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Research Article

Optical Bullets and Domain Walls with Cross Spatio-Dispersion and Having Kudryashov's form of Self-Phase Modulation

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Abstract: In this paper, we investigate the retrieval of optical bullets and domain walls in a novel context, considering the interplay of cross spatio-dispersive effects and employing Kudryashov's proposed form of self-phase modulation. The integration of these two key elements represents a significant advancement in the field of nonlinear optics, enabling the recovery of optical structures with enhanced precision and control. We propose an enhanced Kudryashov's approach, which overcomes the limitations of conventional methods and facilitates the retrieval of optical bullets and domain walls even in the presence of cross spatio-dispersive effects. Kudryashov's self-phase modulation mechanism has proven to be a powerful tool in controlling the nonlinear dynamics of optical systems, offering unique opportunities to manipulate light waves through phase variations induced by the medium's response to intense optical fields. The results presented herein offer new insights into the intricate interplay between self-phase modulation and spatio-dispersive effects in nonlinear optical systems. Moreover, our findings provide a promising avenue for developing novel applications in ultrafast all-optical signal processing, such as the manipulation of light bullets for high-speed data transmission and storage. The enhanced retrieval and control of optical bullets and domain walls are expected to have a profound impact on the advancement of nonlinear optics and its practical implications.

***Keywords*:** bullets, Kudryashov, cross spatio-dispersion, domain walls

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1. Introduction

The key principle of the sustainment of optical solitons, optical dromions, optical bullets and domain walls is the existence of a delicate balance between Chromatic Dispersion (CD) and Self-Phase Modulation (SPM). There has been a deluge of results that have been reported on such forms of pulses which were established based on this fundamental principle [1-10]. The current paper moves a couple of steps further. The governing model, namely the Nonlinear Schrödinger's Equation (NLSE), is extended to (3 + 1)-dimensions and consequently the retrieval of optical bullets is studied in the paper. The CD is along the three spatial directions and has been extended to include the pairwise cross spatio-dispersion effects. Thus, there are six forms of dispersive effects. Next, the SPM effect is also a new form that was proposed by Kudryashov recently and has been studied in various contexts [5]. Therefore, the governing NLSE comes with six dispersive effects along with four forms of SPM coupled together as proposed by Kudryashov.

Two approaches would reveal optical bullets, domain walls and singular soliton solutions in (3 + 1)-dimensions. The first approach is the enhanced Kudryashov's algorithm that would retrieve optical bullets along with other solitons in (3 + 1)-dimensions. The second approach is the well-known G'/G-expansion method that would only reveal domain walls and singular solitons. The details of the two approaches, along with the retrieved results, are exhibited after a quick revisit of the model and a recapitulation of the mathematical principles.

The paper introduces several core contributions and unique innovations in the study of optical solitons, optical dromions, optical bullets, and domain walls. These contributions can be summarized as follows: Extension to (3 + 1)-dimensions, comprehensive dispersion model, novel Self-Phase Modulation (SPM) effect, coupling of dispersion and SPM effects, enhanced Kudryashov's algorithm, G'/G-expansion method and novel optical structures. The paper's core contributions lie in extending the NLSE to (3 + 1)-dimensions with a comprehensive dispersion model, introducing a novel SPM effect, and employing two distinct mathematical approaches to reveal and understand various optical structures. These findings advance the understanding of light propagation in complex media and have potential implications for optical communication, signal processing, and other relevant fields.

The organization of the article is follows. In subsection 1.1, the governing model, the Nonlinear Schrödinger's Equation (NLSE), is presented, extended to (3 + 1)-dimensions. The comprehensive dispersion model, including CD along three spatial directions and pairwise cross spatio-dispersion effects, is described. The new form of SPM proposed by Kudryashov and its inclusion in the NLSE are introduced. In section 2, a recapitulation of the mathematical principles underlying the NLSE is provided. The groundwork for the subsequent sections is set by establishing the theoretical framework. In section 3, the enhanced version of Kudryashov's algorithm is explained. How the enhanced algorithm retrieves optical bullets and other solitons in (3 + 1)-dimensions is described. In section 4, the application of the enhanced Kudryashov's algorithm to reveal and analyze optical bullets and other solitons in (3 + 1)-dimensions is demonstrated. The results obtained using this approach are presented. Also, the utilization of the G'/G-expansion method is detailed. Its effectiveness in revealing domain walls and singular solitons in the extended NLSE is investigated. In section 5, the key findings and contributions of the paper are summarized. The significance of the results obtained from both the enhanced Kudryashov's approach and the G'/G-expansion technique is discussed. Potential applications and future research directions are pointed out.

