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# Optical Solitons for the Dispersive Concatenation Model

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**Abstract:** The study undertakes a comprehensive exploration of optical solitons within the context of the dispersive concatenation model, utilizing three distinct integration algorithms. These approaches, namely the enhanced Kudryashov's method, the Riccati equation expansion approach, and the Weierstrass' expansion scheme, offer distinct perspectives and insights into the behavior of optical solitons. By employing the enhanced Kudryashov's approach, the research uncovers a spectrum of soliton solutions, including straddled, bright, and singular optical solitons. This algorithm not only provides a nuanced understanding of the various soliton types but also highlights the occurrence of singular solitons that exhibit unique characteristics. The Riccati equation expansion approach, on the other hand, yields dark solitons in addition to singular solitons. This particular method expands our comprehension of soliton behavior by encompassing the presence of dark solitons alongside singular ones. This diversification contributes to a more comprehensive grasp of soliton phenomena. Furthermore, the application of the Weierstrass' expansion scheme extends the analysis to encompass bright, singular, and other variations of straddled solitons. This method introduces further complexity and diversity to the optical soliton. Importantly, the study meticulously addresses the parameter constraints that govern the behavior of these solitons. By providing a comprehensive presentation of these constraints, the research enhances the practical applicability of the findings, offering insights into the conditions under which these soliton solutions emerge.

***Keywords*:** solitons, concatenation, Riccati, Weierstrass, Kudryashov

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## 1. Introduction

The mathematical engineering of optical solitons has engulfed the field of nonlinear optics in full capacity. There are several mathematical approaches that yield soliton solutions, extract conservation laws followed by laying down the quasi–monochromatic dynamics of such solitons [1-20]. The hunt for these serendipity results in this field of research is always priceless. One such issue is the establishment of a couple of new models to govern the transmission of solitons across trans-continental and trans-oceanic distances is undoubtedly a mathematical marvel.

During 2014, a concatenation model was proposed that is the conjunction of three well known equations and they are the Nonlinear Schrödinger's Equation (NLSE), Lakshmanan-Porsezian-Daniel (LPD) model and finally the Sasa-Satsuma Equation (SSE) [1, 2]. Subsequently during 2015, another form of the concatenation model was proposed that included the higher-order dispersive effects. This time the proposed model was the conjunction of NLSE, the Schrödinger-Hirota Equation (SHE), LPD and the quintic-order NLSE [5-7, 15]. Its soliton solutions have been recovered and their preliminary analysis have been carried out. The current paper moves further along. The retrieval of a full spectrum of optical solitons is achieved in this paper using three integration approaches. They are the enhanced Kudryashov's approach, the Riccati equation approach and finally with the usage of the Weierstrass' function approach. These approaches yield solitons and periodic solutions to the model. However, the periodic solutions are not listed since the focus of the paper stays confined to the recovery optical solitons. The details are meticulously crafted in the rest of the work.

Our research brings forth several distinctive and novel aspects that differentiate it from existing studies in the field. Unlike prior works that have primarily focused on soliton solutions and conservation laws in nonlinear optics, our study delves into the mathematical engineering of optical solitons, presenting a unique perspective on the subject. A significant departure lies in our aim to establish new models governing the transmission of solitons across vast distances, from trans-continental to trans-oceanic ranges. This ambitious endeavor involves merging mathematical principles with physical phenomena, resulting in the creation of innovative models that expand the boundaries of current research. Furthermore, while earlier investigations have laid the groundwork by introducing concatenation models combining established equations such as the NLSE, LPD model, and SSE, our contribution advances the field. We've extended these models to incorporate higher-order dispersive effects, introducing equations like the SHE and quintic-order NLSE. This expansion enriches the range of phenomena our models can describe. The core novelty of our current study lies in the comprehensive recovery of an array of optical solitons using three distinct integration approaches. By employing the enhanced Kudryashov's approach, the Riccati equation approach, and the Weierstrass' function approach, we secure soliton solutions within our model. Crucially, our focus remains on the recovery of optical solitons, distinguishing our work from others.

### 1.1 *Governing model*

The dispersive concatenation model containe four well-known nonlinear models embedded in it. They are the NLSE, SHE, LPD and the fifth-order NLSE. This is written for the first time as:

$$
\begin{aligned}
& iq_t + aq_{xx} + b|q|^2 q - i\delta_1\left(\sigma_1 q_{xxx} + \sigma_2 |q|^2 q_x\right) \\
& + \delta_2\left[\sigma_3 q_{xxxx} + \sigma_4 |q|^2 q_{xx} + \sigma_5 |q|^4 q + \sigma_6 |q_x|^2 q + \sigma_7 (q_x)^2 q^* + \sigma_8 q^*_{xx} q^2\right] \\
& - i\delta_3\left[\sigma_9 q_{xxxxx} + \sigma_{10}|q|^2 q_{xxx} + \sigma_{11}|q|^4 q_x + \sigma_{12} q q_x q^*_{xx} + \sigma_{13} q^* q_x q_{xx} + \sigma_{14} q q^*_x q_{xx} + \sigma_{15}(q_x)^2 q^*_x\right] = 0 \qquad (1)
\end{aligned}
$$

where $q(x, t)$ is a complex valued function that represents the wave profile and $q^*(x, t)$ is its complex-conjugate while $i = \sqrt{-1}$. The first term represents the linear temporal evolution. The constants $a$ and $b$ are the coefficients of the Chromatic Dispersion (CD) and Self-Phase Modulation (SPM) respectively. The parameters $\sigma_j$, ($j = 1 - 15$) are all real-valued constants. When $\delta_1 = \delta_2 = \delta_3 = 0$, Eq. (1) reduces to the standard NLSE. If, however, $\delta_1 \neq 0$ and $\delta_2 = \delta_3 = 0$, Eq. (1) reduced to SHE. For $\delta_1 = \delta_3 = 0$ with $\delta_2 \neq 0$, Eq. (1) yields the LPD equation. Finally, when $\delta_1 = \delta_2 = 0$ but $\delta_3 \neq 0$, Eq. (1) reduced to quintic-order NLSE. Eq. (1) is thus a true concatenation of the well-known models that describe the soliton transmissin dynamics across trans-continental and trans-oceanic dynamics.

The equation provided is situated within the context of nonlinear optics, a branch of optics that deals with optical phenomena in materials where the response of the medium to light is nonlinear. In this context, nonlinearities arise when the refractive index of a material is no longer directly proportional to the electric field of the incident light. The equation appears to be a generalized model that combines several well-known equations from nonlinear optics. These equations often emerge as simplified descriptions o f specific nonlinear effects in various optical systems. They capture phenomena like self-focusing, self-phase modulation, and soliton propagation, among others. Nonlinear optical systems are of great interest due to their capacity to produce intriguing phenomena and applications. Solitons, for instance, are localized waveforms that maintain their shape and speed while propagating, a behavior that arises from a balance between nonlinearity and dispersion. Such solitons are used for data transmission in optical fiber communication systems. Considering the dispersive concatenation model's inclusion of equations representing various nonlinear behaviors, the equation contributes to the understanding of complex interactions in nonlinear optical systems. It offers insights into how different nonlinear effects combine and influence light propagation in intricate ways, leading to behaviors like soliton formation, pulse compression, and modulation instability. As a result, the equation's placement within the field of nonlinear optics suggests that it aims to provide a unified framework to explore the combined effects of different nonlinear phenomena on light propagation. This leads to a deeper understanding of optical behavior in materials where nonlinearity plays a significant role.

