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Research Article

# Numerical PDE-Based Pricing of Convertible Bonds Under Two-Factor Models

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**Abstract:** Convertible bonds are popular financial instruments by which firms raise capital. Owing to the various features of such bonds, especially the early-exercise call, put, and conversion provisions, they can be valued by numerical techniques only. The price of a convertible bond is driven by both the underlying stock price and the interest rate, and these two factors are correlated. Under the partial differential equation framework, a two-dimensional convection-diffusion-reaction equation containing a mixed derivative must be solved. In this work, we employ an Alternating-Direction-Implicit method, namely the Craig-Sneyd scheme to solve the two-factor pricing equation. Comparison against the commonly employed Crank-Nicolson method shows the merit of the scheme. Besides, we analyze how the different contractual features of a convertible bond affect its price.

***Keywords*:** convertible bonds, two-factor models, Alternating-Direction-Implicit method, Craig-Sneyd scheme

**MSC:** 65M06, 65M22, 91G20, 91G60

## 1. Introduction

Convertible bond are corporate bonds (issued by publicly traded companies) that give the investors the right to exchange the bonds to a given number of shares of the issuer's common stock at some specific time. The most popular type of convertible bonds (CB) are coupon-bearing bonds with a variety of embedded options such as call features (callable) or put features (puttable). Callable bonds have a call feature which permits the issuer to buy back or call all or part of the bond prior to maturity while puttable bonds give the bondholder the right to sell the bond back to the issuer at a predetermined price on specified dates. Thus, a convertible bond is a hybrid fixed-income instrument that combines features of both equities and bonds. The convertibility, callability, and puttability features, which are mostly of Bermudan or American style, make CBs complex financial instruments with no no analytical solution. Moreover, CB are subject to the possibility of events of default since they are issued by firms.

A survey of the theoretical and empirical aspects of convertible bond pricing can be found in [1]. The vast majority of existing convertible bonds pricing frameworks take the bond as a derivative of the underlying equity (see for example the one-factor models of Brennan and Schwartz [2] and Ayache, Forsyth and Vetzal [3]). Two integral equations were derived in [4] under the assumption of constant interest rate to analyze puttable convertible bonds (CB) under the Black-


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Scholes model and a more recent work with the call feature added can be found in Lin and Zhu [5]. Additionally, since most convertible bonds CBs have long maturities, several authors assume interest rates to be stochastic, giving rise to two-dimensional models where the price of the equity price and the interest rate both derive the convertible bond value. A recent work in this direction include pricing of CB of American-style with stochastic volatility of Heston and stochastic interest rate of the Cox-Ingersoll-Ross [6].

Hung and Wang [7] and Chambers and Lu [8] construct two-factor trees for valuing CBs. These trees are complicated to implement because of the multiplicity of branches that model movements in the interest rate and stock price. On the other hand, the models of Barone-Adesi, Bermudez and Hatgionnides [9] and Yigitbasioglu and Alexander [10] are partial differential equations (PDEs) based methods. Besides, in many of the above-mentioned frameworks, the reduced-form method (a stock-value approach based on market information) is used to incorporate default risk. In general, the reduced-from approach is more widely accepted due to the ease of parameter estimation [11]. The numerous literature with the reduced-form approach was classified by Batten et al. [1, 12] into four categories which are the finite difference method, finite element method, lattice model, and simulation model.

In Coonjobeharry et al. [13], we developed a model based on two factors where the underlying stock price and the stochastic interest rates are assumed to follow jump-diffusion processes in addition to incorporating default risk. The CB is priced by solving the proposed partial integro-differential equation using a spectral method along with Clenshaw-Curtis quadratures.

In this work, we are interested in pricing CBs under PDE-based two-factor models. A well-known approach for solving the two-dimensional pricing PDEs is the method-of-lines technique; once the PDEs are discretized in the spatial variables, a popular method for solving the resulting semi-discrete systems is the Crank-Nicolson (CN) scheme. Our recent work in this direction to value interest rate derivatives under short-rate models extended with jumps can be found in [14] and we considered the pricing of equity options using meshless methods under two-dimensional models in [15-18]. In this work, we employ one type of Alternating-Direction-Implicit (ADI) method, namely the Craig-Sneyd (CS) scheme, as an alternative to the Crank-Nicolson scheme for pricing CBs. The CS scheme [19] was developed to obtain the solution of pure-diffusion equations with mixed derivatives of second-order accuracy. In [20] and [21], the authors show that, the CS scheme [22] is unconditionally stable when applied to finite difference discretizations for the solution of two-dimensional convection-diffusion PDEs having mixed derivatives. The authors derived sufficient and necessary conditions on the parameters of the CS scheme for unconditional stability in the presence of mixed derivative terms. As the CB pricing PDE is in the form of a convection-diffusion-reaction equation with an additional term present due to the inclusion of default risk in the model, we are interested in the performance of the CS scheme in solving such an equation.