1.1 *Governing equation*

Within the scope of this work, we explore the extended (3 + 1)-dimensional NLSE, which is structured based on Kudryashov's form of SPM, as described below [1, 2, 3, 9]

$$iq_t - \left(a_1 q_{xx} + a_2 q_{yy} + a_3 q_{zz} + 2a_4 q_{xy} - 2a_5 q_{xz} - 2a_6 q_{yz}\right) + \left(\frac{b_1}{|q|^{2n}} + \frac{b_2}{|q|^n} + b_3 |q|^n + b_4 |q|^{2n}\right) q = 0. \tag{1}$$

The coefficients a_i, (i = 1, 2, 3, 4, 5, 6), and b_j, (j = 1, 2) are constants. When $n = 2$ and both b_1 and b_2 equal zero, the extended (3 + 1)-dimensional NLSE is observed, demonstrating parabolic law nonlinearity. When $a_2 = a_3 = a_4 = a_5 = a_6 = 0$, we arrive at Kudryashov's equation. That being said, if $b_1 = b_2 = 0$, the dual-power law nonlinearity manifests itself in the extended (3 + 1)-dimensional NLSE. Both the field envelope, given as $q = q(x, y, z, t)$, and the chromatic dispersion, represented by a_1, are included in this equation. This equation's linear temporal evolution of the field comes from the first term, with $i = \sqrt{-1}$. The temporal variable is given by t, while the spatial variables are represented by x, y, and z. Lastly, a_2, a_3, a_4, a_5, and a_6 come from the cross spatio-dispersion.

2. Preliminary analysis

The chosen wave variable to solve equation (1) is as follows:

$$q(x, y, z, t) = U(\xi)e^{i\Phi(x, y, z, t)}. \tag{2}$$

The shape of the pulse is denoted by $U(\xi)$, where

$$\xi = k\left(B_1x + B_2y + B_3z - vt\right), \tag{3}$$

and

$$\Phi\left(x, y, z, t\right) = -\kappa_1x - \kappa_2y - \kappa_3z + \omega t + \theta. \tag{4}$$

Along the x, y, and z directions, the wave numbers are represented by κ_j for j = 1, 2, 3, respectively. Additionally, θ represents the phase constant and ω represents the frequency. In the context of an optical bullet, B_j represents the direction ratios that describe the widths along the three spatial components. By inserting equations (2), (3), and (4) into equation (1) and splitting it into real and imaginary parts, we arrive at:

$$v = 2B_1\left(a_1\kappa_1 + a_4\kappa_2 - a_5\kappa_3\right) + 2B_2\left(a_2\kappa_2 + a_4\kappa_1 - a_6\kappa_3\right) + 2B_3\left(a_3\kappa_3 - a_5\kappa_1 - a_6\kappa_2\right) \tag{5}$$

and

$$-Ak^2U'' + \left(-\omega + B\right)U + b_1U^{1-2n} + b_2U^{1-n} + b_3U^{1+n} + b_4U^{1+2n} = 0, \tag{6}$$

where

$$A = a_1B_1^2 + a_2B_2^2 + a_3B_3^2 + 2a_4B_1B_2 - 2a_5B_1B_3 - 2a_6B_2B_3, \tag{7}$$

and

$$B = a_1\kappa_1^2 + a_2\kappa_2^2 + a_3\kappa_3^2 + 2a_4\kappa_1\kappa_2 - 2a_5\kappa_1\kappa_3 - 2a_6\kappa_2\kappa_3. \tag{8}$$

Applying the restriction $U(\xi) = \left\{V(\xi)\right\}^{\frac{1}{n}}$ to equation (6) yields the analytic solution:

$$-nAk^2VV'' - Ak^2(1-n)\left(V'\right)^2 + n^2b_1 + n^2b_2V + n^2\left(-\omega + B\right)V^2 + n^2b_3V^3 + n^2b_4V^4 = 0. \tag{9}$$

3. Enhanced Kudryashov's approach: A succinct overview

Suppose we have a model equation:

$$F(u, u_x, u_y, u_z, u_t, u_{xt}, u_{xx}, ...) = 0. \tag{10}$$

The wave profile is represented by the function $u = u(x, y, z, t)$, where t represents time and x, y, z represent the spatial variables, respectively.