## 2. Mathematical analysis

In order to address the problem posed by Equation (1), we make use of the following amplitude-phase split-up for the wave profile:

$$q(x,t) = \phi(\xi)\exp[i\psi(x,t)],\ \psi(x,t) = -\kappa x + wt + \theta_0,\ \xi = x - Vt, \tag{2}$$

where the soliton frequency is denoted by $w$, the wave number by $\kappa$, the phase constant by $\theta_0$, and the velocity of the soliton by $V$. Additionally, $\phi$ represents a real-valued function that characterizes the amplitude of the wave. By substituting equation (2) into equation (1) and then separating the real and imaginary components, we derive the expression for the real part as follows:

$$\begin{aligned}
&(\delta_2\sigma_3 - 5\delta_3\kappa\sigma_9)\phi^{(4)}(\xi) + (10\delta_3\kappa^3\sigma_9 - 6\delta_2\kappa^2\sigma_3 - 3\delta_1\kappa\sigma_1 + a)\phi''(\xi) + (\delta_1\kappa^3\sigma_1 + \delta_2\kappa^4\sigma_3 \\
&-\delta_3\kappa^5\sigma_9 - a\kappa^2 - w)\phi(\xi) + (2\delta_3\kappa\sigma_{12} - 2\delta_3\kappa\sigma_{13} - 2\delta_3\kappa\sigma_{14} - \delta_3\kappa\sigma_{15} + \delta_2\sigma_6 + \delta_2\sigma_7)\phi(\xi)\phi'^2(\xi) \\
&+(-3\delta_3\kappa\sigma_{10} - \delta_3\kappa\sigma_{12} - \delta_3\kappa\sigma_{13} + \delta_3\kappa\sigma_{14} + \delta_2\sigma_4 + \delta_2\sigma_8)\phi^2(\xi)\phi''(\xi) + (\delta_2\sigma_5 - \delta_3\kappa\sigma_{11})\phi^5(\xi) \\
&+(\delta_3\kappa^3\sigma_{10} + \delta_3\kappa^3\sigma_{12} + \delta_3\kappa^3\sigma_{13} - \delta_3\kappa^3\sigma_{14} - \delta_3\kappa^3\sigma_{15} - \delta_1\kappa\sigma_2 - \delta_2\kappa^2\sigma_4 + \delta_2\kappa^2\sigma_6 \\
&-\delta_2\kappa^2\sigma_7 - \delta_2\kappa^2\sigma_8 + b)\phi^3(\xi) = 0,
\end{aligned} \tag{3}$$

while the imaginary part is given by:

$$-\delta_3\sigma_9\phi^{(5)}(\xi) + (10\delta_3\kappa^2\sigma_9 - 4\delta_2\kappa\sigma_3 - \delta_1\sigma_1)\phi'''(\xi) + (-5\delta_3\kappa^4\sigma_9 + 4\delta_2\kappa^3\sigma_3 + 3\delta_1\kappa^2\sigma_1 - 2a\kappa - V)\phi'(\xi)$$

$$-\delta_3\sigma_{15}\phi'^3(\xi)-(\delta_3\sigma_{12}+\delta_3\sigma_{13}+\delta_3\sigma_{14})\phi(\xi)\phi'(\xi)\phi''(\xi)-\delta_3\sigma_{10}\phi^2(\xi)\phi'''(\xi)$$

$$+(3\delta_3\kappa^2\sigma_{10}-\delta_3\kappa^2\sigma_{12}+3\delta_3\kappa^2\sigma_{13}-\delta_3\kappa^2\sigma_{14}-\delta_3\kappa^2\sigma_{15}-2\delta_2\kappa\sigma_4-2\delta_2\kappa\sigma_7$$

$$+2\delta_2\kappa\sigma_8-\delta_1\sigma_2)\phi^2(\xi)\phi'(\xi)-\delta_3\sigma_{11}\phi^4(\xi)\phi'(\xi)=0. \tag{4}$$

Applying the derivative with respect to $\xi$ to the real part (3) results in:

$$(\delta_2\sigma_3-5\delta_3\kappa\sigma_9)\phi^{(5)}(\xi)+(10\delta_3\kappa^3\sigma_9-6\delta_2\kappa^2\sigma_3-3\delta_1\kappa\sigma_1+a)\phi'''(\xi)$$

$$+(\delta_1\kappa^3\sigma_1+\delta_2\kappa^4\sigma_3-\delta_3\kappa^5\sigma_9-a\kappa^2-w)\phi'(\xi)$$

$$+(2\delta_3\kappa\sigma_{12}-2\delta_3\kappa\sigma_{13}-2\delta_3\kappa\sigma_{14}-\delta_3\kappa\sigma_{15}+\delta_2\sigma_6+\delta_2\sigma_7)\phi'^3(\xi)+(2\delta_3\kappa\sigma_{12}-6\delta_3\kappa\sigma_{13}$$

$$-2\delta_3\kappa\sigma_{14}-2\delta_3\kappa\sigma_{15}+2\delta_2\sigma_6+2\delta_2\sigma_7-6\delta_3\kappa\sigma_{10}+2\delta_2\sigma_4+2\delta_2\sigma_8)\phi(\xi)\phi'(\xi)\phi''(\xi)$$

$$+(-3\delta_3\kappa\sigma_{10}-\delta_3\kappa\sigma_{12}-\delta_3\kappa\sigma_{13}+\delta_3\kappa\sigma_{14}+\delta_2\sigma_4+\delta_2\sigma_8)\phi^2(\xi)\phi'''(\xi)+5(\delta_2\sigma_5-\delta_3\kappa\sigma_{11})\phi^4(\xi)\phi'(\xi)$$

$$+3(\delta_3\kappa^3\sigma_{10}+\delta_3\kappa^3\sigma_{12}+\delta_3\kappa^3\sigma_{13}-\delta_3\kappa^3\sigma_{14}-\delta_3\kappa^3\sigma_{15}-\delta_1\kappa\sigma_2-\delta_2\kappa^2\sigma_4$$

$$+\delta_2\kappa^2\sigma_6-\delta_2\kappa^2\sigma_7-\delta_2\kappa^2\sigma_8+b)\phi^2(\xi)\phi'(\xi)=0. \tag{5}$$

Equations (4) and (5) are considered equivalent when the following conditions are fulfilled:

$$\delta_2\sigma_3+\delta_3\sigma_9(1-5\kappa)=0,$$

$$10\delta_3\kappa^2\sigma_9(\kappa-1)+a+2\sigma_3\delta_2\kappa(2-3\kappa)+\delta_1\sigma_1(1-3\kappa)=0,$$

$$\delta_1\kappa^2\sigma_1(\kappa-3)+\delta_2\sigma_3\kappa^3(\kappa-4)-\delta_3\sigma_9\kappa^4(\kappa-5)-a\kappa^2-w+2a\kappa+V=0,$$

$$2\delta_3\kappa\sigma_{12}-2\delta_3\kappa\sigma_{13}-2\delta_3\kappa\sigma_{14}+\delta_3\sigma_{15}(1-\kappa)+\delta_2\sigma_6+\delta_2\sigma_7=0,$$

$$\delta_3\sigma_{12}(2\kappa+1)+\delta_3\sigma_{13}(1-6\kappa)+\delta_3\sigma_{14}(1-2\kappa)-2\delta_3\kappa\sigma_{15}+2\delta_2\sigma_6$$

$$+2\delta_2\sigma_7-6\delta_3\kappa\sigma_{10}+2\delta_2\sigma_4+2\delta_2\sigma_8=0,$$

$$\delta_3\sigma_{10}(1-3\kappa)-\delta_3\kappa\sigma_{12}-\delta_3\kappa\sigma_{13}+\delta_3\kappa\sigma_{14}+\delta_2\sigma_4+\delta_2\sigma_8=0,$$

$$5\delta_2\sigma_5+\delta_3\sigma_{11}(1-5\kappa)=0,$$

$$3\delta_3\kappa^2\sigma_{10}(\kappa-1)+\delta_3\kappa^2\sigma_{12}(1-3\kappa)+3\delta_3\sigma_{13}\kappa^2(\kappa-1)+\delta_3\kappa^2\sigma_{14}(1-3\kappa)$$