The structure of this paper is as follows. We describe the valuation model for a CB whose value is driven by the equity price as well as the interest rate in §2. In §3, the numerical method of the pricing PDE is discussed. Numerical experiments are conducted in §4 to show the performances of the CS scheme compared to the CN scheme, as well as analyzing how the different features and parameters in the CB model affect the price. Conclusions are given in §5.

## 2. The pricing model

Let the stock price be denoted by $S$ and the interest rate be denoted by $r$, then a convertible bond being a derivative $S$ and $r$ satisfies the stochastic differential equations

$$\frac{dS}{S} = (r - q + h\eta)dt + \sigma_S dZ_S - \eta dP,$$

$$dr = \alpha(\theta - r)dt + \sigma_r \sqrt{r}\, dZ_r,$$

where $q$ represents the continuous dividend yield, $\sigma_s$ and $\sigma_r$ are the volatility of $S$ and $r$ respectively, $\alpha$ denotes the speed of reversion of $r$ about the long time mean of $\theta$, $Z_r$ and $Z_s$ and two Wiener processes correlated with $\rho$, and in the event of a default, $\eta$ which follows a Poisson process $P$ with intensity given by $h$, known as the hazard rate measures the fractional drop in the price of the stock. Note that $r$ follows the square root process of [23], which precludes negative interest rates. Let the maturity of the convertible bond be represented by $T$, $\tau = T - t$ the time to maturity, $F$ the face value, $C$ the coupon, and $k$ the number of shares in which the CB can be converted. In the event that a default occurs, the bondholder can opt for receiving a fractional part $0 \leq R \leq 1$ of $F$, the face value of the bond, or shares of value $kS(1 - \eta)$, whichever is the maximum. Then, following standard hedging arguments (see [10]), letting $\mathcal{V}(S, r, \tau)$ denote the price of the CB price, it can be shown that $\mathcal{V}(S, r, \tau)$ satisfies

$$\frac{\partial \mathcal{V}}{\partial \tau} = \frac{1}{2}\sigma_s^2 S^2 \frac{\partial^2 \mathcal{V}}{\partial S^2} + \rho \sigma_s \sigma_r S\sqrt{r}\frac{\partial^2 \mathcal{V}}{\partial S \partial r} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 \mathcal{V}}{\partial r^2} + (r - q + h\eta)S\frac{\partial \mathcal{V}}{\partial S}$$

$$+\alpha(\theta - r)\frac{\partial \mathcal{V}}{\partial r} - (r + h)\mathcal{V} + h\max(RF,\, kS(1-\eta)), \tag{1}$$

with initial condition $\mathcal{V}(S, r, 0) = \max(kS, F + C)$. It is well know that $\mathcal{V} \to 0$ as $r \to \infty$. Hence for large values of $r$, the bond price $\mathcal{V}$ decrease almost linearly. The boundary conditions $\lim_{S\to\infty} \frac{\partial^2 \mathcal{V}}{\partial S^2} = 0$ and $\lim_{r\to\infty} \frac{\partial^2 \mathcal{V}}{\partial r^2} = 0$ therefore follow from the intuition that for an extremely high price of the stock, the bondholder will prefer to exchange to shares, leading to a linear increase in the CB price with $S$ as $S \to \infty$, and that the CB price behaves almost linearly at extremely high rate of interests. At the left boundaries at $S = 0$ and $r = 0$, we use one-sided approximations which is described in the next section.

