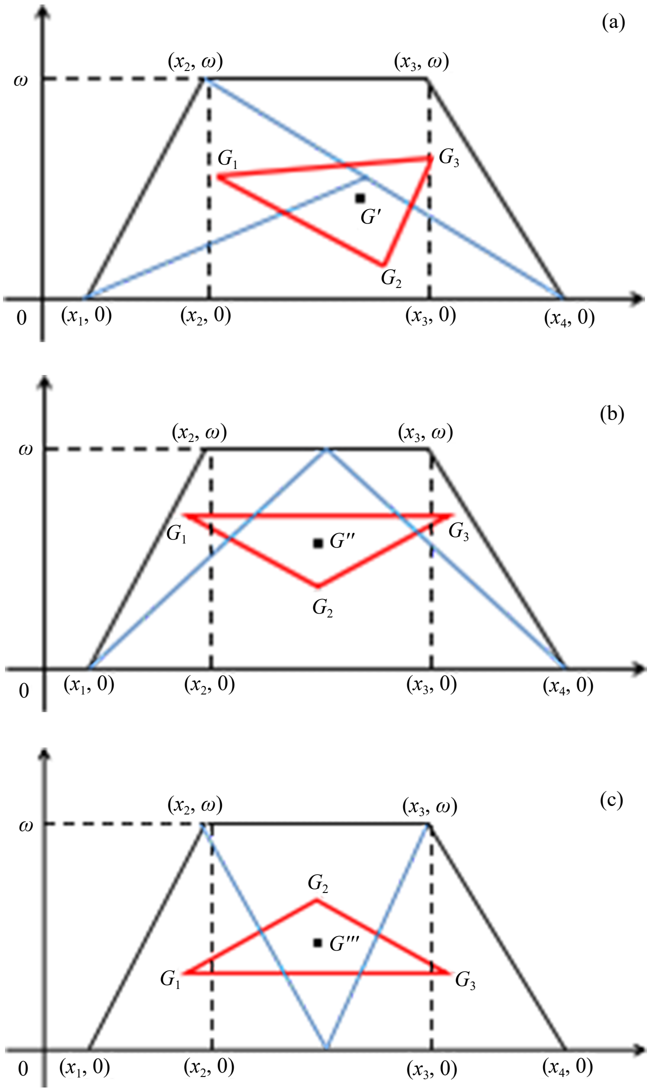

Figure 1. Centroid G' of trapezoid (a), centroid G'' of trapezoid (b), centroid G''' of trapezoid (c)

The three centroids of generalized trapezoidal fuzzy numbers are

$$G' = \left(\frac{4x_1 + 6x_2 + 2x_3 + 6x_4}{18}, \frac{4\omega}{9}\right), G'' = \left(\frac{4x_1 + 5x_2 + 5x_3 + 4x_4}{18}, \frac{5\omega}{9}\right), G''' = \left(\frac{5x_1 + 4x_2 + 4x_3 + 5x_4}{18}, \frac{4\omega}{9}\right).$$

3.1.1 *Centroid point of generalized triangular and trapezoidal fuzzy numbers*

Since the centroid

$$G' = \left(\frac{4x_1 + 6x_2 + 2x_3 + 6x_4}{18}, \frac{4\omega}{9}\right), G'' = \left(\frac{4x_1 + 5x_2 + 5x_3 + 4x_4}{18}, \frac{5\omega}{9}\right) \text{ and } G''' = \left(\frac{5x_1 + 4x_2 + 4x_3 + 5x_4}{18}, \frac{4\omega}{9}\right)$$

of generalized trapezoid are non-collinear, they form a triangle $\Delta G'G''G'''$. The centroid point G of G', G'' and G''' is computed and it is the balancing point of the generalized trapezoid ABCD shown in Figure 2. Thus G is selected as the new centroid point of the generalized trapezoidal fuzzy number $\tilde{A} = (x_1, x_2, x_3, x_4; \omega)$ and is found as

$$\left(x_0,\ y_0\right)=\left(\frac{13x_1+15x_2+11x_3+15x_4}{54},\ \frac{13\omega}{27}\right) \tag{1}$$

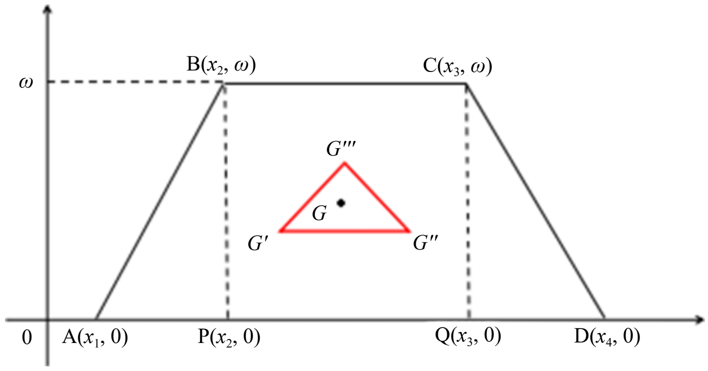

Figure 2. Centroid of generalized trapezoidal fuzzy number

Remark 3.1 If we put $x_2 = x_3$, then (4) becomes a generalized triangular fuzzy number. That is,

$$\left(x_0,\ y_0\right)=\left(\frac{13x_1+26x_2+15x_4}{54},\ \frac{13\omega}{27}\right). \tag{2}$$

3.1.2 *Centroid point of generalized pentagonal fuzzy number*

First, the generalized pentagonal fuzzy number $\tilde{A} = (x_1, x_2, x_3, x_4, x_5; \omega)$ is split into two plane figurestrapezoid ABDE and triangle BCD. The centroids $G' = \left(\frac{13x_1+15x_2+11x_4+15x_5}{54},\ \frac{13\omega}{54}\right)$ and $G'' = \left(\frac{x_2+x_3+x_4}{3},\ \frac{2\omega}{3}\right)$ these two plane figures are its balancing points as shown in Figure 3. Here G' is the centroid of the triangle with the vertices $G_1 = \left(\frac{4x_1+6x_2+2x_4+6x_5}{18},\ \frac{4\omega}{9}\right)$, $G_2 = \left(\frac{4x_1+5x_2+5x_4+4x_5}{18},\ \frac{5\omega}{18}\right)$ $G_3 = \left(\frac{5x_1+4x_2+4x_4+5x_5}{18},\ \frac{2\omega}{9}\right)$ as per the procedure introduced in section 3. The average point G of G' and G'' is the balancing point of the generalized pentagon so that it is considered as a new centroid of the generalized pentagonal fuzzy number $\tilde{A} = (x_1, x_2, x_3, x_4, x_5; \omega)$ and is given as follows:

$$G=\left(x_0,\ y_0\right)=\left(\frac{13x_1+33x_2+18x_3+29x_4+15x_5}{108},\ \frac{49\omega}{108}\right) \tag{3}$$

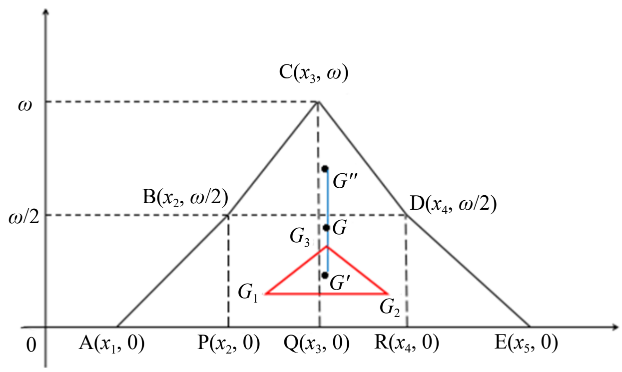

Figure 3. Centroid of generalized pentagonal fuzzy number

3.1.3 *Centroid point of generalized heptagonal fuzzy number*

First, the generalized heptagonal fuzzy number $\tilde{A} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7; \omega)$ shown in Figure 4 is split into two trapezoids ABFG, BCEF and one triangle CDE. The centroids G', G'' and G''' of one triangle CDE and two trapezoids BCEF, and ABFG are the required balancing points. where $G' = \left(\frac{13x_1 + 15x_2 + 11x_6 + 15x_7}{54}, \frac{13\omega}{81}\right)$, $G'' = \left(\frac{13x_2 + 15x_3 + 11x_5 + 15x_6}{54}, \frac{40\omega}{81}\right)$ and $G''' = \left(\frac{x_3 + x_4 + x_5}{3}, \frac{7\omega}{9}\right)$. Here G' is the centroid of the triangle with the vertices

$$G_1 = \left(\frac{4x_1 + 6x_2 + 2x_6 + 6x_7}{18}, \frac{2\omega}{9}\right), G_2 = \left(\frac{4x_1 + 5x_2 + 5x_6 + 4x_7}{18}, \frac{5\omega}{18}\right) \text{ and } G_3 = \left(\frac{5x_1 + 4x_2 + 4x_6 + 5x_7}{18}, \frac{2\omega}{9}\right)$$

G'' is the centroid of the triangle with the vertices $G_4 = \left(\frac{4x_2 + 6x_3 + 2x_5 + 6x_6}{18}, \frac{13\omega}{27}\right)$, $G_5 = \left(\frac{4x_2 + 5x_3 + 5x_5 + 4x_6}{18}, \frac{14\omega}{27}\right)$ and $G_6 = \left(\frac{5x_2 + 4x_3 + 4x_5 + 5x_6}{18}, \frac{40\omega}{81}\right)$ as per the procedure introduced in section 3. Consequently, the average G of G', G'' and G''' is the balancing point of the heptagon. Thus G is selected as a new centroid of the generalized heptagonal fuzzy number $\tilde{A} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7; \omega)$ and is found as follows:

$$(x_0, y_0) = \left(\frac{13x_1 + 28x_2 + 33x_3 + 14x_4 + 29x_5 + 26x_6 + 15x_7}{162}, \frac{116\omega}{243}\right) \tag{4}$$

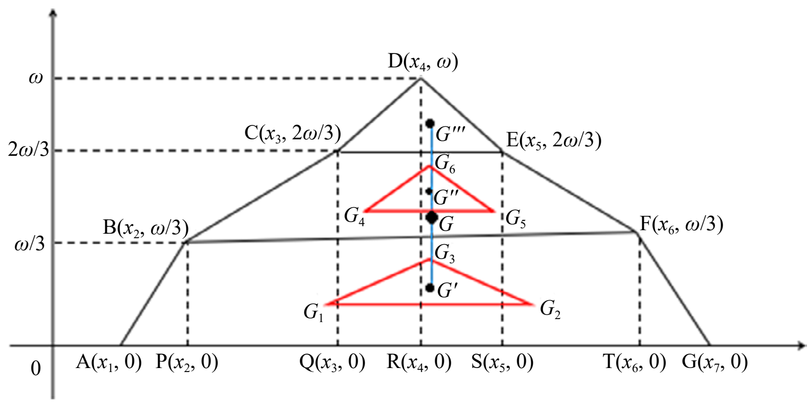

Figure 4. Centroid of generalized heptagonal fuzzy number

Definition 3.1 Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers with many forms of membership functions. The distance between $\tilde{a}$ and $\tilde{b}$ is defined as follows:

Consider the above two fuzzy numbers with triangular, pentagonal and heptagonal shapes of membership functions with centroid points respectively as

$$D(\tilde{a}, \tilde{b}) = \begin{cases} m = \min\{(\alpha_a - \alpha_b)_T, (\alpha_a - \alpha_b)_P, (\alpha_a - \alpha_b)_H\} & \text{if } |\alpha_a - \alpha_b|_i \neq (\alpha_a - \alpha_b)_i \ \forall i \\ M = \max\{(\alpha_a - \alpha_b)_T, (\alpha_a - \alpha_b)_P, (\alpha_a - \alpha_b)_H\} & \text{if } |\alpha_a - \alpha_b|_i = (\alpha_a - \alpha_b)_i \ \forall i \\ -\max\{|m|, M\} & \text{if } |m| > M \\ \max\{m, M\} & \text{if } |m| < M \end{cases}$$

where $m = \min\{(\alpha_a - \alpha_b)_i\}$ if $\alpha_a - \alpha_b < 0$ and $M = \max\{(\alpha_a - \alpha_b)_i\}$ if $\alpha_a - \alpha_b > 0$.