The wave profile

$$u(x, y, z, t) = U(\xi), \ \ \xi = k(x + y + z - \upsilon t), \tag{11}$$

condenses equation (10) to

$$P(U, -k\upsilon U', kU', k^2U'', ...) = 0. \tag{12}$$

In this equation, the wave velocity is indicated by υ, the wave variable is represented by ξ, and the wave width is denoted by k.

The basic procedure of the enhanced Kudryashov's technique is outlined in the following steps:

Step 1: We can find the explicit solution for the reduced model (12) as follows:

$$U(\xi) = \alpha_0 + \sum_{l=1}^{N} \left\{ \alpha_l R(\xi)^l + \beta_l \left(\frac{R'(\xi)}{R(\xi)^l} \right) \right\}, \tag{13}$$

along with the auxiliary equation

$$R'(\xi)^2 = R(\xi)^2 \left(1 - \chi R(\xi)^2\right). \tag{14}$$

The constants α_0, χ, α_l, and β_l (l = 1, ..., N) are involved with the value of N obtained through the balancing technique described in equation (13).

Step 2: Equation (14) gives the soliton wave

$$R(\xi) = \frac{4c}{4c^2 e^{\xi} + \chi e^{-\xi}}. \tag{15}$$

In this context, c remains constant.

Step 3: Inserting equation (13) into equation (12) along with equation (14) gives us the constants we need in equations (11) and (13). Finally, the acquired parametric restrictions can be applied by plugging them into equation (13) along with equation (15). Thus, the resulting solitons are positioned in between, and they can be transformed into bullets, domain walls, or singular solitons.

4. Optical bullets and domain walls

The paper investigates the retrieval of optical bullets and domain walls. Optical bullets are localized, self-guided beams of light that maintain their shape during propagation, while domain walls are interfaces separating different regions of light intensity or phase. The retrieval of these structures is crucial for various applications in ultrafast all-optical signal processing and high-speed data transmission. This sectin will be dedicated to the derivation of optical bullets and domain wall solutions to the governing model given by equation (1) using two approaches.

4.1 *Enhanced Kudryashov's approach*

The implementation of the balance method between VV'' and V^4 in equation (9) allows us $N = 1$, resulting in the following expression for the solution:

$$V(\xi) = \alpha_0 + \alpha_1 R(\xi) + \beta_1 \frac{R'(\xi)}{R(\xi)}. \tag{16}$$

By inserting the expression from equation (16) into equation (9) alongside equation (14), we obtain the subsequent set of algebraic equations:

$$\alpha_1^2 \chi \left\{ Ak^2 (n+1) - 6b_4 \beta_1^2 n^2 \right\} + \beta_1^2 \chi^2 \left\{ b_4 \beta_1^2 n^2 - Ak^2 (n+1) \right\} + \alpha_1^4 b_4 n^2 = 0, \tag{17}$$

$$\alpha_1 \left[2\alpha_0 \left\{ Ak^2 \chi + 2b_4 n \left(\alpha_1^2 - 3\beta_1^2 \chi \right) \right\} + b_3 n \left(\alpha_1^2 - 3\beta_1^2 \chi \right) \right] = 0, \tag{18}$$

$$2\alpha_1 \beta_1 \left\{ Ak^2 (n+1) \chi + 2b_4 n^2 \left(\alpha_1^2 - \beta_1^2 \chi \right) \right\} = 0, \tag{19}$$

$$\alpha_1^2 \left\{ -Ak^2 + 6b_4 n^2 \left(\alpha_0^2 + \beta_1^2 \right) + 3\alpha_0 b_3 n^2 + Bn^2 - n^2 \omega \right\}$$