The CB is convertible into $k$ shares, callable at price $B_c$ and puttable at price $B_p$. Due to the conversion provisions, the convertible bond price is subject to some no-arbitrage constraints. Because of this conversion right,

$$\mathcal{V} \geq kS,$$

and to incorporate the provision for the put feature

$$\mathcal{V} \geq B_p.$$

Combining these two constraints gives

$$\mathcal{V} \geq \max(B_p, kS).$$

Also, since the issuer of the bond is given the right to call back the bond,

$$\mathcal{V} \leq \max(kS, B_c),$$

as when the bond is called back, the holder reserves the right to exchange the bond for shares. We also set $B_c = \infty$ where the bond cannot be called and set $B_p = 0$ when the bond is not puttable. When the bond is callable and puttable at the same time $t$ which is not coinciding with a coupon date $t_c$, the accrued interest, denoted by $A$ on the upcoming coupon $C$, needs to be taken into consideration and this is calculated as

$$A = \frac{t - t_c^i}{t_c^{i+1} - t_c^i} C,$$

with $t_c^i$ being the coupon date preceding $t_c^{i+1}$. The clean call price is denoted by $B_c^{\text{clean}}$ and the clean put price is denoted by $B_p^{\text{clean}}$ satisfy

$$B_c = B_c^{\text{clean}} + A,$$

$$B_p = B_p^{\text{clean}} + A.$$

## 3. Numerical method

We truncate the $S$-domain to $[0, S_{\max}]$ and $r$-domain to $[0, r_{\max}]$. Let $\Delta S = S_{\max}/(m - 1)$ be the step size for $S$ and $\Delta r = r_{\max}/(n - 1)$ be the step size for $r$, then the $S$-grid is constructed as $G_s = \{S_i = (i-1)\Delta S\}_{i=1}^{m}$ and the $r$-grid is constructed as $G_r = \{r_j = (j-1)\Delta r\}_{j=1}^{n}$. The two-dimensional grid is then constructed with the $mn$ grid nodes.

Letting $\mathcal{V}_{i,j}$ denote the value of the CB at the grid node $(S_i, r_j)$ and collecting the $mn$ nodal values as a vector of the bond prices $\mathcal{V} = [\mathcal{V}_1, ..., \mathcal{V}_n]^T \in \mathbb{R}^{mn}$, for

$$\mathcal{V}_i = [\mathcal{V}_{1,i}, \ldots, \mathcal{V}_{m,i}]^T.$$

Let $D_s$ and $D_{ss}$ denote the tridiagonal $m \times m$ matrices

$$D_s = \frac{1}{(2\Delta S)} \text{tridiag}\left[-1, 0, 1\right], \quad D_{ss} = \frac{1}{(\Delta S)^2} \text{tridiag}\left[1, -2, 1\right],$$

arising when a second-order finite difference discretisation is employed for the first-order differential term in $S$

$$\left(\frac{\partial \mathcal{V}}{\partial S}\right)_i \approx \frac{\mathcal{V}_{i+1} - \mathcal{V}_{i-1}}{2\Delta S},$$

and second-order differential terms in $S$

$$\left(\frac{\partial^2 \mathcal{V}}{\partial S^2}\right)_i \approx \frac{\mathcal{V}_{i+1} - 2\mathcal{V}_i + \mathcal{V}_{i-1}}{(\Delta S)^2},$$

and denote by $I_s$ the identity matrix of order $m$. Let $I$ be the identity matrix of order $mn$ and the matrices of order $n$ for $D_r$, $D_{rr}$, and $I_r$ are constructed in the same way.

The boundary condition $\frac{\partial^2 \mathcal{V}}{\partial S^2} = 0$ and $\frac{\partial^2 \mathcal{V}}{\partial r^2} = 0$ at $S_{\max}$ and $r_{\max}$ respectively are implemented by setting the last row of the second order derivative matrices $D_{ss}$ and $D_{rr}$ to zero.

One sided approximations are used at the left boundaries $S_{\min} = 0$ and $r_{\min} = 0$ as follows:

$$\left(\frac{\partial \mathcal{V}}{\partial S}\right)_{i=0} \approx \frac{-3\mathcal{V}_0 + 4\mathcal{V}_1 - \mathcal{V}_2}{(2\Delta S)}, \quad \left(\frac{\partial \mathcal{V}}{\partial r}\right)_{i=0} \approx \frac{-3\mathcal{V}_0 + 4\mathcal{V}_1 - \mathcal{V}_2}{(2\Delta r)},$$

$$\left(\frac{\partial^2 \mathcal{V}}{\partial S^2}\right)_{i=0} \approx \frac{2\mathcal{V}_0 - 5\mathcal{V}_1 + 4\mathcal{V}_2 - \mathcal{V}_3}{(\Delta S)^2}, \quad \left(\frac{\partial^2 \mathcal{V}}{\partial r^2}\right)_{i=0} \approx \frac{2\mathcal{V}_0 - 5\mathcal{V}_1 + 4\mathcal{V}_2 - \mathcal{V}_3}{(\Delta r)^2},$$

and for the first derivatives we use the following one-sided approximations at $S_{\max}$ and $r_{\max}$

$$\left(\frac{\partial \mathcal{V}}{\partial S}\right)_{i=m} \approx \frac{3\mathcal{V}_{m-2} - 4\mathcal{V}_{m-1} + \mathcal{V}_m}{2\Delta S}, \quad \left(\frac{\partial \mathcal{V}}{\partial r}\right)_{i=m} \approx \frac{3\mathcal{V}_{m-2} - 4\mathcal{V}_{m-1} + \mathcal{V}_m}{2\Delta r}.$$