The above distance function will be effective in nature if we include all the shapes of fuzzy numbers.

$$D(\tilde{a}, \tilde{b}) = \begin{cases} m = \min\{(\alpha_a - \alpha_b)_T, (\alpha_a - \alpha_b)_P, (\alpha_a - \alpha_b)_H, (\alpha_a - \alpha_b)_N, \ldots\} & \text{if } |\alpha_a - \alpha_b|_i \neq (\alpha_a - \alpha_b)_i \ \forall i \\ M = \max\{(\alpha_a - \alpha_b)_T, (\alpha_a - \alpha_b)_P, (\alpha_a - \alpha_b)_H, (\alpha_a - \alpha_b)_N, \ldots\} & \text{if } |\alpha_a - \alpha_b|_i = (\alpha_a - \alpha_b)_i \ \forall i \\ -\max\{|m|, M\} & \text{if } |m| > M \\ \max\{m, M\} & \text{if } |m| < M \end{cases}$$

where T, P, H, N, … denotes the triangular, pentagonal, heptagonal, nonagonal, … shapes of fuzzy numbers respectively.

Remark 3.2 The above distance function is reduced as follows:

(i) If $\tilde{a}$ is a fuzzy number with many forms of membership functions and b is a crisp number, the distance function is reduced as

$$D(\tilde{a}, b) = \begin{cases} m = \min\{(\alpha_a - b)_T, (\alpha_a - b)_P, (\alpha_a - b)_H, (\alpha_a - b)_N, \ldots\} & \text{if } |\alpha_a - b|_i \neq (\alpha_a - b)_i \ \forall i \\ M = \max\{(\alpha_a - b)_T, (\alpha_a - b)_P, (\alpha_a - b)_H, (\alpha_a - b)_N, \ldots\} & \text{if } |\alpha_a - b|_i = (\alpha_a - b)_i \ \forall i \\ -\max\{|m|, M\} & \text{if } |m| > M \\ \max\{m, M\} & \text{if } |m| < M \end{cases}$$

(ii) If a and b are crisp numbers, the distance function is reduced as $D(a, b) = a - b$.

Definition 3.2 Let $\tilde{a}$ and $\tilde{b}$ be two symmetric fuzzy numbers with many forms of membership functions. The ordering of the numbers is stated as

$$D(\tilde{a}, \tilde{b}) > 0 \text{ if } \tilde{a} > \tilde{b}$$

$$D(\tilde{a}, \tilde{b}) < 0 \text{ if } \tilde{a} < \tilde{b}$$

$$D(\tilde{a}, \tilde{b}) = 0 \text{ if } \tilde{a} = \tilde{b}$$

The above two fuzzy numbers with triangular, pentagonal and heptagonal shapes of membership functions and its centroid points are consider in the Table 1.

Table 1. Centroid points for triangular, pentagonal and heptagonal shapes

Triangular form of fuzzy number	Centroid Point	Pentagonal form of fuzzy number	Centroid Point	Heptagonal form of fuzzy number	Centroid Point
$\tilde{a}_T = (a_1, a_2, a_3)$	$(\alpha_a, \beta_a)_T$	$\tilde{a}_P = (a_1, a_2, a_3, a_4, a_5)$	$(\alpha_a, \beta_a)_P$	$\tilde{a}_H = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$	$(\alpha_b, \beta_b)_H$
$\tilde{b}_T = (b_1, b_2, b_3)$	$(\alpha_b, \beta_b)_T$	$\tilde{b}_P = (b_1, b_2, b_3, b_4, b_5)$	$(\alpha_b, \beta_b)_P$	$\tilde{b}_H = (b_1, b_2, b_3, b_4, b_5, b_6, b_7)$	$(\alpha_b, \beta_b)_H$

4. Testing of fuzzy hypothesis using fuzzy data with many forms of membership functions

This section discusses the testing of fuzzy one-sided and two-sided hypotheses [14] using symmetric fuzzy data with many forms of membership functions.

Definition 4.1 Let $\widetilde{X}_1^j, \widetilde{X}_2^j, \ldots, \widetilde{X}_n^j$ be a fuzzy form of an uncertain random sample $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n$ with some specific shape of membership function rather than the crisp random sample $x_1, x_2, \ldots, x_n$ of a probability (density or mass) function $f(x; \Omega)$, where Ω is the unknown parameter. The fuzzy point estimation [1] $\widetilde{\Omega}^* = \Omega$ is defined and denoted as $\widetilde{\Omega}^* = u(x_1, x_2, \ldots, x_n)$.

Here, we introduce an algorithm for testing crisp and fuzzy (one and two-sided) hypotheses based on fuzzy data under a symmetric fuzzy environment.

4.1 *Testing of fuzzy hypothesis against fuzzy two-sided hypothesis*

Suppose that we want to test the simple fuzzy hypothesis $H_0 : \Omega$ is approximately Ω_0 ($H_0 : \Omega$ is $\widetilde{H}_0$) against the fuzzy two-sided hypothesis $H_1 : \Omega$ is not approximately Ω_0 ($H_1 : \Omega$ is $\widetilde{H}_1$). Here $\widetilde{H}_0 = \widetilde{\Omega}_0$ is a fuzzy number with many forms of membership functions.

The extension of the decision rule for testing simple crisp hypothesis $H_0 : \Omega = \Omega_0$ against $\widetilde{H}_1 : \Omega \neq \Omega_0$ at the significant level $\xi = 0.05$ under fuzzy data can be defined as given below.

Definition 4.2 New Decision Rule If $D(\widetilde{Z}, \Theta_{\xi/2})$ and $D(\widetilde{Z}, \Theta_{1-\xi/2})$ are the distance between the fuzzy test statistic $\widetilde{Z}$ and the quantiles $\Theta_{\xi/2}$ and $\Theta_{1-\xi/2}$ respectively, then the decision rule for testing simple fuzzy hypothesis $H_0 : \Omega$ is approximately Ω_0 ($H_0 : \Omega$ is $\widetilde{H}_0$) against fuzzy two-sided hypothesis $H_1 : \Omega$ is not approximately Ω_0 ($H_1 : \Omega$ is $\widetilde{H}_1$) is defined as follows:

$$D\left(\widetilde{Z}, \Theta_{\xi/2}\right) \leq 0 \text{ or } D\left(\widetilde{Z}, \Theta_{1-\xi/2}\right) \geq 0 \Rightarrow \text{Accept } H_1$$

$$D\left(\widetilde{Z}, \Theta_{\xi/2}\right) > 0 \text{ or } D\left(\widetilde{Z}, \Theta_{1-\xi/2}\right) < 0 \Rightarrow \text{Accept } H_0$$

4.2 *Testing of fuzzy hypothesis against fuzzy one-sided hypothesis*

Suppose that we want to test the simple fuzzy hypothesis $H_0 : \Omega$ is approximate Ω_0 ($H_0 : \Omega$ is $\widetilde{H}_0$) against fuzzy right one-sided hypothesis $H_1 : \Omega$ is essentially larger than Ω_0 ($H_1 : \Omega$ is $\widetilde{H}_{1L}$). Here $\widetilde{H}_0 = \widetilde{\Omega}_0$ is a fuzzy number with many forms of membership functions.

The extension of the decision rule for testing simple crisp hypothesis $H_0 : \Omega = \Omega_0$ against right one-sided hypothesis $\widetilde{H}_1$: $\Omega \neq \Omega_0$ at the significant level $\xi = 0.05$ under fuzzy data can be defined as given below.

Definition 4.3 New Decision Rule It $D(\widetilde{Z}, \Theta_{1-\xi/2})$ is the distance between the fuzzy test statistic $\widetilde{Z}$ and the quantile $\Theta_{1-\xi/2}$, then the decision rule for testing simple fuzzy hypothesis $H_0 : \Omega$ is approximate Ω_0 ($H_0 : \Omega$ is $\widetilde{H}_0$) against fuzzy right one-sided hypothesis $H_1 : \Omega$ is essentially larger than Ω_0 ($H_1 : \Omega$ is $\widetilde{H}_{1L}$) is defined as follows:

$$D\left(\widetilde{Z}, \Theta_{1-\xi}\right) \geq 0 \Rightarrow \text{Accept } H_1$$

$$D\left(\widetilde{Z}, \Theta_{1-\xi}\right) < 0 \Rightarrow \text{Accept } H_0$$

Remark The above decision rule may be introduced for testing simple fuzzy hypothesis $H_0 : \Omega$ is approximately Ω_0 ($H_0 : \Omega$ is $\widetilde{H}_0$) against fuzzy left one-sided hypothesis $H_1 : \Omega$ is essentially smaller than Ω_0 ($H_1 : \Omega$ is $\widetilde{H}_{1S}$) as

$$D\left(\widetilde{Z}, \Theta_{\xi}\right) \leq 0 \Rightarrow \text{Accept } H_1$$

$$D\left(\widetilde{Z}, \Theta_{1-\xi}\right) > 0 \Rightarrow \text{Accept } H_0$$

Now, we introduce the procedure for testing the fuzzy hypothesis of two-sided and one-sided under fuzzy data $\widetilde{X}_1^i, \widetilde{X}_2^i, \ldots, \widetilde{X}_n^i$ with many forms of membership functions.

Step 1: First, we obtain many forms of fuzzy data from the data observed from a symmetric fuzzy environment using the formula for quantification introduced in Definition 3.1.

Step 2: Fuzzy test statistics are calculated under the various forms of fuzzy data obtained in step 1.

Step 3: We find the distance between the test statistics and the quantile value at ξ level of significance using the proposed distance function.

Step 4: The acceptance of H_0 or the acceptance of H_1 is decided using the decision rule presented in Definitions 4.2 and 4.3.

5. Testing fuzzy hypotheses in the normal distribution

5.1 *Testing fuzzy hypotheses for the mean*

Suppose that we have taken a random sample of size n $N(\Omega, \sigma^2)$ and observed the fuzzy numbers $\widetilde{X}_1^i, \widetilde{X}_2^i, ..., \widetilde{X}_n^i$ with many forms of membership functions rather than the crisp data $x_1, x_2, ..., x_n$. Now, we want to test any one of the following fuzzy hypotheses at the significance level ξ:

$$H_0 : \Omega \text{ is approximately } \Omega_0 \quad (H_0 : \Omega \text{ is } \widetilde{H}_0)$$

$$H_1 : \Omega \text{ is not approximately } \Omega_0 \quad (H_1 : \Omega \text{ is } \widetilde{H}_1)$$

$$H_0 : \Omega \text{ is approximately } \Omega_0 \quad (H_0 : \Omega \text{ is } \widetilde{H}_0)$$

$$H_1 : \Omega \text{ is essentially larger than } \Omega_0 \quad (H_1 : \Omega \text{ is } \widetilde{H}_{1L})$$

$$H_0 : \Omega \text{ is approximately } \Omega_0 \quad (H_0 : \Omega \text{ is } \widetilde{H}_0)$$

$$H_1 : \Omega \text{ is essentially smaller than } \Omega_0 \quad (H_1 : \Omega \text{ is } \widetilde{H}_{1S})$$

For the above hypotheses, we suppose that

$$\widetilde{H}_0 = (a_1, a_2, ..., a_n, \Omega_0, b_1, b_2, b_n); \ \widetilde{H}_1 = \widetilde{H}_0^c;$$

$$\widetilde{H}_{1L} = \left(a_1', a_2', ..., a_n', \Omega_0\right); \ \widetilde{H}_0 = \left(\Omega_0, b_1', b_2', ..., b_n'\right);$$

where $a_1 \le a_1'$, $a_2 \le a_2'$, ..., $a_n \le a_n'$ and $b_1' \le b_1$, $b_2' \le b_2$, ..., $b_n' \le b_n$.