$$-\beta_1^2 n \chi \left\{ -2Ak^2 + 2b_4 n \left(3\alpha_0^2 + \beta_1^2 \right) + 3\alpha_0 b_3 n + Bn - n\omega \right\} = 0, \tag{20}$$

$$\beta_1 \left\{ 2\alpha_0 \left(Ak^2 \chi + b_4 \left(6\alpha_1^2 n - 2\beta_1^2 n \chi \right) \right) + b_3 n \left(3\alpha_1^2 - \beta_1^2 \chi \right) \right\} = 0, \tag{21}$$

$$\alpha_1 \left\{ \alpha_0 \left(-Ak^2 + 12b_4 \beta_1^2 n + 2Bn - 2n\omega \right) + 4\alpha_0^3 b_4 n + 3\alpha_0^2 b_3 n + 3b_3 \beta_1^2 n + b_2 n \right\} = 0, \tag{22}$$

$$\beta_1 \left\{ 4\alpha_0^3 b_4 + 3\alpha_0^2 b_3 + b_3 \beta_1^2 + 2\alpha_0 \left(2b_4 \beta_1^2 + B - \omega \right) + b_2 \right\} = 0, \tag{23}$$

$$\alpha_1 \beta_1 \left\{ -Ak^2 + 4b_4 n \left(3\alpha_0^2 + \beta_1^2 \right) + 6\alpha_0 b_3 n + 2Bn - 2n\omega \right\} = 0, \tag{24}$$

$$-\alpha_0^2 \omega + 6\alpha_0^2 b_4 \beta_1^2 + 3\alpha_0 b_3 \beta_1^2 + \alpha_0^4 b_4 + \alpha_0^3 b_3 + \alpha_0 b_2 + b_4 \beta_1^4 - \beta_1^2 \omega + b_1 + \alpha_0^2 B + \beta_1^2 B = 0. \tag{25}$$

Solving these equations together yields the following results:

Result 1:

$$\alpha_0 = -\frac{b_3 (n+1)}{2b_4 (n+2)},\ \beta_1 = 0,\ k = \frac{n}{\sqrt{-A(n+1)}} \sqrt{\frac{16 b_1 b_4^3 (n+2)^4 + b_3^4 (n+1)^2}{2 b_3^2 b_4 (n+2)^2}},$$

$$\omega = -\frac{5b_3^2(n+1)}{4b_4(n+2)^2} + \frac{4b_1b_4^2(n+2)^2}{b_3^2(n-1)(n+1)^2} + B,\ \alpha_1 = \sqrt{\frac{b_3^4(n+1)^2\chi + 16b_1b_4^3(n+2)^4}{2b_3^2b_4^2(n+2)^2}},$$

$$b_2 = -\frac{(n-2)\left(b_3^4(n-1)(n+1)^3 - 16b_1b_4^3(n+2)^4\right)}{8b_3b_4^2(n+2)^3\left(n^2-1\right)}. \tag{26}$$

Plugging the obtained parameters in equation (26) with equation (15) into equation (16), as a consequence, we get

$$q(x,y,z,t) = \left\{ -\frac{b_3(n+1)}{2b_4(n+2)} + \frac{4c\sqrt{\frac{b_3^4(n+1)^2\chi + 16b_1b_4^3(n+2)^4}{2b_3^2b_4^2(n+2)^2}}\, e^{\sqrt{\frac{16n^2b_1b_4^3(n+2)^4 + b_3^4n^2(n+1)^2}{-2b_3^2b_4A(n+1)(n+2)^2}}(B_1x+B_2y+B_3z-vt)}}{4c^2e^{2\sqrt{\frac{16n^2b_1b_4^3(n+2)^4 + b_3^4n^2(n+1)^2}{-2b_3^2b_4A(n+1)(n+2)^2}}(B_1x+B_2y+B_3z-vt)} + \chi} \right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\kappa_1x - \kappa_2y - \kappa_3z + \left\{-\frac{5b_3^2(n+1)}{4b_4(n+2)^2} + \frac{4b_1b_4^2(n+2)^2}{b_3^2(n-1)(n+1)^2} + B\right\}t + \theta\right)}. \tag{27}$$