Then equation (1) is discretized as

$$\mathcal{V}'(\tau) = A\mathcal{V}(\tau) + \hbar, \tag{2}$$

where the matrix $A \in \mathbb{R}^{mn \times mn}$ is represented by

$$A = \frac{1}{2}\sigma_s^2 S^2 [I_r \otimes D_{ss}] + \rho\sigma_r\sigma_s S\sqrt{r}[I_r \otimes D_s]\cdot[D_r \otimes I_s] + \frac{1}{2} r\sigma_r^2 [D_{rr} \otimes I_s]$$

$$+(r - q + h\eta)S[I_r \otimes D_s] + \alpha(\theta - r)[D_r \otimes I_s] - (r + h)I, \tag{3}$$

with $B \otimes C \in \mathbb{R}^{bc \times bc}$ denoting the Kronecker product of the matrices $B \in \mathbb{R}^{b \times b}$ and $C \in \mathbb{R}^{c \times c}$ and $\hbar$ is the vector of terms $h \max(RF, kS(1 - \eta))$ arising from (1).

## 3.1 *Time-stepping*

**Crank-Nicolson scheme** Denoting the solution vector at time $\tau_j$ by $\mathcal{V}^j$ and letting $\Delta\tau = \tau_{j+1} - \tau_j$ be the uniform time step, applying a CN scheme to equation (2) gives

$$\left(I - \frac{\Delta\tau}{2}A\right)\mathcal{V}^{j+1} = \left(I + \frac{\Delta\tau}{2}A\right)\mathcal{V}^j + \Delta\tau R. \tag{4}$$

To speed up the algorithm, the LU decomposition of the matrix $(I - \frac{\Delta\tau}{2}A)$ is precomputed outside the loop for the time-stepping and equation (4) is solved by using forward/backward substitution. It is well known that the CN time-stepping is unconditionally stable and is of accuracy second-order. With an increasing number of grid nodes $m$ and $n$ as explained in [24], the Crank-Nicolson scheme becomes inefficient. This is because the bandwidth of the left-hand side matrix in (4), and thereafter the LU factorisation matrices, is known to be directly proportional to the minimum of $n$ and $m$.

**Craig-Sneyd scheme** The matrix $A$ is then splitted into the three submatrices $\mathcal{A}_\rho$ corresponding to the mixed derivative term, $\mathcal{A}_s$, the term corresponding to the $S$-derivative and $\mathcal{A}_r$, the terms corresponding to the $r$-derivative, in the form

$$A = \mathcal{A}_\rho + \mathcal{A}_s + \mathcal{A}_r,$$

where

$$\mathcal{A}_\rho = \sigma_r \sigma_s \rho S \sqrt{r}[I_r \otimes D_s][D_r \otimes I_s],$$

$$\mathcal{A}_s = \frac{1}{2}\sigma_s^2 S^2 [I_r \otimes D_{ss}] + (r - q + h\eta) S [I_r \otimes D_s] - \frac{1}{2}(r+h)I,$$

$$\mathcal{A}_r = \frac{1}{2}\sigma_r^2 r [D_{rr} \otimes I_s] + \alpha(\theta - r)[D_r \otimes I_s] - \frac{1}{2}(r+h)I.$$