$$+\delta_3\kappa^2\sigma_{15}(1-3\kappa)+\delta_1\sigma_2(1-3\kappa)+\delta_2\kappa\sigma_4(2-3\kappa)+3\delta_2\kappa^2\sigma_6+\delta_2\kappa\sigma_7(2-3\kappa)$$

$$-\delta_2\kappa\sigma_8(3\kappa+2)+3b=0. \tag{6}$$

Equation (4) can now be rewritten in the following form:

$$\phi^{(5)}(\xi)+\Delta_1\phi'''(\xi)+\Delta_2\phi'(\xi)+\Delta_3\phi'^3(\xi)+\Delta_4\phi(\xi)\phi'(\xi)\phi''(\xi)+\Delta_5\phi^2(\xi)\phi'''(\xi)$$

$$+\Delta_6\phi^2(\xi)\phi'(\xi)+\Delta_7\phi^4(\xi)\phi'(\xi)=0, \tag{7}$$

where

$$\Delta_1=\frac{-1}{\delta_3\sigma_9}(10\delta_3\kappa^2\sigma_9-4\delta_2\kappa\sigma_3-\delta_1\sigma_1),$$

$$\Delta_2=\frac{-1}{\delta_3\sigma_9}(-5\delta_3\kappa^4\sigma_9+4\delta_2\kappa^3\sigma_3+3\delta_1\kappa^2\sigma_1-2a\kappa-V),$$

$$\Delta_3=\frac{\sigma_{15}}{\sigma_9},$$

$$\Delta_4=\frac{1}{\sigma_9}(\sigma_{12}+\sigma_{13}+\sigma_{14}),$$

$$\Delta_5=\frac{\sigma_{10}}{\sigma_9},$$

$$\Delta_6=\frac{-1}{\delta_3\sigma_9}(3\delta_3\kappa^2\sigma_{10}-\delta_3\kappa^2\sigma_{12}+3\delta_3\kappa^2\sigma_{13}-\delta_3\kappa^2\sigma_{14}-\delta_3\kappa^2\sigma_{15}-2\delta_2\kappa\sigma_4-2\delta_2\kappa\sigma_7+2\delta_2\kappa\sigma_8-\delta_1\sigma_2),$$

$$\Delta_7=\frac{\sigma_{11}}{\sigma_9}, \tag{8}$$

provided $\sigma_9$ and $\delta_3$ are both non-zero parameters. The solution to Equation (7) will be derived by utilizing the three methods outlined in the subsequent sections.

## 3. Enhanced Kudryashov's method

A new methodology, known as the updated version of Kudryashov's method, has been devised by Kudryashov, building upon the previous work by Zayed et al. In this section, we utilize this approach to integrate equation (7). We put forward the following solution form for equation (7):

$$\phi(\xi) = \sum_{s=0}^{N} L_l \left[ H(\xi) \right]^l, \tag{9}$$

where $L_l$ (with $l$ ranging from 0 to $N$) are real numbers that can be calculated at a later stage. It is crucial to note that $L_N$ is not equal to zero. Furthermore, the function $H(\xi)$ satisfies the given nonlinear Ordinary Differential Equation (ODE):

$$H'^2(\xi) = H^2(\xi)\left[1 - \zeta H^{2h}(\xi)\right] \ln^2 K, \ 0 < K \neq 1, \tag{10}$$

wherein the value of $\zeta$ can be any non-zero parameter. Equation (10) has the solution presented below:

$$H(\xi) = \left[ \frac{4A}{4A^2 \exp_K(h\xi) + \zeta \exp_K(-h\xi)} \right]^{\frac{1}{h}}, \tag{11}$$

whenever $\exp_K(h\xi) = K^{(h\xi)}$, $A$ represents a non-zero real number, and the value of $h$ is a positive integer.

In Eq. (7), by equating the power series of the largest derivative $\phi^{(5)}$ with the nonlinear term $\phi^4\phi'$, we obtain:

$$5N + h = N + 5h \Rightarrow N = h. \tag{12}$$

**Case 1** By setting $h$ equal to 1, the corresponding value of $N$ becomes 1, leading to the transformation of Equation (9) as follows:

$$\phi(\xi) = L_0 + L_1 H(\xi), \tag{13}$$

where $L_0$ and $L_1$ are parameter values that can be chosen without any restrictions, except for the requirement that $L_1$ should not equal zero.

The algebraic system formed by adding equations (13) and (10) to equation (7) and summing the coefficients of $[H(\xi)]^j[H'(\xi)]^g$, where $j$ ranges from 0 to 4 and $g$ ranges from 0 to 1, can be solved using Maple. The resulting solutions are displayed below:

$$L_0 = \frac{\epsilon}{2}\sqrt{-\frac{(\Delta_4 + 2\Delta_5)\ln^2 K + 2\Delta_6}{\Delta_7}}, \ L_1 = \frac{\epsilon \ln K}{2}\sqrt{\frac{2\zeta(\Delta_4 + 6\Delta_5)}{\Delta_7}}, \tag{14}$$

and

$$\Delta_1 = -\frac{1}{6\Delta_7}\left[(\Delta_4^2 + 7\Delta_4\Delta_5 + 12\Delta_5^2 - 60\Delta_7)\ln^2 K + \Delta_6(\Delta_4 + 3\Delta_5)\right],$$

$$\Delta_2 = \frac{1}{48\Delta_7}\left[(5\Delta_4^2 + 56\Delta_4\Delta_5 + 108\Delta_5^2 - 528\Delta_7)\ln^4 K + 8\Delta_6(\Delta_4 + 6\Delta_5)\ln^2 K + 12\Delta_6^2\right],$$

$$\Delta_3 = -\frac{3(\Delta_4^2 + 8\Delta_4\Delta_5 + 12\Delta_5^2 - 160\Delta_7)}{2(\Delta_4 + 6\Delta_5)} \tag{15}$$

provided

$$\Delta_7\left[\left(\Delta_4+2\Delta_5\right)\ln^2 K+2\Delta_6\right]<0,\ 2\zeta(\Delta_4+6\Delta_5)\Delta_7>0 \text{ and } \epsilon=\pm 1.$$

The straddled soliton solution for equation (1) can be determined by substituting equations (14) and (11) into equation (13) as demonstrated:

$$q(x,t)=\left[\frac{\epsilon}{2}\sqrt{-\frac{(\Delta_4+2\Delta_5)\ln^2 K+2\Delta_6}{\Delta_7}}+\frac{2\epsilon A\ln K\sqrt{\frac{2\zeta(\Delta_4+6\Delta_5)}{\Delta_7}}}{[4A^2\ \exp_K(x-Vt)+\zeta\exp_K(-(x-Vt))]}\right]\times$$

$$\exp i\left[-\kappa x+wt+\theta_0\right]. \tag{16}$$

The creation of the bright soliton solution for equation (1) involves the manipulation of equation (16) with a specific substitution. By setting $\zeta=4A^2$ for the particular case, the desired outcome can be achieved as outlined below:

$$q(x,t)=\left[\frac{\epsilon}{2}\sqrt{-\frac{(\Delta_4+2\Delta_5)\ln^2 K+2\Delta_6}{\Delta_7}}+\epsilon\ln K\sqrt{\frac{(\Delta_4+6\Delta_5)}{2\Delta_7}}\operatorname{sech}[\,(x-Vt)\ln K\,]\right]\times$$

$$\exp i\left[-\kappa x+wt+\theta_0\right], \tag{17}$$

provided

$$\Delta_7\left(\Delta_4+6\Delta_5\right)>0,$$

while the singular soliton solution to equations (1) can be created , if we change the value of $\zeta=-4A^2$ in equation (16) for the particular case as:

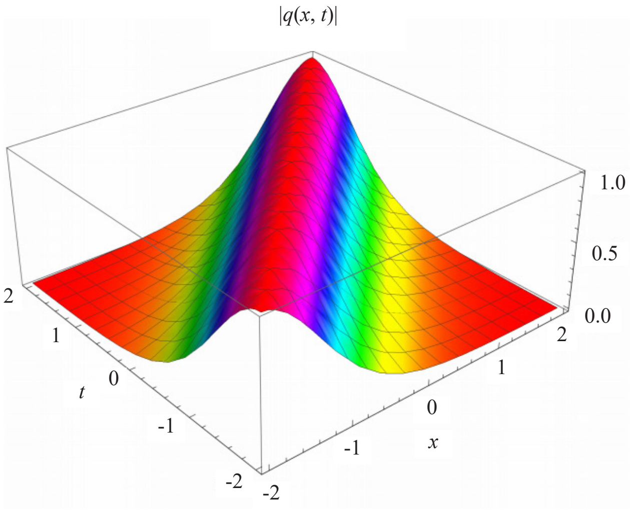
$$q(x,t)=\left[\frac{\epsilon}{2}\sqrt{-\frac{(\Delta_4+2\Delta_5)\ln^2 K+2\Delta_6}{\Delta_7}}+\epsilon\ln K\sqrt{\frac{-(\Delta_4+6\Delta_5)}{2\Delta_7}}\operatorname{csch}[\,(x-Vt)\ln K\,]\right]\times$$

$$\exp i\left[-\kappa x+wt+\theta_0\right], \tag{18}$$

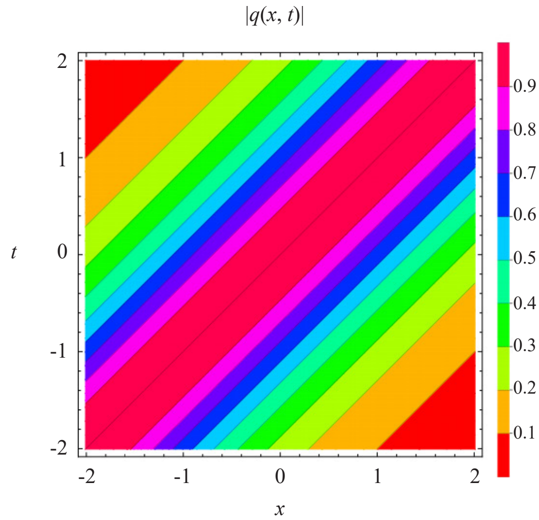
where

$$\Delta_7\left(\Delta_4+6\Delta_5\right)<0.$$

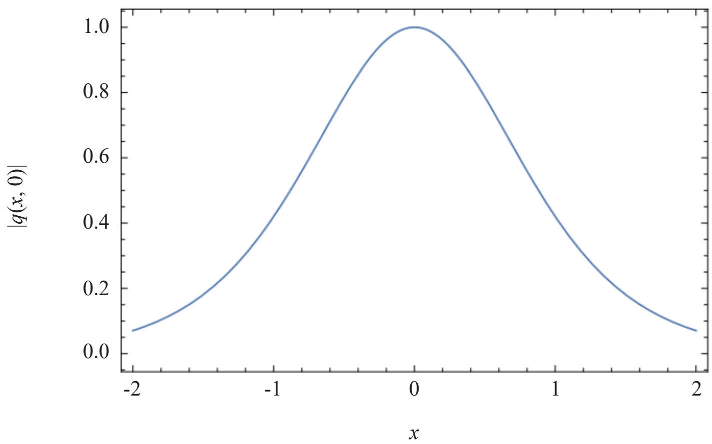
Within Figure 1, there are various plots illustrating the bright soliton solution (17) for the model equation (1). The parameter values employed are given as: $\kappa=1$, $V=1$, $\epsilon=1$, $K=e$, $\sigma_{11}=1$, $\sigma_{12}=1$, $\sigma_{13}=1$, $\sigma_{14}=1$, $\sigma_{15}=-1$, $\sigma_{10}=1$, $\delta_1=1$, $\delta_2=1$, $\delta_3=1$, $\sigma_2=1$, $\sigma_4=1$, $\sigma_7=1$, $\sigma_8=1$, and $\sigma_9=1$.




(a) Surface plot




(b) Contour plot




(c) 2D plot

**Figure 1.** Profile of a bright soliton solution (17)

**Case 2** Assuming $h$ is equal to 2, the value of $N$ is set to 2, leading to the following transformation of Equation (9):

$$\phi(\xi) = L_0 + L_1 H(\xi) + L_2 H^2(\xi), \tag{19}$$

where the parameters $L_0$, $L_1$, and $L_2$ are arbitrary and can assume any value. However, it is important to note that $L_2$ cannot be zero. By incorporating equations (19) and (10) into equation (7) and summing the coefficients of $[H(\xi)]^j[H'(\xi)]^g$, where $j$ varies from 0 to 9 and g varies from 0 to 1, we can solve an algebraic system using Maple. The results obtained are provided below:

$$L_0 = \frac{\epsilon}{2}\sqrt{-\frac{2[2(\Delta_4 + 2\Delta_5)\ln^2 K + \Delta_6]}{\Delta_7}}, \quad L_1 = 0, \quad L_2 = \epsilon \ln K\sqrt{\frac{2\zeta(\Delta_4 + 6\Delta_5)}{\Delta_7}}, \tag{20}$$

and

$$\Delta_1 = -\frac{1}{6\Delta_7}\left[(4\Delta_4^2 + 28\Delta_4\Delta_5 + 48\Delta_5^2 - 240\Delta_7)\ln^2 K + \Delta_6(\Delta_4 + 3\Delta_5)\right],$$

$$\Delta_2 = \frac{1}{12\Delta_7}\left[(20\Delta_4^2 + 224\Delta_4\Delta_5 + 432\Delta_5^2 - 2112\Delta_7)\ln^4 K + 8\Delta_6(\Delta_4 + 6\Delta_5)\ln^2 K + 3\Delta_6^2\right],$$

$$\Delta_3 = -\frac{3(\Delta_4^2 + 8\Delta_4\Delta_5 + 12\Delta_5^2 - 160\Delta_7)}{2(\Delta_4 + 6\Delta_5)}, \tag{21}$$

provided

$$\Delta_7\left[2\left(\Delta_4 + 2\Delta_5\right)\ln^2 K + \Delta_6\right] < 0,\ 2\zeta\left(\Delta_4 + 6\Delta_5\right)\Delta_7 > 0 \text{ and } \epsilon = \pm 1.$$

The straddled soliton solution to equation (1) is obtained by replacing (20) with (11) in equation (19):

$$q(x,t) = \left[\frac{\epsilon}{2}\sqrt{-\frac{2[2(\Delta_4 + 2\Delta_5)\ln^2 K + \Delta_6]}{\Delta_7}} + \frac{4\epsilon A \ln K\sqrt{\dfrac{2\zeta(\Delta_4 + 6\Delta_5)}{\Delta_7}}}{(4A^2\ \exp_K[2(x - Vt)] + \zeta\ \exp_K[-2(x - Vt)])}\right] \times$$

$$\exp i\left[-\kappa x + wt + \theta_0\right]. \tag{22}$$

The bright soliton solution to equations (1) can be created if we set $\zeta = 4A^2$ in equation (22) for the particular case as

$$q(x,t) = \left[\frac{\epsilon}{2}\sqrt{-\frac{2[2(\Delta_4 + 2\Delta_5)\ln^2 K + \Delta_6]}{\Delta_7}} + \epsilon \ln K\sqrt{\frac{2(\Delta_4 + 6\Delta_5)}{\Delta_7}}\operatorname{sech}[\ 2(x - Vt)\ln K\ ]\right] \times$$

$$\exp i\left[-\kappa x + wt + \theta_0\right], \tag{23}$$

provided

$$\Delta_7(\Delta_4 + 6\Delta_5) > 0,$$

while the singular soliton solution to equation (1) can be created , if we assign $\zeta = -4A^2$ in equation (22) for the particular case:

$$q(x,t) = \left[\frac{\epsilon}{2}\sqrt{-\frac{2[2(\Delta_4 + 2\Delta_5)\ln^2 K + \Delta_6]}{\Delta_7}} + \epsilon \ln K\sqrt{-\frac{2(\Delta_4 + 6\Delta_5)}{\Delta_7}}\operatorname{csch}[\ 2(x - Vt)\ln K\ ]\right] \times$$

$$\exp i\left[-\kappa x + wt + \theta_0\right], \tag{24}$$

provided

$$\Delta_7\left(\Delta_4+6\Delta_5\right)<0.$$

By adjusting the parameters $h$ and $N$, it is possible to produce a similar number of solitary wave solutions to equation (1).