Setting $\chi = \pm 4c^2$, leads to optical bullets with $b_1b_4 > 0$, and $A < 0$:

$$q(x,y,z,t) = \left\{ -\frac{b_3(n+1)}{2b_4(n+2)} + \sqrt{\frac{b_3^4c^2(n+1)^2 + 4b_1b_4^3(n+2)^4}{2b_3^2b_4^2c^2(n+2)^2}} \right.$$

$$\left. \times \operatorname{sech}\left[\sqrt{-\frac{16b_1b_4^3n^2(n+2)^4 + b_3^4n^2(n+1)^2}{2Ab_3^2b_4(n+1)(n+2)^2}}\left(B_1x + B_2y + B_3z - vt\right)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\kappa_1x - \kappa_2y - \kappa_3z + \left\{-\frac{5b_3^2(n+1)}{4b_4(n+2)^2} + \frac{4b_1b_4^2(n+2)^2}{b_3^2(n-1)(n+1)^2} + B\right\}t + \theta\right)}, \tag{28}$$

and singular soliton solution with $b_1b_4 > 0$, $A < 0$, and $c^2 < 4b_1b_4^3b_3^4$:

$$q(x,y,z,t) = \left\{ -\frac{b_3(n+1)}{2b_4(n+2)} + \sqrt{\frac{-b_3^4c^2(n+1)^2 + 4b_1b_4^3(n+2)^4}{2b_3^2b_4^2c^2(n+2)^2}} \right.$$

$$\left. \times \operatorname{csch}\left[\sqrt{-\frac{16b_1b_4^3n^2(n+2)^4 + b_3^4n^2(n+1)^2}{2Ab_3^2b_4(n+1)(n+2)^2}}\left(B_1x + B_2y + B_3z - vt\right)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\kappa_1x - \kappa_2y - \kappa_3z + \left\{-\frac{5b_3^2(n+1)}{4b_4(n+2)^2} + \frac{4b_1b_4^2(n+2)^2}{b_3^2(n-1)(n+1)^2} + B\right\}t + \theta\right)}. \tag{29}$$

Optical bullets, also known as light bullets or optical wave bullets, are localized and self-guided light structures that can maintain their shape and intensity during propagation in certain nonlinear optical media. They are distinct from solitons in that they can have complex temporal and spatial profiles, often with multiple intensity peaks. Optical bullets have several important physical interpretations and properties. Optical bullets are fascinating and important phenomena in nonlinear optics. Their complex spatio-temporal profiles, stability, and ability to withstand dispersion make them valuable for various applications in high-speed communication, all-optical signal processing, and fundamental studies in nonlinear optics.

Singular soliton solutions are solitary wave solutions that exhibit singular behavior at one or more points. These solutions are relevant to various nonlinear systems and can provide insights into the dynamics and behavior of nonlinear waves with singularities. Understanding the properties and implications of singular solitons is an active area of research in the study of nonlinear partial differential equations and their applications in physics and engineering. In the context of optics, singular soliton solutions have significant implications for the behavior of light waves in certain nonlinear optical systems. Optical systems are highly relevant to the study of solitons due to the presence of various nonlinear effects that can lead to the formation of solitary wave solutions. Singular soliton solutions in optics can be found in different physical settings, such as fiber optics, nonlinear waveguides, and photonic crystals.

Result 2:

$$\alpha_0 = -\frac{b_3(n+1)}{2b_4(n+2)},\ \alpha_1 = 0,\ k = \sqrt{\frac{4b_4 n^2(n+2)^2\sqrt{b_1 b_4\left(1-n^2\right)} + b_3^2(n-1)n^2(n+1)^2}{4Ab_4^2\left(n^2-1\right)(n+2)^2}},$$

$$\beta_1 = \frac{1}{2}\sqrt{\frac{4b_4(n+2)^2\sqrt{b_1 b_4\left(1-n^2\right)} + b_3^2(n-1)(n+1)^2}{b_4^2(n-1)(n+2)^2}},$$

$$\omega = \frac{2b_1 b_4}{\sqrt{b_1 b_4\left(1-n^2\right)}} - \frac{b_3^2(n+1)}{b_4(n+2)^2} + B,\ b_2 = \frac{b_3(2-n)\sqrt{b_1 b_4^7(n+2)^8\left(1-n^2\right)}}{b_4^4(n-1)(n+2)^5}. \tag{30}$$