The part $(r + h)I$ in (3) is evenly distributed over $A_s$ and $A_r$. The second-order CS scheme for the solution of the ODE (2) is then given by

$$W_0 = \Delta\tau R + (I + \Delta\tau A)\mathcal{V}^j,$$

$$W_1 = \left(I - \frac{\Delta\tau}{2}\mathcal{A}_s\right)^{-1}\left(W_0 - \frac{\Delta\tau}{2}\mathcal{A}_s \mathcal{V}^j\right),$$

$$W_2 = \left(I - \frac{\Delta\tau}{2}\mathcal{A}_r\right)^{-1}\left(W_1 - \frac{\Delta\tau}{2}\mathcal{A}_r \mathcal{V}^j\right),$$

$$W_3 = \frac{\Delta\tau}{2}\left(\mathcal{A}_\rho W_2 - \mathcal{A}_\rho \mathcal{V}^j\right) + W_0,$$

$$W_4 = \left(I - \frac{\Delta\tau}{2}\mathcal{A}_s\right)^{-1}\left(W_3 - \frac{\Delta\tau}{2}\mathcal{A}_s \mathcal{V}^j\right),$$

$$W_5 = \left(I - \frac{\Delta\tau}{2}\mathcal{A}_r\right)^{-1}\left(W_4 - \frac{\Delta\tau}{2}\mathcal{A}_r \mathcal{V}^j\right),$$

$$\mathcal{V}^{j+1} = W_5. \tag{5}$$

The CS scheme was originally developed by Craig and Sneyd [19] in order to solve pure-diffusion equations containing mixed derivatives with second-order accuracy. In in't Hout and Welfert [20-21] the authors showed that the CS scheme is unconditionally stable when applied to the solution of two-dimensional convection-diffusion equations with mixed derivatives. The convergence of the CS was discussed by In't Hout and Wyns [25]. The basic idea of the CS scheme is a first predictor step followed by two unidirectional corrector steps, and then a second predictor step followed by two unidirectional corrector steps.

As with the CN scheme, the matrix-vector equations are solved via the LU decomposition of the matrices $(I - \frac{\Delta\tau}{2}A_r)$ and $(I - \frac{\Delta\tau}{2}A_s)$. However, as pointed out in in't Hout and Foulon [24], the bandwidth of these matrices are now fixed, even as $m$ and $n$ increase, and the number of floating point operations at each time step depends only on the number of grid nodes $mn$. Consequently, for large values of $m$ and $n$, the CS scheme is more efficient than the CN scheme.

## 4. Numerical results

In this section, we provide some numerical examples to compare the performance of CN and CS schemes and we further investigate on how the CB price changes for different parameters of a convertible bond model and different features. Consider a convertible bond with maturity $T = 5$ years and face value $F = 100$ with coupons worth $C = 4$ paid semiannually. The bond can be converted into $k = 1$ share and is callable during the time interval $t \in [2, 3]$ years at a price of $B_c^{\text{clean}} = 110$, and is puttable during the time interval $t \in [2, 3]$ years at a price of $B_p^{\text{clean}} = 105$. The current stock price is given by $S = 100$ and current interest rate is $r = 0.05$. The dividend rate $q$ is assumed to be zero unless stated otherwise. The other parameters are taken as

$$\left(\sigma_s, \sigma_r, \alpha, \theta, \rho, h, R, \eta\right) = \left(0.2, 0.22, 0.5, 0.07, 0.1, 0.2, 0, 0.5\right).$$

Table 1 compares the performances of both schemes in terms of accuracy and computational speed in seconds (Cpu(s)) for $S_{\max} = 200$ and $r_{\max} = 0.5$. Due to memory constraints, we are limited to taking a maximum of 64 grid points only in the $S$ and $r$ directions. With a value of $m = n = 64$, the discretization matrices are of size 4,096 × 4,096. The second-order accuracy of the CS scheme is maintained even when the pricing equation, which is a convection-diffusion-reaction equation, contains the additional term $h \max(kS(1-\eta), RF)$. It is also observed that when the number of grid nodes is small, the CN scheme requires solving only one matrix as seen from equation (4) and therefore performs better than the CS scheme in terms of computational speed. However, as $m$ and $n$ increases, the CS scheme is computationally less expensive which is explained by the fact that the bandwidth of the matrices of the CS method do not grow with increasing $m$ and $n$. As an example, with the CS scheme, the bandwidth of the $L$ matrices obtained from the LU factorizations of matrix $(I - \frac{\Delta\tau}{2} A_s)$ remains fixed at 3 when using finite differences schemes of second-order, in contrast to the bandwidth of the matrix $(I - \frac{\Delta\tau}{2} A)$ of the CN scheme which grows with increasing $m$ and $n$.