## 4. Unified riccati's equation expansion approach

The analysis of the soliton solution to Equation (1) in this section involves the utilization of the UREE technique [14]. This technique relies on the presumptive form of the soliton solutions of Equation (7), which can be expressed as:

$$\phi(\xi)=\sum_{i=0}^{M}R_i\left[Y(\xi)\right]^i. \tag{25}$$

For $R_l$, where $l$ takes on values from 0 to $1-N$, which are real numbers, and $Y(\xi)$, the solution to the subsequent set of Riccati's ODEs is obtained as follows:

$$Y'(\xi)=h_0+h_1Y+h_2Y^2. \tag{26}$$

With $h_l$ being constants for $l$ taking values of 0, 1, and 2, Equation (25) has a formal solution that is determined by the concept of a balance number, given by:

$$\phi(\xi)=R_0+R_1Y(\xi). \tag{27}$$

By inserting Eqs. (26) and (27) into Eq. (7) and collecting the coefficients of $[Y(\xi)]^m$, where $m$ takes on values from 0 to 6, a system of algebraic equations is formulated. This system can be solved effectively using a computer program, resulting in the following obtained results:

$$R_0=\frac{\epsilon}{4\Delta_7}\left[\sqrt{2\Delta_7[(\Delta_4+2\Delta_5)(h_1^2-4h_0h_2)-4\Delta_6]}+h_1\sqrt{-2\Delta_7(\Delta_4+6\Delta_5)}\right],$$

$$R_1=\frac{\epsilon h_2}{2\Delta_7}\sqrt{-2\Delta_7(\Delta_4+6\Delta_5)}, \tag{28}$$

and

$$\Delta_1=\frac{1}{12\Delta_7}\left[(h_1^2-4h_0h_2)(\Delta_4^2+7\Delta_4\Delta_5+12\Delta_5^2-60\Delta_7)-2\Delta_6(\Delta_4+3\Delta_5)\right],$$

$$\Delta_2=-\frac{1}{48\Delta_7}\left[(h_1^2-4h_0h_2)^2(\Delta_4^2+4\Delta_4\Delta_5-192\Delta_7)+4\Delta_6(h_1^2-4h_0h_2)(\Delta_4+6\Delta_5)-12\Delta_6^2\right],$$

$$\Delta_3=\frac{-3(\Delta_4^2+8\Delta_4\Delta_5+12\Delta_5^2-160\Delta_7)}{2(\Delta_4+6\Delta_5)}, \tag{29}$$

provided

$$\Delta_7\left(\Delta_4+6\Delta_5\right)<0 \text{ and } 2\Delta_7\left[\left(\Delta_4+2\Delta_5\right)\left(h_1^2-4h_0h_2\right)-4\Delta_6\right]>0.$$

A few more situations are omitted for the sake of brevity. The Riccati equation (26) is given accurate solutions by:

$$Y(\xi)=\begin{cases}-\dfrac{h_1}{2h_2}-\dfrac{\sqrt{\Delta}}{2h_2}\dfrac{[K_1\tanh(\frac{\sqrt{\Delta}}{2}\xi)+K_2]}{[K_1+K_2\tanh(\frac{\sqrt{\Delta}}{2}\xi)]} & \text{if } \Delta>0 \text{ and } K_1^2+K_2^2\neq 0,\\ -\dfrac{h_1}{2h_2}+\dfrac{\sqrt{-\Delta}}{2h_2}\dfrac{[K_3\tan(\frac{\sqrt{-\Delta}}{2}\xi)-K_4]}{[K_3+K_4\tan(\frac{\sqrt{-\Delta}}{2}\xi)]} & \text{if } \Delta<0 \text{ and } K_3^2+K_4^2\neq 0,\\ -\dfrac{h_1}{2h_2}-\dfrac{1}{h_2\xi+K_5} & \text{if } \Delta=0,\end{cases} \tag{30}$$

where arbitrary real numbers $K_j$ (where $j$ ranges from 1 to 5) are used as coefficients, and the value of $\Delta$ is calculated as $h_1^2-4h_0h_2$, where $h_0$, $h_1$, and $h_2$ are arbitrary real values.

When $\Delta=h_1^2-4h_0h_2>0$, equations (27), (28) and (30) lead to write the solutions of Eq. (7) as

$$\phi(\xi)=\frac{\epsilon}{4\Delta_7}\sqrt{2\Delta_7[(\Delta_4+2\Delta_5)\Delta-4\Delta_6]}$$

$$-\frac{\epsilon\sqrt{\Delta}}{4\Delta_7}\sqrt{-2\Delta_7(\Delta_4+6\Delta_5)}\left[\frac{K_1\tanh(\frac{\sqrt{\Delta}}{2}\xi)+K_2}{K_1+K_2\tanh(\frac{\sqrt{\Delta}}{2}\xi)}\right], \tag{31}$$

provided

$$\Delta_7\left(\Delta_4+6\Delta_5\right)<0, \text{ and } 2\Delta_7\left[\left(\Delta_4+2\Delta_5\right)\Delta-4\Delta_6\right]>0.$$

In this regard, the solution that satisfies Eq. (1) represents a solitary wave:

$$q(x,t)=\left[\frac{\epsilon}{4\Delta_7}\sqrt{2\Delta_7[(\Delta_4+2\Delta_5)\Delta-4\Delta_6]}\right.$$

$$\left.-\frac{\epsilon\sqrt{\Delta}}{4\Delta_7}\sqrt{-2\Delta_7(\Delta_4+6\Delta_5)}\left\{\frac{K_1\tanh[\frac{\sqrt{\Delta}}{2}(x-Vt)]+K_2}{K_1+K_2\tanh[\frac{\sqrt{\Delta}}{2}(x-Vt)]}\right\}\right]\exp i\left[-\kappa x+wt+\theta_0\right]. \tag{32}$$

Especially when $K_1$ is unequal to zero and $K_2$ is equal to zero in Equation (32), the occurrence of dark solitons can be observed as:

$$q(x,t)=\left[\frac{\epsilon}{4\Delta_7}\sqrt{2\Delta_7[(\Delta_4+2\Delta_5)\Delta-4\Delta_6]}\right.$$

$$\left.-\frac{\epsilon\sqrt{\Delta}}{4\Delta_7}\sqrt{-2\Delta_7(\Delta_4+6\Delta_5)}\tanh\left[\frac{\sqrt{\Delta}}{2}(x-Vt)\right]\right]\exp i\left[-\kappa x+wt+\theta_0\right], \tag{33}$$

while when $K_1 = 0$ and $K_2 \neq 0$ in Eq. (32) the singular solution is:

$$q(x,t)=\left[\frac{\epsilon}{4\Delta_7}\sqrt{2\Delta_7[(\Delta_4+2\Delta_5)\Delta-4\Delta_6]}\right.$$

$$\left.-\frac{\epsilon\sqrt{\Delta}}{4\Delta_7}\sqrt{-2\Delta_7(\Delta_4+6\Delta_5)}\coth\left[\frac{\sqrt{\Delta}}{2}(x-Vt)\right]\right]\exp i\left[-\kappa x+wt+\theta_0\right]. \tag{34}$$

Figures 2 presents a selection of plots showcasing the dark soliton solution (33) to the model equation (1). The parameter values used are as follows: $\kappa = 1$, $V = 1$, $\epsilon = 1$, $h_0 = -1$, $h_1 = 1$, $h_2 = 1$, $\sigma_2 = 1$, $\sigma_4 = 1$, $\sigma_7 = 1$, $\sigma_8 = 1$, $\sigma_9 = 1$, $\sigma_{11} = 1$, $\sigma_{12} = 1$, $\sigma_{13} = 1$, $\sigma_{14} = 1$, $\sigma_{15} = 1$, $\sigma_{10} = -1$, $\delta_1 = 1$, $\delta_2 = 1$, and $\delta_3 = 1$.