The usual point estimation for Ω is $\Omega^* = \bar{x}$. By substituting $\widetilde{X}_i$, $i = 1, ..., n$, for x_i in the point estimation, the fuzzy point estimation $\widetilde{\widetilde{X}}$ is found as

$$\widetilde{\widetilde{X}} = \frac{1}{n}\sum_{i=1}^{n}\widetilde{X}_i$$

Under the crisp hypothesis $H_0 : \Omega = \Omega_0$, the value of the crisp test statistic is $z_0 = \dfrac{\Omega^* - \Omega_0}{\sigma/\sqrt{n}}$. By substituting the fuzzy point estimation $\widetilde{\widetilde{X}}$ for Ω^* in z_0 and using the fuzzy arithmetic, the fuzzy test statistic is obtained as follows:

$$\widetilde{Z} = \frac{\widetilde{\widetilde{X}} - \widetilde{H}_0}{\sigma/\sqrt{n}}$$

Example 5.1 In a town the usage of LED TV in the houses are observed. The following data shows that the lifetime of the TV on a randomly selected 25 houses that were monitored. But, in practice measuring the lifetime of a TV not in the exact year. The TV may work properly for a minimum period but the usage of time and rest time of the TV will affected the lifetime of the TV. So, such data may be reported as imprecise quantities. Assume that, the lifetimes of TV are considered as fuzzy numbers with a random sample of size $n = 25$ from a population $N(\Omega, \sigma^2 = 9)$. Suppose the lifetime of the TV with an unknown mean θ, we observe the fuzzy data in Table 2.

Table 2. The fuzzy data from a normal population

$\widetilde{X}_i = (x_i, \delta_i)$				
(5.98, 0.6)	(5.99, 0.6)	(6.11, 0.61)	(6.38, 0.64)	(5.41, 0.54)
(5.64, 0.56)	(5.69, 0.57)	(6.22, 0.62)	(5.58, 0.56)	(5.94, 0.59)
(5.73, 0.57)	(6, 0.6)	(5.85, 0.59)	(6.3, 0.63)	(6.52, 0.65)
(6.21, 0.62)	(5.9, 0.59)	(5.95, 0.6)	(6.1, 0.61)	(6.27, 0.63)
(6.32, 0.63)	(6.2, 0.62)	(6.25, 0.63)	(6.23, 0.62)	(5.74, 0.57)

A) Suppose that we want to test the following hypotheses, at the significance level $\xi = 0.05$.

$$\begin{cases} H_0 : \Omega \text{ is a approximately } 6 \\ H_{1L} : \Omega \text{ is a essentially larger than } 6 \end{cases} = \begin{cases} H_0 : \Omega \text{ is } \widetilde{H}_0 \\ H_0 : \Omega \text{ is } \widetilde{H}_{1L} \end{cases}$$

To find a Fuzzy Test Statistic based on Fuzzy data with the triangular form of the membership function.
Let $\widetilde{H}_0 = (5.75, 6, 6.25)$ and $\widetilde{H}_{1L} = (5.80, 6)$.

Table 3. The triangular fuzzy data from normal population

$\widetilde{X}_i^T = (x_i - \delta_i, x_i, x_i + \delta_i)_T$				
(5.38, 5.98, 6.58)	(5.39, 5.99, 6.59)	(5.5, 6.11, 6.72)	(5.74, 6.38, 7.02)	(4.87, 5.41, 5.95)
(5.08, 5.64, 6.2)	(5.12, 5.69, 6.26)	(5.6, 6.22, 6.84)	(5.02, 5.58, 6.14)	(5.35, 5.94, 6.53)
(5.16, 5.73, 6.3)	(5.4, 6, 6.6)	(5.26, 5.85, 6.44)	(5.67, 6.3, 6.93)	(5.87, 6.52, 7.17)
(5.59, 6.21, 6.83)	(5.31, 5.9, 6.49)	(5.35, 5.95, 6.55)	(5.49, 6.1, 6.71)	(5.64, 6.27, 6.9)
(5.69, 6.32, 6.95)	(5.58, 6.2, 6.82)	(5.62, 6.25, 6.88)	(5.61, 6.23, 6.85)	(5.17, 5.74, 6.31)

Now, Based on the triangular fuzzy data $\widetilde{X}_i$, $i = 1, 2, \ldots, n$, (Table 3) obtained from Table 2, the fuzzy point estimation $\widetilde{\widetilde{X}}$ is found as

$$\widetilde{\widetilde{X}} = \frac{1}{n}\sum_{i=1}^{n}\widetilde{X}_i$$

$$= \frac{1}{25}(135.46,\ 150.51,\ 165.56)$$

$$= (5.4184, 6.0204, 6.6224)$$

The fuzzy point estimation is $\widetilde{\widetilde{X}} = (5.4184, 6.0204, 6.6224)$.
The fuzzy test statistic is

$$\tilde{Z} = \frac{\tilde{\tilde{X}} - \tilde{H}_0}{\sigma/\sqrt{n}}$$

$$= \frac{1}{3/\sqrt{25}}\left[(5.4184,\ 6.0204,\ 6.6224) - (5.75,\ 6,\ 6.25)\right]$$

$$= \frac{5}{3}(-0.8316,\ 0.0204,\ 0.8724)$$

$$= (-1.3860,\ 0.0340,\ 1.4540)$$

The fuzzy test statistic is $\tilde{Z} = (-1.3860, 0.0340, 1.4540)$.
Using the proposed centroid in equation (2), $x_{\tilde{Z}} = 0.0866$.
To find a Fuzzy Test Statisticbased on Fuzzy data with a Pentagonal form of membership function.
Let $\tilde{H}_0 = (5.5, 5.75, 6, 6.25, 6.5)$ and $\tilde{H}_{1L} = (5.55, 5.80, 6)$.

Table 4. The pentagonal fuzzy data from normal population

$\tilde{X}_i^P = (x_i - 2\delta_i,\ x_i - \delta_i,\ x_i,\ x_i + \delta_i,\ x_i + 2\delta_i)_P$		
(4.78, 5.38, 5.98, 6.58, 7.18)	(4.89, 5.5, 6.11, 6.72, 7.33)	(4.33, 4.87, 5.41, 5.95, 6.49)
(4.52, 5.08, 5.64, 6.2, 6.76)	(4.98, 5.6, 6.22, 6.84, 7.46)	(4.76, 5.35, 5.94, 6.53, 7.12)
(4.59, 5.16, 5.73, 6.3, 6.87)	(4.67, 5.26, 5.85, 6.44, 7.03)	(5.22, 5.87, 6.52, 7.17, 7.82)
(4.97, 5.59, 6.21, 6.83, 7.45)	(4.75, 5.35, 5.95, 6.55, 7.15)	(4.6, 5.17, 5.74, 6.31, 6.88)
(5.06, 5.69, 6.32, 6.95, 7.58)	(4.99, 5.62, 6.25, 6.88, 7.51)	(5.01, 5.64, 6.27, 6.9, 7.53)
(4.79, 5.39, 5.99, 6.59, 7.19)	(5.1, 5.74, 6.38, 7.02, 7.66)	-
(4.55, 5.12, 5.69, 6.26, 6.83)	(4.46, 5.02, 5.58, 6.14, 6.7)	-
(4.8, 5.4, 6, 6.6, 7.2)	(5.04, 5.67, 6.3, 6.93, 7.56)	-
(4.72, 5.31, 5.9, 6.49, 7.08)	(4.88, 5.49, 6.1, 6.71, 7.32)	-
(4.96, 5.58, 6.2, 6.82, 7.44)	(4.99, 5.61, 6.23, 6.85, 7.47)	-

Now, Based on the pentagonal fuzzy data $\tilde{X}_i$ $i = 1, 2, \ldots, n$, (Table 4) obtained from Table 2, the fuzzy point estimation $\tilde{\tilde{X}}$ is found as

$$\tilde{\tilde{X}} = \frac{1}{n}\sum_{i=1}^{n}\tilde{X}_i$$

$$= \frac{1}{25}(120.41,\ 135.46,\ 150.51,\ 165.56,\ 180.61)$$

$$= (4.8164,\ 5.4184,\ 6.0204,\ 6.6224,\ 7.2244)$$

The fuzzy point estimation is $\tilde{\tilde{X}} = (4.8164, 5.4184, 6.0204, 6.6224, 7.2244)$.
The fuzzy test statistic is

$$\widetilde{Z} = \frac{\widetilde{\widetilde{X}} - \widetilde{H}_0}{\sigma/\sqrt{n}}$$

$$= \frac{1}{3/\sqrt{25}}\left[(4.8164,\ 5.4184,\ 6.0204,\ 6.6224,\ 7.2244) - (5.5,\ 5.75,\ 6,\ 6.25,\ 6.5)\right]$$

$$= \frac{5}{3}(-1.6836,\ -0.8316,\ 0.0204,\ 0.8724,\ 1.7244)$$

$$= (-2.8060,\ -1.3860,\ 0.0340,\ 1.4540,\ 2.8740)$$

The fuzzy test statistic is $\widetilde{Z} = (-2.8060, -1.3860, 0.0340, 1.4540, 2.8740)$.
Using the proposed centroid in equation (3), $x_{\widetilde{Z}} = 0.0340$.
To find a Fuzzy Test Statistic based on Fuzzy data with a Heptagonal form of membership.
Let $\widetilde{H}_0 = (5.25, 5.5, 5.75, 6, 6.25, 6.5, 6.75)$ and $\widetilde{H}_{1L} = (5.30, 5.55, 5.80, 6)$.

Table 5. The heptagonal fuzzy data from normal population

$\widetilde{X}_i^H = (x_i - 3\delta_i,\ x_i - 2\delta_i,\ x_i - \delta_i,\ x_i,\ x_i + \delta_i,\ x_i + 2\delta_i,\ x_i + 3\delta_i)_H$	
(4.18, 4.78, 5.38, 5.98, 6.58, 7.18, 7.78)	(4.15, 4.75, 5.35, 5.95, 6.55, 7.15, 7.75)
(3.96, 4.52, 5.08, 5.64, 6.2, 6.76, 7.32)	(4.36, 4.99, 5.62, 6.25, 6.88, 7.51, 8.14)
(4.02, 4.59, 5.16, 5.73, 6.3, 6.87, 7.44)	(4.46, 5.1, 5.74, 6.38, 7.02, 7.66, 8.3)
(4.35, 4.97, 5.59, 6.21, 6.83, 7.45, 8.07)	(3.9, 4.46, 5.02, 5.58, 6.14, 6.7, 7.26)
(4.43, 5.06, 5.69, 6.32, 6.95, 7.58, 8.21)	(4.41, 5.04, 5.67, 6.3, 6.93, 7.56, 8.19)
(4.19, 4.79, 5.39, 5.99, 6.59, 7.19, 7.79)	(4.27, 4.88, 5.49, 6.1, 6.71, 7.32, 7.93)
(3.98, 4.55, 5.12, 5.69, 6.26, 6.83, 7.4)	(4.37, 4.99, 5.61, 6.23, 6.85, 7.47, 8.09)
(4.2, 4.8, 5.4, 6, 6.6, 7.2, 7.8)	(3.79, 4.33, 4.87, 5.41, 5.95, 6.49, 7.03)
(4.13, 4.72, 5.31, 5.9, 6.49, 7.08, 7.67)	(4.17, 4.76, 5.35, 5.94, 6.53, 7.12, 7.71)
(4.34, 4.96, 5.58, 6.2, 6.82, 7.44, 8.06)	(4.57, 5.22, 5.87, 6.52, 7.17, 7.82, 8.47)
(4.28, 4.89, 5.5, 6.11, 6.72, 7.33, 7.94)	(4.38, 5.01, 5.64, 6.27, 6.9, 7.53, 8.16)
(4.36, 4.98, 5.6, 6.22, 6.84, 7.46, 8.08)	(4.03, 4.6, 5.17, 5.74, 6.31, 6.88, 7.45)
(4.08, 4.67, 5.26, 5.85, 6.44, 7.03, 7.62)	-