The convertibility, callability, and puttability features make CBs complex financial instruments for which no closed-form pricing formulas exist and therefore it is difficult to show the order of convergence numerically. We therefore report on the relative errors and the relative convergence rate which has been calculated using the formula

$$\text{Relative Convergence Rate} = \frac{\log\left(\dfrac{\text{Rel. Error}_{m,n}^{t_s}}{\text{Rel. Error}_{2m,2n}^{2t_s}}\right)}{\log 2},$$

where

$$\text{Rel. Error}_{m,n}^{t_s} = \left|\text{Price}_{m,n}^{t_s} - \text{Price}_{2m,2n}^{2t_s}\right|.$$

A second-order convergence rate is shown for both the Crank-Nicolson and Craig-Sneyd. Note that the relative errors are the same for both scheme since the yield the same convertible price. The difference is in terms of running time only.

Table 1. CB prices computed using the Crank-Nicolson scheme and the Craig-Sneyd scheme

| $m = n$ | Timesteps ($t_s$) | Crank-Nicolson (CN) | | Craig-Sneyd (CS) | | CN/CS | |
|---|---|---|---|---|---|---|---|
| | | CB Price | Cpu(s) | CB Price | Cpu(s) | Rel Error | Rel Conv. |
| 16 | 1250 | 112.0198 | 0.13 | 112.0198 | 0.29 | - | - |
| 32 | 2500 | 112.1809 | 0.98 | 112.1809 | 1.59 | 0.171 | - |
| 64 | 5000 | 112.2335 | 21.17 | 112.2335 | 15.59 | 0.043 | 2.002 |

Figure 1 shows the graph of the convertible bond price as an increasing monotone function of $S$ and as a decreasing monotone function of $r$ using the CS scheme.

Having shown the benefit of the CS scheme over the CN scheme, the CB prices in the remaining examples are computed by the CS scheme using $m = n = 64$ and 5,000 timesteps.

In Table 2, we investigate how the call and put features impact on the convertible bond price. The callable property of the CB is advantageous to the issuer, which justifies why the callable bond price falls. The put option is beneficial to the bondholder, hence justifying the higher price for puttable bonds. However, the above two effects are not symmetric, with the dominant call feature since it spans over a larger period of 3 years in contrast to the put feature which spans over a duration of 1 year only.

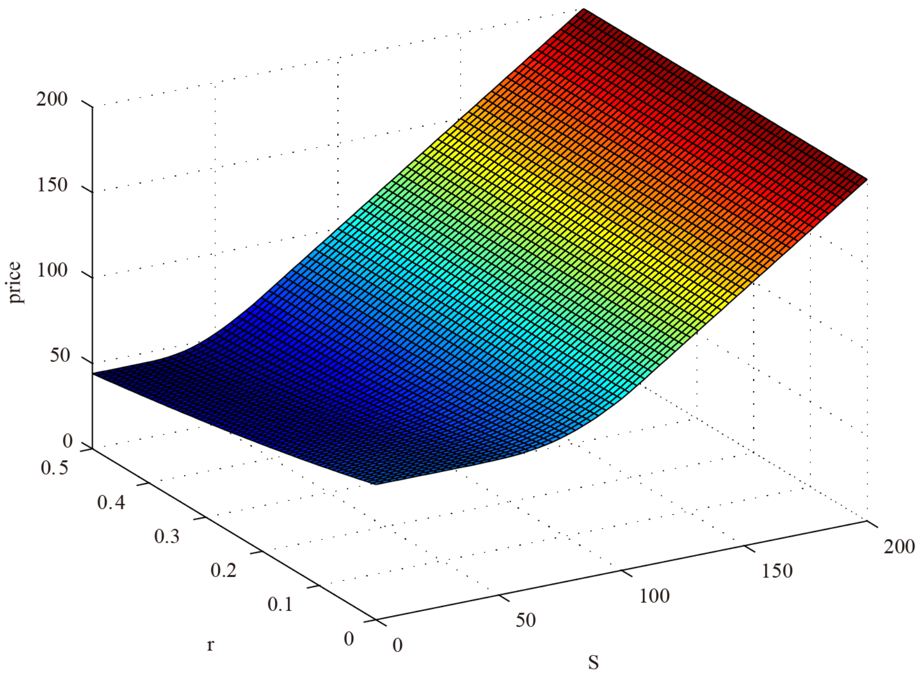


Figure 1. CB price as a function of stock price and interest rate computed using the CS scheme

**Table 2.** CB prices for different contractual features

| CB features | | Price |
|---|---|---|
| Callable | Puttable | |
| No | No | 120.4246 |
| Yes | No | 111.1670 |
| No | Yes | 121.1034 |
| Yes | Yes | 112.2335 |