Plugging the obtained parameters in equation (30) with equation (15) into equation (16), as a consequence, we get

$$q(x,y,z,t) = \left\{-\frac{b_3(n+1)}{2b_4(n+2)} + \frac{1}{2}\sqrt{\frac{4b_4(n+2)^2\sqrt{b_1 b_4\left(1-n^2\right)} + b_3^2(n-1)(n+1)^2}{b_4^2(n-1)(n+2)^2}}\right.$$

$$\left.\times\left(\frac{\chi - 4c^2 e^{2\sqrt{\frac{4b_4 n^2(n+2)^2\sqrt{b_1 b_4\left(1-n^2\right)}+b_3^2(n-1)n^2(n+1)^2}{4Ab_4^2\left(n^2-1\right)(n+2)^2}}\left(B_1 x+B_2 y+B_3 z-vt\right)}}{4c^2 e^{2\sqrt{\frac{4b_4 n^2(n+2)^2\sqrt{b_1 b_4\left(1-n^2\right)}+b_3^2(n-1)n^2(n+1)^2}{4Ab_4^2\left(n^2-1\right)(n+2)^2}}\left(B_1 x+B_2 y+B_3 z-vt\right)} + \chi}\right)\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\kappa_1 x-\kappa_2 y-\kappa_3 z+\left\{\frac{2b_1 b_4}{\sqrt{b_1 b_4\left(1-n^2\right)}}-\frac{b_3^2(n+1)}{b_4(n+2)^2}+B\right\}t+\theta\right)}. \tag{31}$$

Setting $\chi = \pm 4c^2$, leads to domain walls and singular soliton solutions with A, $b_4 > 0$, $b_1 < 0$ and $n > 1$ as:

$$q(x,y,z,t) = \left\{ -\frac{b_3(n+1)}{2b_4(n+2)} - \frac{1}{2}\sqrt{\frac{4b_4(n+2)^2\sqrt{b_1 b_4\left(1-n^2\right)} + b_3^2(n-1)(n+1)^2}{b_4^2(n-1)(n+2)^2}} \right.$$

$$\left. \times \tanh\left[\sqrt{\frac{4b_4 n^2(n+2)^2\sqrt{b_1 b_4\left(1-n^2\right)} + b_3^2(n-1)n^2(n+1)^2}{4Ab_4^2\left(n^2-1\right)(n+2)^2}}\left(B_1 x + B_2 y + B_3 z - vt\right)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\kappa_1 x - \kappa_2 y - \kappa_3 z + \left\{\frac{2b_1 b_4}{\sqrt{b_1 b_4\left(1-n^2\right)}} - \frac{b_3^2(n+1)}{b_4(n+2)^2} + B\right\}t + \theta\right)}, \quad (32)$$

and

$$q(x,y,z,t) = \left\{ -\frac{b_3(n+1)}{2b_4(n+2)} - \frac{1}{2}\sqrt{\frac{4b_4(n+2)^2\sqrt{b_1 b_4\left(1-n^2\right)} + b_3^2(n-1)(n+1)^2}{b_4^2(n-1)(n+2)^2}} \right.$$

$$\left. \times \coth\left[\sqrt{\frac{4b_4 n^2(n+2)^2\sqrt{b_1 b_4\left(1-n^2\right)} + b_3^2(n-1)n^2(n+1)^2}{4Ab_4^2\left(n^2-1\right)(n+2)^2}}\left(B_1 x + B_2 y + B_3 z - vt\right)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\kappa_1 x - \kappa_2 y - \kappa_3 z + \left\{\frac{2b_1 b_4}{\sqrt{b_1 b_4\left(1-n^2\right)}} - \frac{b_3^2(n+1)}{b_4(n+2)^2} + B\right\}t + \theta\right)}, \quad (33)$$

respectively.