Now, based on the heptagonal fuzzy data $\widetilde{X}_i$ $i = 1, 2,\ldots, n$, (Table 5) obtained from Table 2, the fuzzy point estimation $\widetilde{\widetilde{X}}$ is found as

$$\widetilde{\widetilde{X}} = \frac{1}{n}\sum_{i=1}^{n}\widetilde{X}_i$$

$$= \frac{1}{25}(105.36,\ 120.41,\ 135.46,\ 150.51,\ 165.56,\ 180.61,\ 195.66)$$

$$= (4.2144,\ 4.8164,\ 5.4184,\ 6.0204,\ 6.6224,\ 7.2244,\ 7.8264)$$

The fuzzy point estimation is $\widetilde{\widetilde{X}}$ = (4.2144, 4.8164, 5.4184, 6.0204, 6.6224, 7.2244, 7.8264).
The fuzzy test statistic is

$$\widetilde{Z} = \frac{\widetilde{\widetilde{X}} - \widetilde{H}_0}{\sigma/\sqrt{n}}$$

$$= \frac{1}{3/\sqrt{25}}\left[(4.2144,\ 4.8164,\ 5.4184,\ 6.0204,\ 6.6224,\ 7.2244,\ 7.8264) - (5.25,\ 5.5,\ 5.75,\ 6,\ 6.25,\ 6.5,\ 6.75)\right]$$

$$= \frac{5}{3}(-2.5356,\ -1.6836,\ -0.8316,\ 0.0204,\ 0.8724,\ 1.7244,\ 2.5764)$$

$$= (-4.2260,\ -2.8060,\ -1.3860,\ 0.0340,\ 1.4540,\ 2.8740,\ 4.2940)$$

The fuzzy test statistic is $\widetilde{Z}$ = (−4.2260, −2.8060, − 1.3860, 0.0340, 1.4540, 2.8740, 4.2940).
Using the proposed centroid in equation (4), $x_{\widetilde{Z}} = 0.0165$.
Hence, based on the new distance function, we get

$$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = \min\left\{\left(x_{\widetilde{Z}} - Z_{1-\varepsilon}\right)_T,\ \left(x_{\widetilde{Z}} - Z_{1-\varepsilon}\right)_P,\ \left(x_{\widetilde{Z}} - Z_{1-\varepsilon}\right)_H\right\}$$

$$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = \min\left\{(0.0866 - 1.6449)_T,\ (0.0340 - 1.6449)_P,\ (0.0165 - 1.6449)_H\right\}$$

$$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = \min\{-1.5583,\ -1.6109,\ -1.6284\} = -1.6284$$

Since $D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = -1.6284 < 0$. Hence, based on the decision rule 4.3, the hypothesis H_0 is accepted.
In this example, suppose that we want to test the following hypotheses, at the significance level $\xi = 0.05$.

$$\begin{Bmatrix} H_0 : \Omega \text{ is a approximately } 6 \\ H_{1L} : \Omega \text{ is always from } 6 \end{Bmatrix} = \begin{Bmatrix} H_0 : \Omega \text{ is } \widetilde{H}_0 \\ H_0 : \Omega \text{ is } \widetilde{H}_{1L} \end{Bmatrix}$$

Based on the above fuzzy test statistics and its centroids, we have

$$D\left(\widetilde{Z}, Z_{\varepsilon/2}\right) = \max\left\{\left(x_{\widetilde{Z}} - Z_{\varepsilon/2}\right)_T,\ \left(x_{\widetilde{Z}} - Z_{\varepsilon/2}\right)_P,\ \left(x_{\widetilde{Z}} - Z_{\varepsilon/2}\right)_H\right\}$$

$$D\left(\widetilde{Z}, Z_{\varepsilon/2}\right) = \max\left\{(0.0866 + 1.9600)_T,\ (0.0340 + 1.9600)_P,\ (0.0165 + 1.9600)_H\right\}$$

$$= \max\{2.0466,\ 1.9940,\ 1.9765\} = 2.0466 > 0$$

$$D\left(\widetilde{Z}, Z_{1-\varepsilon/2}\right) = \min\left\{\left(x_{\widetilde{Z}} - Z_{1-\varepsilon/2}\right)_T,\ \left(x_{\widetilde{Z}} - Z_{1-\varepsilon/2}\right)_P,\ \left(x_{\widetilde{Z}} - Z_{1-\varepsilon/2}\right)_H\right\}$$

$$D\left(\widetilde{Z}, Z_{1-\varepsilon/2}\right) = \min\left\{(0.0866 - 1.9600)_T,\ (0.0340 - 1.9600)_P,\ (0.0165 - 1.9600)_H\right\}$$

$$= \min\{-1.8734,\ -1.9260,\ -1.9435\} = -1.9435 < 0$$

Since $D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = 2.0466 > 0$ and $D\left(\widetilde{Z}, Z_{1-\varepsilon/2}\right) = -1.9435 < 0$. Hence, based on the decision rule 4.2 hypothesis

H_0 is accepted, at the significance level $\xi = 0.05$.

B) Now suppose that we want to test the following crisp hypotheses (which are equivalent to the case $a_1 = \Omega_0 = a_3$ in the above fuzzy hypotheses):

$$\begin{cases} H_0 : \Omega = 6 \\ H_1 : \Omega > 6 \end{cases}$$

To find a Fuzzy Test Statistic based on Triangular Fuzzy Number.

The fuzzy statistic is

$$\tilde{Z} = \frac{\tilde{\bar{X}} - \Omega_0}{\sigma/\sqrt{n}}$$

$$= \frac{1}{3/\sqrt{25}}\left[(5.4184,\ 6.0204,\ 6.6224) - 6\right]$$

$$= \frac{1}{3/\sqrt{25}}\left[(5.4184,\ 6.0204,\ 6.6224) - (6,\ 6,\ 6)\right]$$

$$= \frac{5}{3}(-0.5816,\ 0.0204,\ 0.6224)$$

$$= (-0.9693,\ 0.0340,\ 1.0373)$$

Hence the fuzzy test statistic is $\tilde{Z} = (-0.9693, 0.0340, 1.0373)$.

Using the proposed centroid in equation (2), $x_{\tilde{Z}} = 0.0712$.

To find Fuzzy Test Statistic based on the Pentagonal Fuzzy Number.

The fuzzy test statistic is

$$\tilde{Z} = \frac{\tilde{\bar{X}} - \Omega_0}{\sigma/\sqrt{n}}$$

$$= \frac{1}{3/\sqrt{25}}\left[(4.8164,\ 5.4184,\ 6.0204,\ 6.6224,\ 7.2244) - 6\right]$$

$$= \frac{1}{3/\sqrt{25}}\left[(4.8164,\ 5.4184,\ 6.0204,\ 6.6224,\ 7.2244) - (6,\ 6,\ 6,\ 6,\ 6)\right]$$

$$= \frac{5}{3}(-1.1836,\ -0.5816,\ 0.0204,\ 0.6224,\ 1.2244)$$

$$= (-1.9727,\ -0.9693,\ 0.0340,\ 1.0373,\ 2.0407)$$

The fuzzy test statistic is $\tilde{Z} = (-1.9727, -0.9693, 0.0340, 1.0373, 2.0407)$.

Using the proposed centroid in equation (3), $x_{\tilde{Z}} = 0.0340$.

To find a Fuzzy Test Statistic based on a Heptagonal Fuzzy Number.

The fuzzy test statistic is

$$\tilde{Z} = \frac{\tilde{\bar{X}} - \Omega_0}{\sigma/\sqrt{n}}$$

$$= \frac{1}{3/\sqrt{25}}\left[(4.2144,\ 4.8164,\ 5.4184,\ 6.0204,\ 6.6224,\ 7.2244,\ 7.2244) - 6\right]$$

$$= \frac{1}{3/\sqrt{25}}\left[(4.2144,\ 4.8164,\ 5.4184,\ 6.0204,\ 6.6224,\ 7.2244,\ 7.2244) - (6,\ 6,\ 6,\ 6,\ 6,\ 6,\ 6)\right]$$

$$= \frac{5}{3}(-1.7856,\ -1.1836,\ -0.5816,\ 0.0204,\ 0.6224,\ 1.2244,\ 1.8264)$$

$$= (-2.9760,\ -1.9727,\ -0.9693,\ 0.0340,\ 1.0373,\ 2.0407,\ 3.0440)$$

The fuzzy test statistic is $\widetilde{Z} = (-2.9760, -1.9727, -0.9693, 0.0340, 1.0373, 2.0407, 3.0440)$.
Using the proposed centroid in equation (4), $x_{\widetilde{Z}} = 0.0216$.
Hence, based on the new distance function, we get

$$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = \min\left\{\left(x_{\widetilde{Z}} - Z_{1-\varepsilon}\right)_T,\ \left(x_{\widetilde{Z}} - Z_{1-\varepsilon}\right)_P,\ \left(x_{\widetilde{Z}} - Z_{1-\varepsilon}\right)_H\right\}$$

$$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = \min\left\{(0.0712 - 1.6449)_T,\ (0.0340 - 1.6449)_P,\ (0.0216 - 1.6449)_H\right\}$$

$$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = \min\{-1.5737,\ -1.6109,\ -1.6233\} = -1.6233 < 0$$

Since $D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = -1.6233 < 0$. Hence, based on the new decision rule 4.3, H_0 is accepted.
In this example, suppose that we want to test the following hypotheses, at the significance level $\xi = 0.05$.

$$\begin{cases} H_0 : \Omega \neq 6 \\ H_1 : \Omega \neq 6 \end{cases}$$

Based on the above fuzzy test statistics and its centroids, we have

$$D\left(\widetilde{Z}, Z_{\varepsilon/2}\right) = \max\left\{\left(x_{\widetilde{Z}} - Z_{\varepsilon/2}\right)_T,\ \left(x_{\widetilde{Z}} - Z_{\varepsilon/2}\right)_P,\ \left(x_{\widetilde{Z}} - Z_{\varepsilon/2}\right)_H\right\}$$

$$D\left(\widetilde{Z}, Z_{\varepsilon/2}\right) = \max\left\{(0.0712 + 1.9600)_T,\ (0.0340 + 1.9600)_P,\ (0.0216 + 1.9600)_H\right\}$$

$$= \max\{2.0312,\ 1.9940,\ 1.9816\} = 2.0312 > 0$$

$$D\left(\widetilde{Z}, Z_{1-\varepsilon/2}\right) = \min\left\{\left(x_{\widetilde{Z}} - Z_{1-\varepsilon/2}\right)_T,\ \left(x_{\widetilde{Z}} - Z_{1-\varepsilon/2}\right)_P,\ \left(x_{\widetilde{Z}} - Z_{1-\varepsilon/2}\right)_H\right\}$$

$$D\left(\widetilde{Z}, Z_{1-\varepsilon/2}\right) = \max\left\{(0.0712 - 1.9600)_T,\ (0.0340 - 1.9600)_P,\ (0.0216 - 1.9600)_H\right\}$$

$$= \min\{-1.8888,\ -1.9260,\ -1.9384\} = -1.9384 < 0$$

Since $D\left(\widetilde{Z}, Z_{\varepsilon/2}\right) = 2.0312 > 0$ and $D\left(\widetilde{Z}, Z_{1-\varepsilon/2}\right) = -1.9384 < 0$. Hence, based on the decision rule 4.2 hypothesis H_0 is accepted, at the significance level $\xi = 0.05$.

The Table 6 shows the Comparison of the proposed with the method introduced by Arefi [29] to the fuzzy hypotheses for the mean.