## 5. Weierstrass' type riccati equation expansion scheme

In this technique we assume the soliton solutions of Eq. (7) as the form:

$$\phi(\xi)=A_0+\sum_{i=1}^{M}X^{i-1}(\xi)\left[A_iX(\xi)+B_iY(\xi)\right], \tag{35}$$

where the Weierstrass elliptic functions $X(\xi)$ and $Y(\xi)$ are the solutions that satisfy the projective Riccati equation:

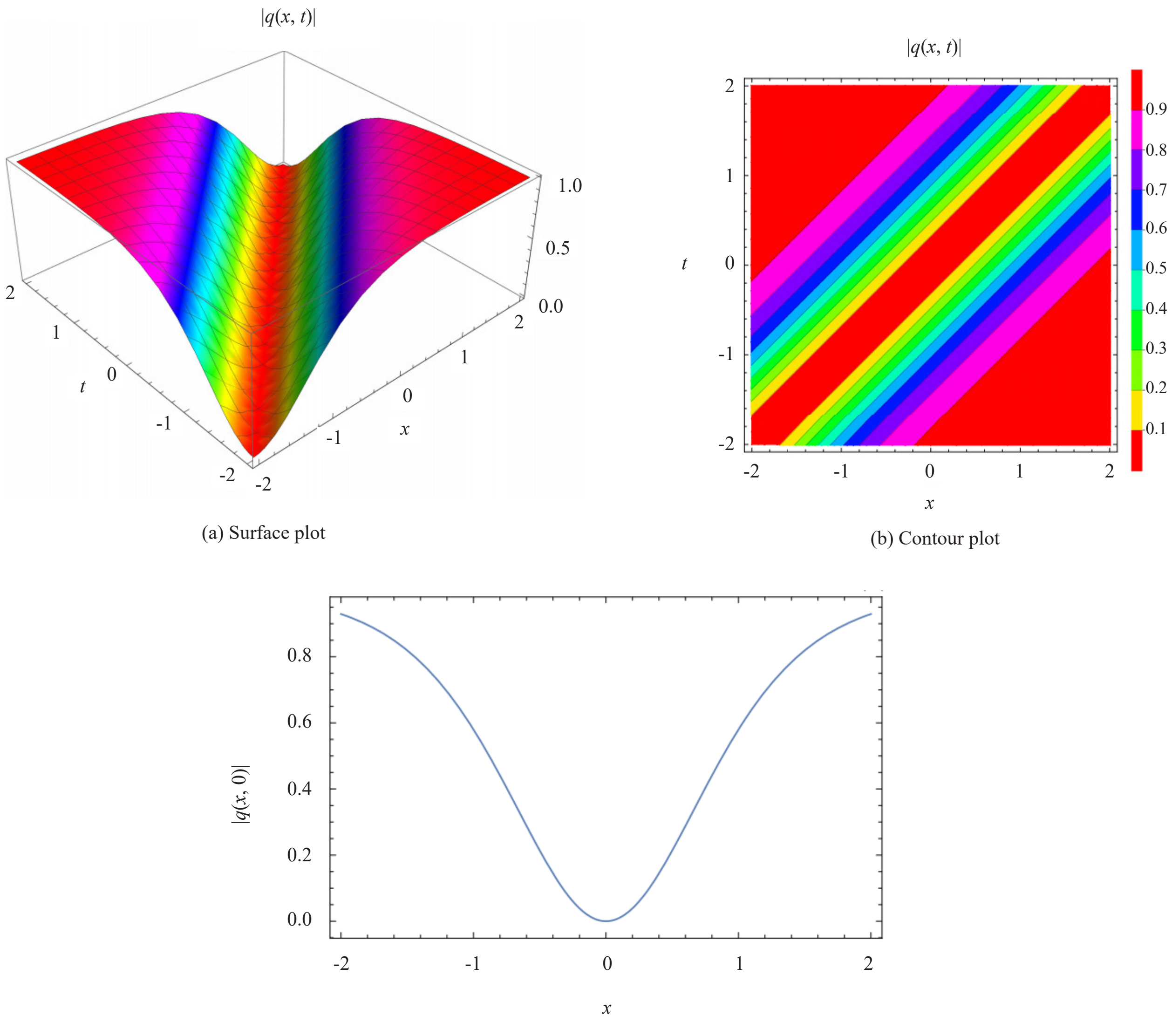
$$X'(\xi)=pX(\xi)Y(\xi),$$

$$Y'(\xi)=q+pY^2(\xi)-rX(\xi). \tag{36}$$

Assume that $p$, $q$, and $r$ are constant parameters in this scenario. To determine the value of the positive integer $M$, one must balance the influence of the highest nonlinear terms and the highest order derivatives of $\phi$ in Equation (7). The undetermined constants $A_0$, $A_i$, and $B_i$ (where $i$ ranges from 1 to $M$) are carefully selected to ensure that $A_M^2 + B_M^2$ does not equal zero. It is widely recognized [16] that the equations given by (36) possess sets of solutions expressed in terms of Weierstrass' elliptic functions.

**Set 1** Under the condition $Y^2(\xi)=-\frac{q}{p}+\frac{2r}{p}X(\xi)$, Eqs. (36) satisfy:

$$Y(\xi)=\frac{12\wp'(\xi,g_2,g_3)}{p\left[pq+12\wp(\xi,g_2,g_3)\right]},\ X(\xi)=\frac{q}{6r}+\frac{2}{pr}\wp(\xi,g_2,g_3). \tag{37}$$




(a) Surface plot

(b) Contour plot

(c) 2D plot

**Figure 2.** Profile of a dark soliton solution (33)

**Set 2** By employing $Y^2(\xi) = -\frac{q}{p} + \frac{2r}{p} X(\xi) - \frac{24r^2}{25pq} X^2(\xi)$ in Eqs. (36), one arrives at:

$$Y(\xi) = \frac{-q\wp'(\xi, g_2, g_3)}{[qp + 12\wp(\xi, g_2, g_3)]\wp(\xi, g_2, g_3)},\ X(\xi) = \frac{5q}{6r} + \frac{5q^2 p}{27r\wp(\xi, g_2, g_3)}. \tag{38}$$

**Set 3** In the presence of $Y^2(\xi) = -\frac{q}{p} + \frac{2r}{p} X(\xi) - \frac{r^2(q+4)}{p(q+2)^2} X^2(\xi)$, Eqs. (36) hold:

$$Y(\xi) = \frac{\wp'(\xi, g_2, g_3)}{\left(\wp(\xi, g_2, g_3) + \frac{p}{2} + \frac{pq}{12}\right)^2 - \frac{p^2}{4}},\ X(\xi) = \frac{(2+q)[pq + 12\wp(\xi, g_2, g_3)]}{r[12p + pq + 12\wp(\xi, g_2, g_3)]}. \tag{39}$$