The next example investigates on how the two events (the decrease in the stock price and recovering a proportion of the bond's face value), which occur upon default, affect the bond price. The prices for the different values of the parameters $R$ and $\eta$ are reported in Table 3. Reading accross the rows of the table, it can be observed that the price increases as the recovery factor rises from 0.25 to 0.5, since then the bondholder is able to get back a higher portion of his investment when a default occurs. However, when $R$ increases from 0 to 0.25, the CB price remains unchanged, as a recovery of one-fourth of the face value, which pertains to the bond part of the CB, is not significant when compared to the equity part of the CB. Reading down the columns of the table, it is observed that the price falls when the fractional drop in the stock price $\eta$ gets larger. This is explained by the fact that the equity component of the convertible bond loses value.

**Table 3.** Convertible bond prices for different default scenarios

| $\eta$ | $R$ | | |
|---|---|---|---|
| | 0 | 0.25 | 0.5 |
| 0 | 116.8815 | 116.8815 | 116.8824 |
| 0.25 | 114.1426 | 114.1426 | 114.1531 |
| 0.5 | 112.2335 | 112.2335 | 112.6338 |

Finally, we investigate how the $S$- and $r$-related parameters impact on the CB price. In Table 4, by varying $\sigma_r$ and $\alpha$, we note that price changes due to the stochastic nature of $r$ are small. In particular, on setting $\sigma_r = 0$ and $\alpha = 0$, we obtain a single-factor model in which the CB price is modelled by the stock price only, and we can see that switching off the interest rate factor results in a minor change in the convertible bond price. This is in line with the conclusion of [2] that "for a reasonable range of interest rates, the errors from the [non-stochastic] interest rate model are likely to be slight". The effect of variations in $\sigma_s$ are studied in Table 5 and note that as $\sigma_s$ varies, significant variation in the convertible bond price is observed. The results of Tables 4 and 5 helps to conclude that the stock price dynamics plays a much more crucial role in convertible bond valuation than the interest rate's dynamics.

**Table 4.** Effect of $r$-related parameters

| $\sigma_r$ | $\alpha$ | CB price |
|---|---|---|
| 0 | 0 | 112.2829 |
| 0.12 | 0.5 | 112.0811 |
| 0.22 | 0.5 | 112.2335 |
| 0.32 | 0.5 | 112.4120 |
| 0.22 | 0.1 | 112.5877 |
| 0.22 | 0.9 | 112.0231 |

**Table 5.** Effect of stock price volatility

| $\sigma_s$ | CB price |
|---|---|
| 0.1 | 109.8354 |
| 0.2 | 112.2335 |
| 0.4 | 118.8819 |

## 5. Conclusion

A CB is a hybrid financial instrument consisting of both an equity part and a fixed-income part. The convertibility, callability, and puttability features make CBs complex financial instruments for which no closed-form pricing formulas exist. The simplest valuation models are one-factor models which assume that the CB is a derivative of the underlying equity only, and take the interest rate as constant or a deterministic function of time. However, since most CBs have long maturities, it is more realistic to assume stochastic interest rates when pricing such financial instruments. In this work, we considered the pricing of CBs under two-factor models. The Craig-Sneyd time-stepping was employed along with a second-order finite difference scheme for the spatial discretisation. Comparisons against the Crank-Nicolson scheme showed that while both schemes yield the same accuracy, the Craig-Sneyd scheme turns out to be more efficient when the number of spatial grid points is increased. In addition, we also explored how the different features of a CB affect its price and provided some interpretations. Convertible bonds can be valued in a more elaborate setting where the hazard rate is stochastic. The firm's credit spread (either the convertible bond's credit spread or a credit default swap spread) is an indicative measure of the default risk of the firm. Thus, as a further research, a more elaborate convertible bonds valuation model would be a three-factor model with the stock price, the interest rate, and the credit spread being stochastic.

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## Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## References


[1] Batten JA, Khaw KLH, Young MR. Convertible bond pricing models. *Journal of Economic Surveys*. 2014; 28: 775-803.

[2] Brennan MJ, Schwartz ES. Analyzing convertible bonds. *Journal of Financial and Quantitative Analysis*. 1980; 15: 907-929.

[3] Ayache E, Forsyth P, Vetzal K. The valuation of convertible bonds with credit risk. *Journal of Derivatives*. 2003; 11: 9-29.

[4] Zhu SP, Lin S, Lu X. Pricing puttable convertible bonds with integral equation approaches. *Computers and Mathematics with Applications*. 2018; 75(8): 2757-2781.