Table 6. Comparative analysis of testing fuzzy hypotheses for the mean

	Proposed Method		Arefi [29]
	General Fuzzy Data	Triangular Fuzzy Data	Triangular Fuzzy Data
Fuzzy Hypothesis			
One sided hypothesis	$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = -1.6284 < 0$	$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = -1.5583 < 0$	$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = -2.0842 < 0$
Two sided hypothesis	$D\left(\widetilde{Z}, Z_{\varepsilon/2}\right) = 2.0466 > 0$	$D\left(\widetilde{Z}, Z_{\varepsilon/2}\right) = 2.0466 > 0$	$D\left(\widetilde{Z}, Z_{\varepsilon/2}\right) = 0.9906 > 0$
	$D\left(\widetilde{Z}, Z_{1-\varepsilon/2}\right) = -1.9345 < 0$	$D\left(\widetilde{Z}, Z_{1-\varepsilon/2}\right) = -1.8734 < 0$	$D\left(\widetilde{Z}, Z_{1-\varepsilon/2}\right) = -2.9294 < 0$
Crisp Hypothesis			
One sided hypothesis	$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = -1.6233 < 0$	$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = -1.5737 < 0$	$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = -1.9453 < 0$
Two sided hypothesis	$D\left(\widetilde{Z}, Z_{\varepsilon/2}\right) = 2.0312 > 0$	$D\left(\widetilde{Z}, Z_{\varepsilon/2}\right) = 2.0312 > 0$	$D\left(\widetilde{Z}, Z_{\varepsilon/2}\right) = -1.6596 > 0$
	$D\left(\widetilde{Z}, Z_{1-\varepsilon/2}\right) = -1.9384 < 0$	$D\left(\widetilde{Z}, Z_{1-\varepsilon/2}\right) = -1.8888 < 0$	$D\left(\widetilde{Z}, Z_{1-\varepsilon/2}\right) = -2.2604 < 0$

5.2 *Testing fuzzy hypotheses for the variance*

Suppose that we have taken a random sample of size n from a population $N(\mu, \Omega)$ (μ is unknown) and observed the fuzzy numbers $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n$. The fuzzy point estimation Ω is

$$\Omega^* = s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$$

Under the crisp hypothesis $H_0 : \Omega = \Omega_0$, the value of the crisp test statistic is $\Theta_0 = \frac{(n-1)s^2}{\Omega_0}$ distributed according to $\chi^2_{(n-1)}$ with $\Theta_0 = \frac{(n-1)\Omega^*}{\Omega_0} = \frac{(n-1)s^2}{\Omega_0}$. By substituting the fuzzy point estimation $\tilde{s}^2$ for $\Omega^* = s^2$ and using the fuzzy arithmetic, the fuzzy test statistic is obtained as follows:

$$\widetilde{Z} = \frac{(n-1)\tilde{s}^2}{\widetilde{H}_0}$$

Based on the above fuzzy test statistic, we can test the null hypothesis $H_0 : \Omega = \Omega_0$ against the hypothesis H_1 as follows:

Example 5.2 A sample of size $n = 28$ $N(\mu, \Omega)$ is given in Table 7 as fuzzy observations.

Table 7. The fuzzy data from a normal population

$\widetilde{X}_i = (x_i, \delta_i)$			
(3.03, 0.61)	(2.6, 0.52)	(3.01, 0.6)	(3.1, 0.62)
(3.45, 0.69)	(2.95, 0.59)	(3.3, 0.66)	(3.04, 0.61)
(2.9, 0.58)	(2.87, 0.57)	(3.21, 0.64)	(3.5, 0.7)
(2.83, 0.57)	(3.52, 0.7)	(3.39, 0.68)	(3, 0.6)
(3.3, 0.66)	(2.9, 0.58)	(2.43, 0.49)	(2.79, 0.56)
(3.59, 0.72)	(2.77, 0.55)	(2.69, 0.54)	(2.63, 0.53)
(2.54, 0.51)	(2.89, 0.58)	(2.83, 0.57)	(2.72, 0.54)

A) We test the following fuzzy hypotheses, at the significance level $\xi = 0.05$.

$$\begin{Bmatrix} H_0 : \Omega \text{ is approximately } 3, \\ H_{1L} : \Omega \text{ is essentially larger than } 3, \end{Bmatrix} \equiv \begin{Bmatrix} H_0 : \Omega \text{ is } \widetilde{H}_0, \\ H_1 : \Omega \text{ is } \widetilde{H}_{1L}. \end{Bmatrix}$$

To find a Fuzzy Test Statistic based on a Triangular Fuzzy Number.
Let $\widetilde{H}_0 = (2.5, 3, 3.5)$ and $\widetilde{H}_{1L} = (2.55, 3)$.

Table 8. The Triangular fuzzy data from normal population

$\widetilde{X}_i^T = (x_i - \delta_i, x_i, x_i + \delta_i)_T$			
(2.42, 3.03, 3.64)	(2.08, 2.6, 3.12)	(2.41, 3.01, 3.61)	(2.48, 3.1, 3.72)
(2.76, 3.45, 4.14)	(2.36, 2.95, 3.54)	(2.64, 3.3, 3.96)	(2.43, 3.04, 3.65)
(2.32, 2.9, 3.48)	(2.3, 2.87, 3.44)	(2.57, 3.21, 3.85)	(2.8, 3.5, 4.2)
(2.26, 2.83, 3.4)	(2.82, 3.52, 4.22)	(2.71, 3.39, 4.07)	(2.4, 3, 3.6)
(2.64, 3.3, 3.96)	(2.32, 2.9, 3.48)	(1.94, 2.43, 2.92)	(2.23, 2.79, 3.35)
(2.87, 3.59, 4.31)	(2.22, 2.77, 3.32)	(2.15, 2.69, 3.23)	(2.1, 2.63, 3.16)
(2.03, 2.54, 3.05)	(2.31, 2.89, 3.47)	(2.26, 2.83, 3.4)	(2.18, 2.72, 3.26)

Now, based on the triangular fuzzy data $\widetilde{X}_i$, $i = 1, 2,\dots, n$, (Table 8) obtained from Table 7, the fuzzy mean $\widetilde{\overline{X}}$ is found as

$$\widetilde{\overline{X}} = \frac{1}{n}\sum_{i=1}^{n} \widetilde{X}_i$$

$$= \frac{1}{28}(67.01,\ 83.78,\ 100.55)$$

$$= (2.3932,\ 2.9921,\ 3.5911)$$

The fuzzy point estimation is

$$\tilde{s}^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(\widetilde{X}_i - \widetilde{\overline{X}}\right)^2$$

$$= \frac{1}{27}(-37.6104,\ 2.6728,\ 60.0715)$$

$$= (-1.3930,\ 0.0990,\ 2.2249)$$

Thus the fuzzy test statistic is

$$\widetilde{Z} = \frac{(n-1)\tilde{s}^2}{\widetilde{H}_0}$$

$$= \frac{27(-1.3930,\ 0.0990,\ 2.2249)}{(2.5,\ 3,\ 3.5)}$$

$$= (-15.0442,\ 0.8909,\ 24.0286)$$

The fuzzy test statistic is $\widetilde{Z} = (-15.0442, 0.8909, 24.0286)$.
Using the proposed centroid in equation (2), $x_{\widetilde{Z}} = 3.4818$.
To find Fuzzy Test Statistic based on Pentagonal Fuzzy Number.
Let $\widetilde{H}_0 = (2, 2.5, 3, 3.5, 4)$ and $\widetilde{H}_{1L} = (2.05, 2.55, 3)$.

Table 9. The pentagonal fuzzy data from normal population

$\widetilde{X}_i^P = (x_i - 2\delta_i,\ x_i - \delta_i,\ x_i,\ x_i + \delta_i,\ x_i + 2\delta_i)_P$		
(1.81, 2.42, 3.03, 3.64, 4.25)	(2.12, 2.82, 3.52, 4.22, 4.92)	(1.69, 2.26, 2.83, 3.4, 3.97)
(2.07, 2.76, 3.45, 4.14, 4.83)	(1.74, 2.32, 2.9, 3.48, 4.06)	(1.86, 2.48, 3.1, 3.72, 4.34)
(1.74, 2.32, 2.9, 3.48, 4.06)	(1.67, 2.22, 2.77, 3.32, 3.87)	(1.82, 2.43, 3.04, 3.65, 4.26)
(1.69, 2.26, 2.83, 3.4, 3.97)	(1.73, 2.31, 2.89, 3.47, 4.05)	(2.1, 2.8, 3.5, 4.2, 4.9)
(1.98, 2.64, 3.3, 3.96, 4.62)	(1.81, 2.41, 3.01, 3.61, 4.21)	(1.8, 2.4, 3, 3.6, 4.2)
(2.15, 2.87, 3.59, 4.31, 5.03)	(1.98, 2.64, 3.3, 3.96, 4.62)	(1.67, 2.23, 2.79, 3.35, 3.91)
(1.52, 2.03, 2.54, 3.05, 3.56)	(1.93, 2.57, 3.21, 3.85, 4.49)	(1.57, 2.1, 2.63, 3.16, 3.69)
(1.56, 2.08, 2.6, 3.12, 3.64)	(2.03, 2.71, 3.39, 4.07, 4.75)	(1.64, 2.18, 2.72, 3.26, 3.8)
(1.77, 2.36, 2.95, 3.54, 4.13)	(1.45, 1.94, 2.43, 2.92, 3.41)	-
(1.73, 2.3, 2.87, 3.44, 4.01)	(1.61, 2.15, 2.69, 3.23, 3.77)	-

Now, based on the pentagonal fuzzy data $\widetilde{X}_i$, $i = 1, 2, \ldots, n$, (Table 9) obtained from Table 7 the fuzzy mean $\widetilde{\widetilde{X}}$ is found as

$$\widetilde{\widetilde{X}} = \frac{1}{n}\sum_{i=1}^{n}\widetilde{X}_i$$

$$= \frac{1}{28}(50.24,\ 67.01,\ 83.78,\ 100.55,\ 117.32)$$

$$= (1.7943,\ 2.3932,\ 2.9921,\ 3.5911,\ 4.19)$$

The fuzzy point estimation is

$$\tilde{s}^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(\widetilde{X}_i - \widetilde{\widetilde{X}}\right)^2$$

$$= \frac{1}{27}(-158.4516,\ -37.6104,\ 2.6728,\ 60.0715,\ 198.0286)$$

$$= (-5.8686,\ -1.3930,\ 0.0990,\ 2.2249,\ 7.3344)$$

Thus the fuzzy test statistic is

$$\tilde{Z} = \frac{(n-1)\tilde{s}^2}{\tilde{H}_0}$$

$$= \frac{27(-5.8686,\ -1.3930,\ 0.0990,\ 2.2249,\ 7.3344)}{(2,\ 2.5,\ 3,\ 3.5,\ 4)}$$

$$= (-79.2258,\ -15.0442,\ 0.8909,\ 24.0286,\ 99.0143)$$

The fuzzy test statistic is $\tilde{Z}$ = (−79.2258, −15.0442, 0.8909, 24.0286, 99.0143).
Using the proposed centroid in equation (3), $x_{\tilde{Z}}$ = 6.2193.
To find a Fuzzy Test Statistic based on a Heptagonal Fuzzy Number.
Let $\tilde{H}_0$ = (1.5, 2, 2.5, 3, 3.5, 4, 4.5) $\tilde{H}_{1L}$ = (1.55, 2.05, 2.55, 3).