**Set 4** Eqs. (36) satisfy the following form when $Y^2(\xi) = -\frac{q}{p} + \frac{2r}{p}X(\xi) - \frac{pr^2(p^2+4)}{q(p+2)^2}X^2(\xi)$ is used:

$$Y(\xi) = \frac{q\wp'(\xi, g_2, g_3)}{\left[p\wp(\xi, g_2, g_3) + \frac{q}{2} + \frac{p^2 q}{12}\right]^2 - \frac{q^2}{4}},\ X(\xi) = \frac{q(p^2+2)[pq + 12\wp(\xi, g_2, g_3)]}{pr[12q + p^2 q + 12p\wp(\xi, g_2, g_3)]}, \tag{40}$$

where

$$g_2 = \frac{p^2 q^2}{12},\ g_3 = \frac{p^3 q^3}{216}. \tag{41}$$

With the aim of achieving this, we balance $\phi^{(5)}$ with $\phi^4\phi'$ in Eq. (7). The resulting balance number is $M = 1$, leading to the solitary solution in the following form based on (35):

$$\phi(\xi) = A_0 + A_1 X(\xi) + B_1 Y(\xi),\ A_1^2 + B_1^2 \neq 0. \tag{42}$$

**Set 1** Substituting (42) and (36) into Eq. (7), with $Y^2(\xi) = -\frac{q}{p} + \frac{2r}{p}X(\xi)$, and then collecting all the coefficients of $X^i(\xi)Y^j(\xi)$ ($i = 1, ..., 4,\ j = 0, 1$) and setting them to zero, results in a system of algebraic equations. The Maple software package can be used to solve this system, yielding the following outcomes:

$$A_0 = \frac{\epsilon}{4\Delta_7}\sqrt{-2\Delta_7[pq(\Delta_4 + 2\Delta_5) + 4\Delta_6]},\ A_1 = 0,\ B_1 = \frac{\epsilon p}{4\Delta_7}\sqrt{-2\Delta_7(\Delta_4 + 6\Delta_5)}, \tag{43}$$

and

$$\Delta_1 = \frac{-1}{12\Delta_7}\left[pq(\Delta_4^2 + 7\Delta_4\Delta_5 + 12\Delta_5^2 - 60\Delta_7) + 2\Delta_6(\Delta_4 + 3\Delta_5)\right],$$

$$\Delta_2 = -\frac{1}{48\Delta_7}\left[p^2q^2(\Delta_4^2 + 4\Delta_4\Delta_5 - 192\Delta_7) - 4pq\Delta_6(\Delta_4 + 6\Delta_5) - 12\Delta_6^2\right],$$

$$\Delta_3 = \frac{-3(\Delta_4^2 + 8\Delta_4\Delta_5 + 12\Delta_5^2 - 160\Delta_7)}{2(\Delta_4 + 6\Delta_5)}, \tag{44}$$

provided

$$\Delta_7\left[pq(\Delta_4 + 2\Delta_5) + 4\Delta_6\right] < 0,\ 2\Delta_7\left(\Delta_4 + 6\Delta_5\right) < 0 \text{ and } \epsilon = \pm 1.$$

The solutions of Weierstrass' elliptic function for Eq. (1) can be determined by incorporating equations (37), (42), and (43), resulting in the following expressions:

$$q(x,t)=\left[\frac{\epsilon}{4\Delta_7}\sqrt{-2\Delta_7[pq(\Delta_4+2\Delta_5)+4\Delta_6]}+\frac{12\epsilon p\sqrt{-2\Delta_7(\Delta_4+6\Delta_5)}\wp'\left(x-Vt,\frac{p^2q^2}{12},\frac{p^3q^3}{216}\right)}{4\Delta_7 p\left[pq+12\wp\left(x-Vt,\frac{p^2q^2}{12},\frac{p^3q^3}{216}\right)\right]}\right]\times$$

$$\exp i\left[-\kappa x+wt+\theta_0\right]. \tag{45}$$

By employing the following forms, one can transform the Weierstrass elliptic function into degenerate forms similar to hyperbolic functions and trigonometric functions:

$$\wp\left(\xi,\frac{\theta^2}{12},-\frac{\theta^3}{216}\right)=\frac{\theta}{12}-\frac{\theta}{4}\operatorname{sech}^2\left(\frac{\sqrt{\theta}}{2}\xi\right),\ \theta>0,$$

$$\wp\left(\xi,\frac{\theta^2}{12},-\frac{\theta^3}{216}\right)=\frac{\theta}{12}+\frac{\theta}{4}\operatorname{csch}^2\left(\frac{\sqrt{\theta}}{2}\xi\right),\ \theta>0,$$

$$\wp\left(\xi,\frac{\theta^2}{12},-\frac{\theta^3}{216}\right)=\frac{\theta}{12}-\frac{\theta}{4}\sec^2\left(\frac{\sqrt{-\theta}}{2}\xi\right),\ \theta<0,$$

$$\wp\left(\xi,\frac{\theta^2}{12},-\frac{\theta^3}{216}\right)=\frac{\theta}{12}-\frac{\theta}{4}\csc^2\left(\frac{\sqrt{-\theta}}{2}\xi\right),\ \theta<0. \tag{46}$$

Obtaining the dark and singular soliton solutions from the Weierstrass elliptic solution (45) when $\theta=-pq$ is achieved using the conversion formula (46), resulting in the following solutions, respectively:

$$q(x,t)=\left[\frac{\epsilon}{4\Delta_7}\sqrt{-2\Delta_7[pq(\Delta_4+2\Delta_5)+4\Delta_6]}-\frac{\epsilon\sqrt{-2\Delta_7(\Delta_4+6\Delta_5)}\sqrt{-pq}}{4\Delta_7}\tanh\left(\frac{\sqrt{-pq}}{2}(x-Vt)\right)\right]\times$$

$$\exp i\left[-\kappa x+wt+\theta_0\right], \tag{47}$$

$$q(x,t)=\left[\frac{\epsilon}{4\Delta_7}\sqrt{-2\Delta_7[pq(\Delta_4+2\Delta_5)+4\Delta_6]}-\frac{\epsilon\sqrt{-2\Delta_7(\Delta_4+6\Delta_5)}\sqrt{-pq}}{4\Delta_7}\coth\left(\frac{\sqrt{-pq}}{2}(x-Vt)\right)\right]\times$$

$$\exp i\left[-\kappa x+wt+\theta_0\right], \tag{48}$$

provided $pq<0$.

**Set 2** By employing equations (42) and (36) and substituting $Y^2(\xi)=-\frac{q}{p}+\frac{2r}{p}X(\xi)-\frac{24r^2}{25pq}X^2(\xi)$ into Eq. (7), and then collecting all the coefficients of $X^i(\xi)Y^j(\xi)$ ($j=0,1$, $i=1,\ldots,6$) and setting them to zero, we obtain a system of algebraic equations. This system can be solved using the Maple software package, yielding the following results:

$$A_0 = -\epsilon\sqrt{\frac{(288p^2q^2 + 337\Delta_1 pq + 288\Delta_2)}{7(9\Delta_5 pq + 16\Delta_6)}},\ A_1 = \frac{48\epsilon r}{25q}\sqrt{\frac{(288p^2q^2 + 337\Delta_1 pq + 288\Delta_2)}{7(9\Delta_5 pq + 16\Delta_6)}},\ B_1 = 0, \tag{49}$$

and

$$\Delta_3 = \frac{2(16p^2q^2 + 9\Delta_1 pq + 16\Delta_2)(9\Delta_5 pq + 16\Delta_6)}{qp(288p^2q^2 + 337\Delta_1 pq + 288\Delta_2)},$$

$$\Delta_4 = -\frac{(-363p^2q^2 + 288\Delta_1 pq + 512\Delta_2)(9\Delta_5 pq + 16\Delta_6)}{8pq(288p^2q^2 + 337\Delta_1 pq + 288\Delta_2)},$$

$$\Delta_7 = \frac{175(9\Delta_5 pq + 16\Delta_6)(144\Delta_5 p^3q^3 + 256(\Delta_1\Delta_5 - \Delta_6)p^2q^2 + 144pq(\Delta_2\Delta_5 - \Delta_1\Delta_6) - 256\Delta_2\Delta_6)}{16(288p^2q^2 + 337\Delta_1 pq + 288\Delta_2)^2}, \tag{50}$$

provided

$$\left(288p^2q^2 + 337\Delta_1 pq + 288\Delta_2)(9\Delta_5 pq + 16\Delta_6\right) > 0 \text{ and } \varepsilon = \pm 1.$$