[5] Lin S, Zhu SP. Pricing callable-puttable convertible bonds with an integral equation approach. *Journal of Futures Markets*. 2022; 42: 1856-1911.

[6] Lin S, Zhu SP. Numerically pricing convertible bonds under stochastic volatility or stochastic interest rate with an ADI-based predictor corrector scheme. *Computers and Mathematics with Applications*. 2020; 79(5): 1393-1419.

[7] Hung M, Wang J. Pricing convertible bonds subject to default risk. *Journal of Derivatives*. 2002; 10(2): 75-87. Available from: doi:10.3905/jod.2002.319197.

[8] Chambers DR, Lu Q. A tree model for pricing convertible bonds with equity, interest rate and default risk. *Journal of Derivatives*. 2007; 14: 25-46.

[9] Barone-Adesi G, Bermudez A, Hatgioannides J. Two-factor convertible bonds valuation using the method of characteristics/finite elements. *Journal of Economic Dynamics and Control*. 2003; 27: 1801-1831.

[10] Yigitbasioglu AB, Alexander C. Pricing and hedging convertible bonds: Delayed calls and uncertain volatility. *International Journal of Theoretical and Applied Finance*. 2006; 9: 415-453.

[11] Kim BJ, Jang BG. Convertible bond valuation with regime switching. *Chaos, Solitons and Fractals*. 2021; 150: 111201.

[12] Batten JA, Khaw KH, Young MR. Pricing convertible bonds. *Journal of Banking and Finance*. 2018; 92: 216-236.

[13] Coonjobeharry RK, Tangman DY, Bhuruth M. A two-factor jump-diffusion model for pricing convertible bonds with default risk. *International Journal of Theoretical and Applied Finance*. 2016; 19(6): 1-26.

[14] Coonjobeharry RK, Tangman DY, Bhuruth M. A novel partial integrodifferen-tial equation-based framework for pricing interest rate derivatives under jump-extended short-rate models. *Journal of Computational Finance*. 2015; 18(4): 129-161.

[15] Thakoor N, Tangman DY, Bhuruth M. RBF-FD schemes for option valuation under models with price-dependent and stochastic volatility. *Engineering Analysis with Boundary Elements*. 2018; 92: 207-217.

[16] Tour G, Thakoor N, Ma J, Tangman DY. A spectral element method for option pricing under regime-switching with jumps. *Journal of Scientific Computing*. 2020; 83: 61. Available from: doi:10.1007/s10915-020-01252-7.

[17] Thakoor N. Localised radial basis functions for no-arbitrage pricing of options under stochastic-alpha-beta-rho dynamics. *The ANZIAM Journal*. 2021; 63(2): 203-227. Available from: doi:10.1017/S1446181121000237.

[18] Narsoo J, Thakoor N, Tangman DY, Bhuruth M. High-order Gaussian RBF-FD methods for real estate index derivatives with stochastic volatility. *Engineering Analysis with Boundary Elements*. 2023; 146: 869-879.

[19] Craig IJD, Sneyd AD. An alternating-direction implicit scheme for parabolic equations with mixed derivatives. *Computers and Mathematics with Applications*. 1988; 16: 341-350.

[20] in't Hout KJ, Welfert BD. Stability of ADI schemes applied to convection-diffusion equations with mixed derivative terms. *Applied Numerical Mathematics*. 2007; 57: 19-35.

[21] in't Hout KJ, Welfert BD. Unconditional stability of second-order ADI schemes applied to multi-dimensional diffusion equations with mixed derivative terms. *Applied Numerical Mathematics*. 2009; 59(3-4): 677-692. Available from: doi:10.1016/j.apnum.2008.03.016.

[22] Craig IJD, Sneyd AD. An alternating-direction implicit scheme for parabolic equations with mixed derivatives. *Computers and Mathematics with Applications*. 1988; 16: 341-350.

[23] Cox JC, Ingersoll JE, Ross SA. A theory of the term structure of interest rates. *Econometrica*. 1985; 53: 385-407.

[24] in't Hout KJ, Foulon S. ADI finite difference schemes for option pricing in the Heston model with correlation. *International Journal of Numerical Analysis and Modeling*. 2010; 7: 303-320.

[25] in 't Hout KJ, Wyns M. Convergence of the Modified Craig-Sneyd scheme for two-dimensional convection-diffusion equations with mixed derivative term. *Journal of Computational and Applied Mathematics*. 2016; 296: 170-180.