Table 10. The Heptagonal fuzzy data from normal population

$\tilde{X}_i^H = (x_i - 3\delta_i,\ x_i - 2\delta_i,\ x_i - \delta_i,\ x_i,\ x_i + \delta_i,\ x_i + 2\delta_i,\ x_i + 3\delta_i)_H$	
(1.2, 1.81, 2.42, 3.03, 3.64, 4.25, 4.86)	(1.21, 1.81, 2.41, 3.01, 3.61, 4.21, 4.81)
(1.38, 2.07, 2.76, 3.45, 4.14, 4.83, 5.52)	(1.32, 1.98, 2.64, 3.3, 3.96, 4.62, 5.28)
(1.16, 1.74, 2.32, 2.9, 3.48, 4.06, 4.64)	(1.29, 1.93, 2.57, 3.21, 3.85, 4.49, 5.13)
(1.12, 1.69, 2.26, 2.83, 3.4, 3.97, 4.54)	(1.35, 2.03, 2.71, 3.39, 4.07, 4.75, 5.43)
(1.32, 1.98, 2.64, 3.3, 3.96, 4.62, 5.28)	(0.96, 1.45, 1.94, 2.43, 2.92, 3.41, 3.9)
(1.43, 2.15, 2.87, 3.59, 4.31, 5.03, 5.75)	(1.07, 1.61, 2.15, 2.69, 3.23, 3.77, 4.31)
(1.01, 1.52, 2.03, 2.54, 3.05, 3.56, 4.07)	(1.12, 1.69, 2.26, 2.83, 3.4, 3.97, 4.54)
(1.04, 1.56, 2.08, 2.6, 3.12, 3.64, 4.16)	(1.24, 1.86, 2.48, 3.1, 3.72, 4.34, 4.96)
(1.18, 1.77, 2.36, 2.95, 3.54, 4.13, 4.72)	(1.21, 1.82, 2.43, 3.04, 3.65, 4.26, 4.87)
(1.16, 1.73, 2.3, 2.87, 3.44, 4.01, 4.58)	(1.4, 2.1, 2.8, 3.5, 4.2, 4.9, 5.6)
(1.42, 2.12, 2.82, 3.52, 4.22, 4.92, 5.62)	(1.2, 1.8, 2.4, 3, 3.6, 4.2, 4.8)
(1.16, 1.74, 2.32, 2.9, 3.48, 4.06, 4.64)	(1.11, 1.67, 2.23, 2.79, 3.35, 3.91, 4.47)
(1.12, 1.67, 2.22, 2.77, 3.32, 3.87, 4.42)	(1.04, 1.57, 2.1, 2.63, 3.16, 3.69, 4.22)
(1.15, 1.73, 2.31, 2.89, 3.47, 4.05, 4.63)	(1.1, 1.64, 2.18, 2.72, 3.26, 3.8, 4.34)

Now, based on the heptagonal fuzzy data $\tilde{X}_i$, i = 1, 2,…, n, (Table 10) obtained from Table 7 the fuzzy mean $\tilde{\tilde{X}}$ is found as

$$\tilde{\tilde{X}} = \frac{1}{n}\sum_{i=1}^{n}\tilde{X}_i$$

$$= \frac{1}{28}(33.47,\ 50.24,\ 67.01,\ 83.78,\ 100.55,\ 117.32,\ 134.09)$$

$$= (1.1954,\ 1.7943,\ 2.3932,\ 2.9921,\ 3.5911,\ 4.19,\ 4.7889)$$

The fuzzy point estimation is

$$\tilde{s}^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(\widetilde{X}_i - \widetilde{\widetilde{X}}\right)^2$$

$$= \frac{1}{27}(-359.8513,\ -158.4516,\ -37.6104,\ 2.6728,\ 60.0715,\ 198.0286,\ 416.5442)$$

$$= (-13.3278,\ -5.8686,\ -1.3930,\ 0.0990,\ 2.2249,\ 7.3344,\ 15.4276)$$

Thus the fuzzy test statistic is

$$\widetilde{Z} = \frac{(n-1)\tilde{s}^2}{\widetilde{H}_0}$$

$$= \frac{27(-13.3278,\ -5.8686,\ -1.3930,\ 0.0990,\ 2.2249,\ 7.3344,\ 15.4276)}{(1.5,\ 2,\ 2.5,\ 3,\ 3.5,\ 4,\ 4.5)}$$

$$= (-239.9009,\ -79.2258,\ -15.0442,\ 0.8909,\ 24.0286,\ 99.0143,\ 277.6961)$$

The fuzzy test statistic is $\widetilde{Z} = (-239.9009, -79.2258, -15.0442, 0.8909, 24.0286, 99.0143, 277.6961)$.
Using the proposed centroid in equation (4), $x_{\widetilde{Z}} = 9.9950$.
Hence, based on the new distance function, we get

$$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = \min\left\{\left(x_{\widetilde{Z}} - Z_{\chi^2_{n-1,1-\varepsilon}}\right)_T,\ \left(x_{\widetilde{Z}} - Z_{\chi^2_{n-1,1-\varepsilon}}\right)_P,\ \left(x_{\widetilde{Z}} - Z_{\chi^2_{n-1,1-\varepsilon}}\right)_H\right\}$$

$$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = \min\left\{(3.4818 - 40.113)_T,\ (6.2193 - 40.113)_P,\ (9.9950 - 40.113)_H\right\}$$

$$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = \min\{-36.6312,\ -33.8937,\ -30.1180\} = -36.6312 < 0$$

Since $D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = -36.6312 < 0$. Hence, based on the decision rule 4.3, H_0 is accepted.

B) Now suppose that we want to test the following crisp hypotheses (which are equivalent to the case $a_1 = \Omega_0 = a_3$ in the above fuzzy hypotheses):

$$\begin{cases} H_0 : \Omega = 3 \\ H_1 : \Omega > 3 \end{cases}$$

To find a Fuzzy Test Statistic based on Triangular Fuzzy Number.
The fuzzy test statistic is

$$\widetilde{Z} = \frac{(n-1)\tilde{s}^2}{\Omega_0}$$

$$= \frac{27(-1.3930,\ 0.0990,\ 2.2249)}{(3,\ 3,\ 3)}$$

$$= (-12.5368,\ 0.8909,\ 20.0238)$$

Using the proposed centroid in equation (2), $x_{\tilde{Z}} = 2.9730$.
To find Fuzzy Test Statistic based on Pentagonal Fuzzy Number.
The fuzzy test statistic is

$$\tilde{Z} = \frac{(n-1)\tilde{s}^2}{\Omega_0}$$

$$= \frac{27(-5.8686,\ -1.3930,\ 0.0990,\ 2.2249,\ 7.3344)}{(3,\ 3,\ 3,\ 3,\ 3)}$$

$$= (-52.8172,\ -12.5368,\ 0.8909,\ 20.0238,\ 66.0095)$$

Using the proposed centroid in equation (3), $x_{\tilde{Z}} = 4.5049$.
To find a Fuzzy Test Statistic based on a Heptagonal Fuzzy Number.
The fuzzy test statistic is

$$\tilde{Z} = \frac{(n-1)\tilde{s}^2}{\Omega_0}$$

$$= \frac{27(-13.3278,\ -5.8686,\ -1.3930,\ 0.0990,\ 2.2249,\ 7.3344,\ 15.4276)}{(3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3)}$$

$$= (-119.9504,\ -52.8172,\ -12.5368,\ 0.8909,\ 20.0238,\ 66.0095,\ 138.8481)$$

Using the proposed centroid in equation (4), $x_{\tilde{Z}} = 5.8256$.
Hence, based on the new distance function, we get

$$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = \min\left\{\left(x_{\tilde{Z}} - Z_{\chi^2_{n-1,1-\varepsilon}}\right)_T,\ \left(x_{\tilde{Z}} - Z_{\chi^2_{n-1,1-\varepsilon}}\right)_P,\ \left(x_{\tilde{Z}} - Z_{\chi^2_{n-1,1-\varepsilon}}\right)_H\right\}$$

$$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = \min\left\{(2.9730 - 40.113)_T,\ (4.5049 - 40.113)_P,\ (5.8256 - 40.113)_H\right\}$$

$$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = \min\{-37.1400,\ -35.6081,\ -34.2874\} = -37.1400 < 0$$

Since $D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = -37.1400 < 0$. Hence, based on the decision rule 4.3, H_0 is accepted.

The Table 11 shows the Comparison of the proposed with the method introduced by Arefi [29] to the fuzzy hypotheses for the variance.

Table 11. Comparative analysis of testing fuzzy hypotheses for the variance

	Proposed Method		Arefi [29]
	General Fuzzy Data	Triangular Fuzzy Data	Triangular Fuzzy Data
Fuzzy Hypothesis			
One sided hypothesis	$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = -36.6312 < 0$	$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = -36.6312 < 0$	$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = -35.7647 < 0$
Crisp Hypothesis			
One sided hypothesis	$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = -37.1400 < 0$	$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = -37.1400 < 0$	$D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = -34.6820 < 0$

5.3 *Hypothesis for difference between means of two normal populations with known variances*

Suppose that we have taken two independent random samples of sizes n_1 and n_2 from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ respectively (with σ_1^2 and σ_2^2 known) and observed the fuzzy numbers $\widetilde{X}_1, \widetilde{X}_2, ..., \widetilde{X}_{n_1}$ and $\widetilde{Y}_1, \widetilde{Y}_2, ..., \widetilde{Y}_{n_2}$ instead of crisp numbers x_1, x_2, x_{n_1} and y_1, y_2, y_{n_2}. The usual point estimation $\Omega = \mu_1 - \mu_2$ is $\Omega^* = \bar{x} - \bar{y}$.

Under the crisp hypothesis $H_0 : \mu_1 - \mu_2 = \Omega_0$, the value of the crisp test statistic is $z_0 = \dfrac{(\bar{x} - \bar{y}) - \Omega_0}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} = \dfrac{\Omega^* - \Omega_0}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$ (distributed according to standard normal $N(0, 1)$). By substituting the fuzzy point estimation $\widetilde{\bar{X}} - \widetilde{\bar{Y}}$ for Ω^* in z_0 and using the fuzzy arithmetic, the fuzzy test statistic is obtained as follows

$$\widetilde{Z} = \frac{\widetilde{\Omega}^* - \widetilde{H}_0}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$$

Example 5.3 Assume that based on two independent random samples of sizes $n_1 = 25$ $n_2 = 16$ and, $N(\mu_1, 9)$ and $N(\mu_2, 4)$ we obtain the symmetric normal fuzzy numbers given in Table 12.

Table 12. The fuzzy data from a normal population

$\widetilde{X}_i = (x_i, \delta_i)$			$\widetilde{Y}_i = (y_i, \eta_i)$	
(10.5, 1.04)	(12.43, 1.24)	(9.55, 0.96)	(6.32, 0.63)	(4.9, 0.49)
(6.49, 0.65)	(13.71, 1.37)	(11.33, 1.13)	(2.56, 0.26)	(5.27, 0.53)
(12.5, 1.25)	(12.77, 1.28)	(10.91, 1.09)	(4.76, 0.48)	(4.29, 0.43)
(6.25, 0.63)	(10.99, 1.1)	(12.22, 1.22)	(4.25, 0.43)	(5.24, 0.52)
(9.82, 0.98)	(8.25, 0.83)	(10.67, 1.07)	(6.27, 0.63)	(3.29, 0.33)
(7.83, 0.78)	(10.52, 1.05)	(12.47, 1.25)	(3.25, 0.33)	(7.1, 0.71)
(12, 1.2)	(10.2, 1.02)	(9.99, 1)	(6.2, 0.62)	(5.26, 0.53)
(12.5, 1.25)	(16.2, 1.62)	(8.57, 0.86)	(5.27, 0.52)	(4.77, 0.48)
(11.79, 1.18)	-	-	-	-

A) We wish to test the following hypotheses at the significance level for $\xi = 0.05$.

$$\begin{cases} H_0 : \Omega \text{ is approximately } 5, \\ H_{1L} : \Omega \text{ is essentially larger than } 5, \end{cases} \equiv \begin{cases} H_0 : \Omega \text{ is } \widetilde{H}_0, \\ H_1 : \Omega \text{ is } \widetilde{H}_{1L}. \end{cases}$$

To find a Fuzzy Test Statistic based on a Triangular Fuzzy Number.
Let $\widetilde{H}_0 = (4.8, 5, 5.2)$ and $\widetilde{H}_{1L} = (4.85, 5)$.