The Weierstrass elliptic wave solution for Eq. (1) can be obtained by utilizing equations (38), (42), and (49), resulting in:

$$q(x,t) = \frac{1}{105}\sqrt{\frac{7(288p^2q^2 + 337\Delta_1 pq + 288\Delta_2)}{(9\Delta_5 pq + 16\Delta_6)}}\left[\frac{2qp + 9\wp\left(x - Vt, \frac{p^2q^2}{12}, \frac{p^3q^3}{216}\right)}{\wp\left(x - Vt, \frac{p^2q^2}{12}, \frac{p^3q^3}{216}\right)}\right]\times$$

$$\exp i\left[-\kappa x + wt + \theta_0\right]. \tag{51}$$

The conversion formula (46) leads to bright and singular optical solitons that emerge from the Weierstrass elliptic solution (51) when $\theta = -pq$ as:

$$q(x,t) = -\frac{1}{35}\sqrt{\frac{7(288p^2q^2 + 337\Delta_1 pq + 288\Delta_2)}{(9\Delta_5 pq + 16\Delta_6)}}\left[\frac{5 + 9\operatorname{sech}^2\left[\frac{\sqrt{-pq}}{2}(x - Vt)\right]}{1 - 3\operatorname{sech}^2\left[\frac{\sqrt{-pq}}{2}(x - Vt)\right]}\right]\times$$

$$\exp i\left[-\kappa x + wt + \theta_0\right],\ \ pq < 0. \tag{52}$$

$$q(x,t) = -\frac{1}{35}\sqrt{\frac{7(288p^2q^2 + 337\Delta_1 pq + 288\Delta_2)}{(9\Delta_5 pq + 16\Delta_6)}}\left[\frac{5 - 9\operatorname{csch}^2\left[\frac{\sqrt{-pq}}{2}(x - Vt)\right]}{1 + 3\operatorname{csch}^2\left[\frac{\sqrt{-pq}}{2}(x - Vt)\right]}\right]\times$$

$$\exp i\left[-\kappa x+wt+\theta_0\right],\ pq<0. \tag{53}$$

**Set 3** By substituting (42) and (36) into Eq. (7), and considering $Y^2(\xi)=-\frac{q}{p}+\frac{2r}{p}X(\xi)-\frac{r^2(q+4)}{p(q+2)^2}X^2(\xi)$, we can collect all the coefficients of $X^i(\xi)Y^j(\xi)$ ($j$ = 0, 1, $i$ = 1, ..., 10) in Eq. (7) and equate them to zero. Solving this system of algebraic equations using the Maple software package provides the following outcomes:

$$A_0=A_1=0,\ B_1=\frac{\epsilon p}{4\Delta_6}\sqrt{6\Delta_6(-5pq^3+\Delta_1q^2-20pq^2+4\Delta_1q+100pq-4\Delta_1)}, \tag{54}$$

and

$$\Delta_2=-\frac{1}{2}pq(\Delta_1-13pq),$$

$$\Delta_3=\frac{\Delta_6(-5pq^2+\Delta_1q-20pq+4\Delta_1-80p)}{p(-5pq^3+\Delta_1q^2-20pq^2+4\Delta_1q+100pq-4\Delta_1)},$$

$$\Delta_4=\frac{2\Delta_6(-5pq^2+\Delta_1q-20pq+4\Delta_1-240p)}{3p(-5pq^3+\Delta_1q^2-20pq^2+4\Delta_1q+100pq-4\Delta_1)},$$

$$\Delta_5=-\frac{2\Delta_6(-5pq^2+\Delta_1q-20pq+4\Delta_1)}{3p(-5pq^3+\Delta_1q^2-20pq^2+4\Delta_1q+100pq-4\Delta_1)},$$

$$\Delta_7=\frac{40\Delta_6^2(-5pq^2+\Delta_1q-20pq+4\Delta_1+48p)}{9p(-5pq^3+\Delta_1q^2-20pq^2+4\Delta_1q+100pq-4\Delta_1)^2}. \tag{55}$$

Eq. (1) can be solved for its Weierstrass elliptic wave solution by employing equations (39), (42), and (54), resulting in the following expression:

$$q(x,t)=\frac{\epsilon p}{4\Delta_6}\sqrt{6\Delta_6(-5pq^3+\Delta_1q^2-20pq^2+4\Delta_1q+100pq-4\Delta_1)}$$

$$\left[\frac{\wp'\left(x-Vt,\frac{p^2q^2}{12},\frac{p^3q^3}{216}\right)}{\left[\left(\wp\left(x-Vt,\frac{p^2q^2}{12},\frac{p^3q^3}{216}\right)+\frac{p}{2}+\frac{pq}{12}\right)^2-\frac{p^2}{4}\right]}\right]\exp i\left[-\kappa x+wt+\theta_0\right], \tag{56}$$

provided

$$\Delta_6\left(-5pq^3+\Delta_1q^2-20pq^2+4\Delta_1q+100pq-4\Delta_1\right)>0 \text{ and } \epsilon=\pm1.$$

The conversion formula (46) leads to obtain many kinds of the traveling wave solutions from the Weierstrass' elliptic solution (56) when $\theta = -pq$ as:

$$q(x,t) = \frac{\epsilon p^2 q\sqrt{-pq}}{16\Delta_6}\sqrt{6\Delta_6(-5pq^3 + \Delta_1 q^2 - 20pq^2 + 4\Delta_1 q + 100pq - 4\Delta_1)} \times$$

$$\left[\frac{\operatorname{sech}^2\left[\frac{1}{2}\sqrt{-pq}(x-Vt)\right]\tanh\left[\frac{1}{2}\sqrt{-pq}(x-Vt)\right]}{\left[\left(\frac{1}{4}pq\operatorname{sech}^2\left[\frac{1}{2}\sqrt{-pq}(x-Vt)\right] + \frac{p}{2}\right)^2 - \frac{p^2}{4}\right]}\right]\exp i\left[-\kappa x + wt + \theta_0\right], \tag{57}$$

$$q(x,t) = -\frac{\epsilon p^2 q\sqrt{-pq}}{16\Delta_6}\sqrt{6\Delta_6(-5pq^3 + \Delta_1 q^2 - 20pq^2 + 4\Delta_1 q + 100pq - 4\Delta_1)} \times$$

$$\left[\frac{\operatorname{csch}^2\left[\frac{1}{2}\sqrt{-pq}(x-Vt)\right]\coth\left[\frac{1}{2}\sqrt{-pq}(x-Vt)\right]}{\left[\left(-\frac{1}{4}pq\operatorname{csch}^2\left[\frac{1}{2}\sqrt{-pq}(x-Vt)\right] + \frac{p}{2}\right)^2 - \frac{p^2}{4}\right]}\right]\exp i\left[-\kappa x + wt + \theta_0\right], \tag{58}$$

as long as in (57) and (58) $pq < 0$.

The results from Set 4 are withheld here.

## 6. Conclusions

This paper successfully ventured and recovered optical soliton solutions to the dispersive concatenation model with linear CD and Kerr form of SPM. The results were recovered with the usage of three integration schemes. A full spectrum of soliton solutions was recovered and exhibited, along with the parameter restrictions or constraints. These included the straddled solitons too. The results are thus tremendously promising and lead to the avenues of further research in this arena. Later, the model will be studied with differential group delay followed by the consideration of the model with dispersion-flattened fibers. Moreover, the model will be addressed numerically with Laplace-Adomian decomposition scheme which would give a visual perspective to the soliton solutions. The model is yet to be addressed to retrieve gap solitons, quiescent solitons and the consideration of the model in magneto-optic waveguides would also be an asset. Thus, an avalanche of work lies ahead. The results obtained from these research undertakings would be in accordance with previous studies and would be disseminated over a period of time [21-24].

## Conflict of interest

The authors declare no conflict of interest.

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