Domain walls refer to interfaces that separate different regions of light intensity or phase in the optical field. Domain walls arise in nonlinear optical systems due to the interplay of self-phase modulation and spatio-dispersive effects. Domain walls represent intriguing interfaces that form due to nonlinear optical effects in certain media. Their presence can significantly influence the dynamics of optical bullets and other localized structures, offering opportunities for applications in all-optical signal processing and the manipulation of light waves in nonlinear optical systems.

Result 3:

$$\alpha_0 = -\frac{b_3(n+1)}{2b_4(n+2)},\ \alpha_1 = \beta_1\sqrt{-\chi},\ k = \sqrt{\frac{4b_4 n^2(n+2)^2\sqrt{b_1 b_4\left(1-n^2\right)} + b_3^2(n-1)n^2(n+1)^2}{Ab_4^2(n+1)(n-1)(n+2)^2}},$$

$$\beta_1 = \frac{1}{2}\sqrt{\frac{4b_4(n+2)^2\sqrt{b_1 b_4\left(1-n^2\right)} + b_3^2(n-1)(n+1)^2}{b_4^2(n-1)(n+2)^2}},$$

$$\omega = \frac{2b_1 b_4}{\sqrt{b_1 b_4 \left(1-n^2\right)}} - \frac{b_3^2 (n+1)}{b_4 (n+2)^2} + B,\ b_2 = \frac{b_3 (n-2)\sqrt{b_1 b_4 \left(1-n^2\right)}}{b_4 (n-1)(n+2)}. \tag{34}$$

Plugging the obtained parameters in equation (34) with equation (15) into equation (16), as a consequence, we get

$$q(x,y,z,t) = \left\{ -\frac{b_3 (n+1)}{2b_4 (n+2)} + \frac{1}{2}\sqrt{\frac{4b_4 (n+2)^2 \sqrt{b_1 b_4 \left(1-n^2\right)} + b_3^2 (n-1)(n+1)^2}{b_4^2 (n-1)(n+2)^2}} \right.$$

$$\times \left(\frac{4c\sqrt{-\chi}\, e^{\sqrt{\frac{4b_4 n^2 (n+2)^2 \sqrt{b_1 b_4 \left(1-n^2\right)} + b_3^2 (n-1) n^2 (n+1)^2}{A b_4^2 (n+1)(n-1)(n+2)^2}} \left(B_1 x + B_2 y + B_3 z - tv\right)}}{4c^2 e^{2\sqrt{\frac{4b_4 n^2 (n+2)^2 \sqrt{b_1 b_4 \left(1-n^2\right)} + b_3^2 (n-1) n^2 (n+1)^2}{A b_4^2 (n+1)(n-1)(n+2)^2}} \left(B_1 x + B_2 y + B_3 z - tv\right)} + \chi} \right.$$

$$\left. \left. + \frac{\chi - 4c^2 e^{2\sqrt{\frac{4b_4 n^2 (n+2)^2 \sqrt{b_1 b_4 \left(1-n^2\right)} + b_3^2 (n-1) n^2 (n+1)^2}{A b_4^2 (n+1)(n-1)(n+2)^2}} \left(B_1 x + B_2 y + B_3 z - tv\right)}}{4c^2 e^{2\sqrt{\frac{4b_4 n^2 (n+2)^2 \sqrt{b_1 b_4 \left(1-n^2\right)} + b_3^2 (n-1) n^2 (n+1)^2}{A b_4^2 (n+1)(n-1)(n+2)^2}} \left(B_1 x + B_2 y + B_3 z - tv\right)} + \chi} \right) \right\}^{\frac{1}{n}}$$

$$\times\, e^{i\left(-\kappa_1 x - \kappa_2 y - \kappa_3 z + \left\{\frac{2b_1 b_4}{\sqrt{b_1 b_4 \left(1-n^2\right)}} - \frac{b_3^2 (n+1)}{b_4 (n+2)^2} + B\right\} t + \theta\right)}. \tag{35}$$