Table 13. The triangular fuzzy data from normal population

$\widetilde{X}_i^T = (x_i - \delta_i, x_i, x_i + \delta_i)_T$			$\widetilde{Y}_i^T = (y_i - \eta_i, y_i, y_i + \eta_i)_T$	
(9.46, 10.5, 11.54)	(11.19, 12.43, 13.67)	(8.59, 9.55, 10.51)	(5.69, 6.32, 6.95)	(4.41, 4.9, 5.39)
(5.84, 6.49, 7.14)	(12.34, 13.71, 15.08)	(10.2, 11.33, 12.46)	(2.3, 2.56, 2.82)	(4.74, 5.27, 5.8)
(11.25, 12.5, 13.75)	(11.49, 12.77, 14.05)	(9.82, 10.91, 12)	(4.28, 4.76, 5.24)	(3.86, 4.29, 4.72)
(5.62, 6.25, 6.88)	(9.89, 10.99, 12.09)	(11, 12.22, 13.44)	(3.82, 4.25, 4.68)	(4.72, 5.24, 5.76)
(8.84, 9.82, 10.8)	(7.42, 8.25, 9.08)	(9.6, 10.67, 11.74)	(5.64, 6.27, 6.9)	(2.96, 3.29, 3.62)
(7.05, 7.83, 8.61)	(9.47, 10.52, 11.57)	(11.22, 12.47, 13.72)	(2.92, 3.25, 3.58)	(6.39, 7.1, 7.81)
(10.8, 12, 13.2)	(9.18, 10.2, 11.22)	(8.99, 9.99, 10.99)	(5.58, 6.2, 6.82)	(4.73, 5.26, 5.79)
(11.25, 12.5, 13.75)	(14.58, 16.2, 17.82)	(7.71, 8.57, 9.43)	(4.75, 5.27, 5.79)	(4.29, 4.77, 5.25)
(10.61, 11.79, 12.97)	-	-	-	-

Now, Based on triangular fuzzy data $\widetilde{X}_i$, $i = 1, 2, \ldots, n_1$ and $\widetilde{Y}_j$, $j = 1, 2, \ldots, n_2$ (Table 13) obtained from Table 12, the fuzzy means $\widetilde{\overline{X}}$ and $\widetilde{\overline{Y}}$ are found as

$$\widetilde{\overline{X}} = \frac{1}{n_1}\sum_{i=1}^{n_1}\widetilde{X}_i$$

$$= \frac{1}{25}(243.4,\ 270.46,\ 297.52)$$

$$= (9.7360,\ 10.8184,\ 11.9008)$$

$$\widetilde{\overline{Y}} = \frac{1}{n_2}\sum_{j=1}^{n_2}\widetilde{Y}_j$$

$$= \frac{1}{16}(71.07,\ 79,\ 86.93)$$

$$= (4.4419,\ 4.9375,\ 5.4331)$$

The fuzzy point estimation is

$$\tilde{\tau}^* = \widetilde{\overline{X}} - \widetilde{\overline{Y}}$$

$$= \frac{1}{n_1}\sum_{i=1}^{n_1}\widetilde{X}_i - \frac{1}{n_2}\sum_{j=1}^{n_2}\widetilde{Y}_j$$

$$= (9.7360,\ 10.8184,\ 11.9008) - (4.4419,\ 4.9375,\ 5.4331)$$

$$= (4.3029,\ 5.8809,\ 7.4589)$$

Thus the fuzzy test statistic is

$$\widetilde{Z} = \frac{\widetilde{\Omega}^{*} - \widetilde{H}_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$= \frac{(4.3029,\ 5.8809,\ 7.4589) - (4.8,\ 5,\ 5.2)}{\sqrt{\frac{9}{25} + \frac{4}{16}}}$$

$$= \frac{(-0.8971,\ 0.8809,\ 2.6589)}{0.7810}$$

$$= (-1.1487,\ 1.1279,\ 3.4045)$$

Using the proposed centroid in equation (2), $x_{\widetilde{Z}} = 1.2122$.
To find Fuzzy Test Statistic based on the Pentagonal Fuzzy Number.
Let $\widetilde{H}_0 = (4.6, 4.8, 5, 5.2, 5.4)$ and $\widetilde{H}_{1L} = (4.65, 4.85, 5)$.

Table 14. The pentagonal fuzzy data from normal population

$\widetilde{X}_i^P = (x_i - 2\delta_i, x_i - \delta_i, x_i, x_i + \delta_i, x_i + 2\delta_i)_P$		$\widetilde{Y}_i^P = (y_i - 2\eta_i, y_i - \eta_i, y_i, y_i + \eta_i, y_i + 2\eta_i)_P$
(8.42, 9.46, 10.5, 11.54, 12.58)	(12.96, 14.58, 16.2, 17.82, 19.44)	(5.06, 5.69, 6.32, 6.95, 7.58)
(5.19, 5.84, 6.49, 7.14, 7.79)	(7.63, 8.59, 9.55, 10.51, 11.47)	(2.04, 2.3, 2.56, 2.82, 3.08)
(10, 11.25, 12.5, 13.75, 15)	(9.07, 10.2, 11.33, 12.46, 13.59)	(3.8, 4.28, 4.76, 5.24, 5.72)
(4.99, 5.62, 6.25, 6.88, 7.51)	(8.73, 9.82, 10.91, 12, 13.09)	(3.39, 3.82, 4.25, 4.68, 5.11)
(7.86, 8.84, 9.82, 10.8, 11.78)	(8.53, 9.6, 10.67, 11.74, 12.81)	(5.01, 5.64, 6.27, 6.9, 7.53)
(6.27, 7.05, 7.83, 8.61, 9.39)	(8.53, 9.6, 10.67, 11.74, 12.81)	(2.59, 2.92, 3.25, 3.58, 3.91)
(9.6, 10.8, 12, 13.2, 14.4)	(9.97, 11.22, 12.47, 13.72, 14.97)	(4.96, 5.58, 6.2, 6.82, 7.44)
(10, 11.25, 12.5, 13.75, 15)	(7.99, 8.99, 9.99, 10.99, 11.99)	(4.23, 4.75, 5.27, 5.79, 6.31)
(9.43, 10.61, 11.79, 12.97, 14.15)	(6.85, 7.71, 8.57, 9.43, 10.29)	(3.92, 4.41, 4.9, 5.39, 5.88)
(9.95, 11.19, 12.43, 13.67, 14.91)	-	(4.21, 4.74, 5.27, 5.8, 6.33)
(10.97, 12.34, 13.71, 15.08, 16.45)	-	(3.43, 3.86, 4.29, 4.72, 5.15)
(10.21, 11.49, 12.77, 14.05, 15.33)	-	(4.2, 4.72, 5.24, 5.76, 6.28)
(8.79, 9.89, 10.99, 12.09, 13.19)	-	(2.63, 2.96, 3.29, 3.62, 3.95)
(6.59, 7.42, 8.25, 9.08, 9.91)	-	(5.68, 6.39, 7.1, 7.81, 8.52)
(8.42, 9.47, 10.52, 11.57, 12.62)	-	(4.2, 4.73, 5.26, 5.79, 6.32)
(8.16, 9.18, 10.2, 11.22, 12.24)	-	(3.81, 4.29, 4.77, 5.25, 5.73)

Now, based on pentagonal fuzzy data $\widetilde{X}_i$, $i = 1, 2, \ldots, n_1$ and $\widetilde{Y}_j$, $j = 1, 2, \ldots, n_2$, (Table 14) obtained from Table 12, the fuzzy means $\widetilde{\overline{X}}$ and $\widetilde{\overline{Y}}$ are found as

$$\widetilde{\widetilde{X}} = \frac{1}{n_1}\sum_{i=1}^{n_1}\widetilde{X}_i$$

$$= \frac{1}{25}(216.34,\ 243.4,\ 270.46,\ 297.52,\ 324.58)$$

$$= (8.6536,\ 9.7360,\ 10.8184,\ 11.9008,\ 12.9832)$$

$$\widetilde{\widetilde{Y}} = \frac{1}{n_2}\sum_{j=1}^{n_2}\widetilde{Y}_j$$

$$= \frac{1}{16}(63.14,\ 71.07,\ 79,\ 86.93,\ 94.86)$$

$$= (3.9463,\ 4.4419,\ 4.9375,\ 5.4331,\ 5.9288)$$

The fuzzy point estimation is

$$\tilde{\tau}^* = \widetilde{\widetilde{X}} - \widetilde{\widetilde{Y}}$$

$$= \frac{1}{n_1}\sum_{i=1}^{n_1}\widetilde{X}_i - \frac{1}{n_2}\sum_{j=1}^{n_2}\widetilde{Y}_j$$

$$= (8.6536,\ 9.7360,\ 10.8184,\ 11.9008,\ 12.9832) - (3.9463,\ 4.4419,\ 4.9375,\ 5.4331,\ 5.9288)$$

$$= (2.7248,\ 4.3029,\ 5.8809,\ 7.4589,\ 9.0369)$$

Thus the fuzzy test statistic is

$$\widetilde{Z} = \frac{\widetilde{\Omega}^* - \widetilde{H}_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$= \frac{(2.7248,\ 4.3029,\ 5.8809,\ 7.4589,\ 9.0369) - (4.6,\ 4.8,\ 5,\ 5.2,\ 5.4)}{\sqrt{\frac{9}{25} + \frac{4}{16}}}$$

$$= \frac{(-2.6752,\ -0.8971,\ 0.8809,\ 2.6589,\ 4.4369)}{0.7810}$$

$$= (-3.4254,\ -1.1487,\ 1.1279,\ 3.4045,\ 5.6811)$$

Using the proposed centroid in equation (3), $x_{\widetilde{Z}} = 1.1279$.
To find a Fuzzy Test Statistic based on a Heptagonal Fuzzy Number.
Let $\widetilde{H}_0 = (4.4, 4.6, 4.8, 5, 5.2, 5.4, 5.6)$ $\widetilde{H}_{1L} = (4.45, 4.65, 4.85, 5)$.

Table 15. The heptagonal fuzzy data from normal population

$\widetilde{X}_i^H = (x_i - 3\delta_i, x_i - 2\delta_i, x_i - \delta_i, x_i, x_i + \delta_i, x_i + 2\delta_i, x_i + 3\delta_i)_H$	$\widetilde{Y}_i^H = (y_i - 3\eta_i, y_i - 2\eta_i, y_i - \eta_i, y_i, y_i + \eta_i, y_i + 2\eta_i, y_i + 3\eta_i)_H$
(7.38, 8.42, 9.46, 10.5, 11.54, 12.58, 13.62)	(4.43, 5.06, 5.69, 6.32, 6.95, 7.58, 8.21)
(4.54, 5.19, 5.84, 6.49, 7.14, 7.79, 8.44)	(1.78, 2.04, 2.3, 2.56, 2.82, 3.08, 3.34)
(8.75, 10, 11.25, 12.5, 13.75, 15, 16.25)	(3.32, 3.8, 4.28, 4.76, 5.24, 5.72, 6.2)
(4.36, 4.99, 5.62, 6.25, 6.88, 7.51, 8.14)	(2.96, 3.39, 3.82, 4.25, 4.68, 5.11, 5.54)
(6.88, 7.86, 8.84, 9.82, 10.8, 11.78, 12.76)	(4.38, 5.01, 5.64, 6.27, 6.9, 7.53, 8.16)
(5.49, 6.27, 7.05, 7.83, 8.61, 9.39, 10.17)	(2.26, 2.59, 2.92, 3.25, 3.58, 3.91, 4.24)
(8.4, 9.6, 10.8, 12, 13.2, 14.4, 15.6)	(4.34, 4.96, 5.58, 6.2, 6.82, 7.44, 8.06)
(8.75, 10, 11.25, 12.5, 13.75, 15, 16.25)	(3.71, 4.23, 4.75, 5.27, 5.79, 6.31, 6.83)
(8.25, 9.43, 10.61, 11.79, 12.97, 14.15, 15.33)	(3.43, 3.92, 4.41, 4.9, 5.39, 5.88, 6.37)
(8.71, 9.95, 11.19, 12.43, 13.67, 14.91, 16.15)	(3.68, 4.21, 4.74, 5.27, 5.8, 6.33, 6.86)
(9.6, 10.97, 12.34, 13.71, 15.08, 16.45, 17.82)	(3, 3.43, 3.86, 4.29, 4.72, 5.15, 5.58)
(8.93, 10.21, 11.49, 12.77, 14.05, 15.33, 16.61)	(3.68, 4.2, 4.72, 5.24, 5.76, 6.28, 6.8)
(7.69, 8.79, 9.89, 10.99, 12.09, 13.19, 14.29)	(2.3, 2.63, 2.96, 3.29, 3.62, 3.95, 4.28)
(5.76, 6.59, 7.42, 8.25, 9.08, 9.91, 10.74)	(4.97, 5.68, 6.39, 7.1, 7.81, 8.52, 9.23)
(7.37, 8.42, 9.47, 10.52, 11.57, 12.62, 13.67)	(3.67, 4.2, 4.73, 5.26, 5.79, 6.32, 6.85)
(7.14, 8.16, 9.18, 10.2, 11.22, 12.24, 13.26)	(3.33, 3.81, 4.29, 4.77, 5.25, 5.73, 6.21)
(11.34, 12.96, 14.58, 16.2, 17.82, 19.44, 21.06)	-
(6.67, 7.63, 8.59, 9.55, 10.51, 11.47, 12.43)	-
(7.94, 9.07, 10.2, 11.33, 12.46, 13.59, 14.72)	-
(7.64, 8.73, 9.82, 10.91, 12, 13.09, 14.18)	-
(8.56, 9.78, 11, 12.22, 13.44, 14.66, 15.88)	-
(7.46, 8.53, 9.6, 10.67, 11.74, 12.81, 13.88)	-
(8.72, 9.97, 11.22, 12.47, 13.72, 14.97, 16.22)	-
(6.99, 7.99, 8.99, 9.99, 10.99, 11.99, 12.99)	-
(5.99, 6.85, 7.71, 8.57, 9.43, 10.29, 11.15)	-

Now, based on heptagonal fuzzy data $\widetilde{X}_i$, $i = 1, 2, \ldots, n_1$ and $\widetilde{Y}_j$, $j = 1, 2, \ldots, n_2$, (Table 15) obtained from Table 12, the fuzzy means $\widetilde{\overline{X}}$ and $\widetilde{\overline{Y}}$ are found as

$$\widetilde{\overline{X}} = \frac{1}{n_1}\sum_{i=1}^{n_1}\widetilde{X}_i$$

$$= \frac{1}{25}(189.28,\ 216.34,\ 243.4,\ 270.46,\ 297.52,\ 324.58,\ 351.64)$$

$$= (7.5712,\ 8.6536,\ 9.7360,\ 10.8184,\ 11.9008,\ 12.9832,\ 14.0656)$$

$$\widetilde{\widetilde{Y}} = \frac{1}{n_2}\sum_{j=1}^{n_2}\widetilde{Y}_j$$

$$= \frac{1}{16}(55.21,\ 63.14,\ 71.07,\ 79,\ 86.93,\ 94.86,\ 102.79)$$

$$= (3.4506,\ 3.9463,\ 4.4419,\ 4.9375,\ 5.4331,\ 5.9288,\ 6.4244)$$

The fuzzy point estimation is

$$\widetilde{\Omega}^* = \widetilde{\widetilde{X}} - \widetilde{\widetilde{Y}}$$

$$= \frac{1}{n_1}\sum_{i=1}^{n_1}\widetilde{X}_i - \frac{1}{n_2}\sum_{j=1}^{n_2}\widetilde{Y}_j$$

$$= (7.5712,\ 8.6536,\ 9.7360,\ 10.8184,\ 11.9008,\ 12.9832,\ 14.0656)$$

$$-(3.4506,\ 3.9463,\ 4.4419,\ 4.9375,\ 5.4331,\ 5.9288,\ 6.4244)$$

$$= (1.1468,\ 2.7248,\ 4.3029,\ 5.8809,\ 7.4589,\ 9.0369,\ 10.6150)$$

Thus the fuzzy test statistic is

$$\widetilde{Z} = \frac{\widetilde{\Omega}^* - \widetilde{H}_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$= \frac{(1.1468,\ 2.7248,\ 4.3029,\ 5.8809,\ 7.4589,\ 9.0369,\ 10.6150) - (4.4,\ 4.6,\ 4.8,\ 5,\ 5.2,\ 5.4,\ 5.6)}{\sqrt{\frac{9}{25} + \frac{4}{16}}}$$

$$= \frac{(-4.4532,\ -2.6752,\ -0.8971,\ 0.8809,\ 2.6589,\ 4.4369,\ 6.2150)}{0.7810}$$

$$= (-5.7019,\ -3.4254,\ -1.1487,\ 1.1279,\ 3.4045,\ 5.6811,\ 7.9577)$$

Using the proposed centroid in equation (4), $x_{\widetilde{Z}} = 1.0998$.
Hence, based on the new distance function, we get

$$D(\widetilde{Z}, Z_{1-\varepsilon}) = \min\{(x_{\widetilde{Z}} - Z_{1-\varepsilon})_T,\ (x_{\widetilde{Z}} - Z_{1-\varepsilon})_P,\ (x_{\widetilde{Z}} - Z_{1-\varepsilon})_H\}$$

$$D(\widetilde{Z}, Z_{1-\varepsilon}) = \min\{(1.2122 - 1.6449)_T,\ (1.1279 - 1.6449)_P,\ (1.0998 - 1.6449)_H\}$$

$$D(\widetilde{Z}, Z_{1-\varepsilon}) = \min\{-0.4327,\ -0.5170,\ -0.5451\} = -0.5451 < 0$$

Since $D\left(\widetilde{Z}, Z_{1-\varepsilon}\right) = -0.5451 < 0$. Hence, based on the decision rule 4.3, H_0 is accepted.

B) Now suppose that we want to test the following crisp hypotheses (which are equivalent to the case $a_1 = \Omega_0 = a_3$ in the above fuzzy hypotheses):

$$\begin{cases} H_0 : \Omega = 5, \\ H_1 : \Omega > 5. \end{cases}$$

To find a Fuzzy Test Statistic based on Triangular Fuzzy Number.

The fuzzy test statistic is

$$\widetilde{Z} = \frac{\widetilde{\Omega}^* - \Omega_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$= \frac{(4.3029,\ 5.8809,\ 7.4589) - 5}{\sqrt{\frac{9}{25} + \frac{4}{16}}}$$

$$= \frac{(4.3029,\ 5.8809,\ 7.4589) - (5,\ 5,\ 5)}{\sqrt{\frac{9}{25} + \frac{4}{16}}}$$

$$= \frac{(-0.6971,\ 0.8809,\ 2.4589)}{0.7810}$$

$$= (-0.8926,\ 1.1279,\ 3.1484)$$

Using the proposed centroid in equation (2), $x_{\widetilde{Z}} = 1.2027$.

To find Fuzzy Test Statistic based on the Pentagonal Fuzzy Number.

The fuzzy test statistic is

$$\widetilde{Z} = \frac{\widetilde{\Omega}^* - \Omega_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$= \frac{(2.7248,\ 4.3029,\ 5.8809,\ 7.4589,\ 9.0369) - 5}{\sqrt{\frac{9}{25} + \frac{4}{16}}}$$

$$= \frac{(2.7248,\ 4.3029,\ 5.8809,\ 7.4589,\ 9.0369) - (5,\ 5,\ 5,\ 5,\ 5)}{\sqrt{\frac{9}{25} + \frac{4}{16}}}$$

$$= \frac{(-2.2752,\ -0.6971,\ 0.8809,\ 2.4589,\ 4.0369)}{0.7810}$$

$$= (-2.9132,\ -0.8926,\ 1.1279,\ 3.1484,\ 5.1689)$$

Using the proposed centroid in equation (3), $x_{\widetilde{Z}} = 1.1279$.
To find a Fuzzy Test Statistic based on a Heptagonal Fuzzy Number.
The fuzzy test statistic is

$$\widetilde{Z} = \frac{\widetilde{\Omega}^* - \Omega_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$= \frac{(1.1468,\ 2.7248,\ 4.3029,\ 5.8809,\ 7.4589,\ 9.0369,\ 10.6150) - 5}{\sqrt{\frac{9}{25} + \frac{4}{16}}}$$

$$= \frac{(1.1468,\ 2.7248,\ 4.3029,\ 5.8809,\ 7.4589,\ 9.0369,\ 10.6150) - (5,\ 5,\ 5,\ 5,\ 5,\ 5,\ 5)}{\sqrt{\frac{9}{25} + \frac{4}{16}}}$$

$$= \frac{(-3.8532,\ -2.2752,\ -0.6971,\ 0.8809,\ 2.4589,\ 4.0369,\ 5.6150)}{0.7810}$$

$$= (-4.9337,\ -2.9132,\ -0.8926,\ 1.1279,\ 3.1484,\ 5.1689,\ 7.1895)$$

Using the proposed centroid in equation (4), $x_{\widetilde{Z}} = 1.1029$.
Hence, based on the new distance function, we get

$$D(\widetilde{Z}, Z_{1-\varepsilon}) = \min\{(x_{\widetilde{Z}} - Z_{1-\varepsilon})_T,\ (x_{\widetilde{Z}} - Z_{1-\varepsilon})_P,\ (x_{\widetilde{Z}} - Z_{1-\varepsilon})_H\}$$

$$D(\widetilde{Z}, Z_{1-\varepsilon}) = \min\{(1.2027 - 1.6449)_T,\ (1.1279 - 1.6449)_P,\ (1.1029 - 1.6449)_H\}$$

$$D(\widetilde{Z}, Z_{1-\varepsilon}) = \min\{-0.4422,\ -0.5170,\ -0.5420\} = -0.5420 < 0$$

Since $D(\widetilde{Z}, Z_{1-\varepsilon}) = -0.5420 < 0$. Hence, based on the decision rule 4.3, H_0 is accepted.

The Table 16 shows the Comparison of the proposed with the method introduced by Arefi [29] to the fuzzy hypothesis for difference between means.

Table 16. Comparative analysisof testing fuzzy hypothesis for difference between means

	Proposed Method		Arefi [29]
	General fuzzy data	Triangular fuzzy data	Triangular fuzzy data
Fuzzy Hypothesis			
One sided hypothesis	$D(\widetilde{Z}, Z_{1-\varepsilon}) = -0.5451 < 0$	$D(\widetilde{Z}, Z_{1-\varepsilon}) = -0.4327 < 0$	$D(\widetilde{Z}, Z_{1-\varepsilon}) = -1.2759 < 0$
Crisp Hypothesis			
One sided hypothesis	$D(\widetilde{Z}, Z_{1-\varepsilon}) = -0.5420 < 0$	$D(\widetilde{Z}, Z_{1-\varepsilon}) = -0.4422 < 0$	$D(\widetilde{Z}, Z_{1-\varepsilon}) = -1.1905 < 0$

6. Conclusion

In testing the hypothesis under a fuzzy environment, the data are collected in the form of some specific shape of fuzzy numbers. Moreover, the methodologies introduced by the researchers for testing statistical or fuzzy hypotheses using fuzzy data are applied only to some specific shapes of membership functions of fuzzy numbers at a time. To overcome this problem, we consider the symmetric fuzzy environment to introduce the methodology for testing statistical and fuzzy hypotheses using fuzzy data with various shapes of membership functions in this proposed approach. The proposed method is especially suitable for testing statistical and fuzzy hypotheses for mean and variance when the data are collected in the form of fuzzy numbers with various shapes of membership functions. The main advantage of this proposed method is that it is a method for natural observation of data in normal approach. Finally, some numerical examples have been given to illustrate the suggested method. The intended shows that the testing of hypothesis under fuzzy environment will be effective when we collect the fuzzy data with many shapes of membership functions from a symmetric fuzzy environment. The same idea may be extended to other types of fuzzy numbers with suitable modification of the methodology involved.

Conflict of interests

The authors declare no competing financial interest.

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