Setting $\chi = \pm 4c^2$, leads to complexiton and domain walls with A, $b_4 > 0$, $b_1 < 0$ and $n > 1$ as:

$$q(x,y,z,t) = \left\{ -\frac{b_3 (n+1)}{2b_4 (n+2)} - \frac{1}{2}\sqrt{\frac{4b_4 (n+2)^2 \sqrt{b_1 b_4 \left(1-n^2\right)} + b_3^2 (n-1)(n+1)^2}{b_4^2 (n-1)(n+2)^2}} \right.$$

$$\times \left(\tanh\left[\sqrt{\frac{4b_4 n^2 (n+2)^2 \sqrt{b_1 b_4 \left(1-n^2\right)} + b_3^2 (n-1) n^2 (n+1)^2}{A b_4^2 (n+1)(n-1)(n+2)^2}} \left(B_1 x + B_2 y + B_3 z - vt\right) \right] \right.$$

$$\left. \left. - i\,\mathrm{sech}\left[\sqrt{\frac{4b_4 n^2 (n+2)^2 \sqrt{b_1 b_4 \left(1-n^2\right)} + b_3^2 (n-1) n^2 (n+1)^2}{A b_4^2 (n+1)(n-1)(n+2)^2}} \left(B_1 x + B_2 y + B_3 z - vt\right) \right] \right) \right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\kappa_1 x-\kappa_2 y-\kappa_3 z+\left\{\frac{2b_1 b_4}{\sqrt{b_1 b_4\left(1-n^2\right)}}-\frac{b_3^2(n+1)}{b_4(n+2)^2}+B\right\}t+\theta\right)}, \tag{36}$$

and

$$q(x,y,z,t)=\left\{-\frac{b_3(n+1)}{2b_4(n+2)}-\frac{1}{2}\sqrt{\frac{4b_4(n+2)^2\sqrt{b_1 b_4\left(1-n^2\right)}+b_3^2(n-1)(n+1)^2}{b_4^2(n-1)(n+2)^2}}\right.$$

$$\left.\times\tanh\left[\frac{1}{2}\sqrt{\frac{4b_4 n^2(n+2)^2\sqrt{b_1 b_4\left(1-n^2\right)}+b_3^2(n-1)n^2(n+1)^2}{Ab_4^2(n+1)(n-1)(n+2)^2}}\left(B_1 x+B_2 y+B_3 z-tv\right)\right]\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\kappa_1 x-\kappa_2 y-\kappa_3 z+\left\{\frac{2b_1 b_4}{\sqrt{b_1 b_4\left(1-n^2\right)}}-\frac{b_3^2(n+1)}{b_4(n+2)^2}+B\right\}t+\theta\right)}, \tag{37}$$

respectively.

4.2 *G′/G-Expansion technique*

The technique [4] will be extensively explained in this section, showcasing its application in obtaining domain walls and singular soliton solutions for equation (1). Following the principles of the homogeneous balance method, equation (9) is found to have a solution represented by:

$$V(\xi)=A_0+A_1\left(\frac{G'(\xi)}{G(\xi)}\right). \tag{38}$$

Here the function $G(\xi)$ is governed by a second-order linear ODE:

$$G''(\xi)+\lambda G'(\xi)+\mu G(\xi)=0, \tag{39}$$

The determination of the real constants λ and μ is necessary to obtain the exact solutions. For this purpose, we assume:

$$b_1=\frac{(1-n)(n+2)^2 b_2^2 b_4}{(n-2)^2(n+1)b_3^2},\ k=1. \tag{40}$$

Upon substituting equations (38) and (39) into equation (9), and setting the coefficients of each power of G'/G to zero, we solve a system of nonlinear algebraic equations, resulting in:

$$A_0=\pm\frac{\lambda}{2n}\sqrt{\frac{A(n+1)}{b_4}}-\frac{(n+1)b_3}{2(n+2)b_4}, \tag{41}$$

$$A_1 = \pm\frac{1}{n}\sqrt{\frac{A(n+1)}{b_4}}, \tag{42}$$

$$\mu = \frac{n^2(n+1)^2(2-n)b_3^3 - 4n^2(n+2)^3 b_2 b_4^2 + (n+2)^2(n+1)(n-2)A\lambda^2 b_3 b_4}{4Ab_3 b_4(n+1)(n+2)^2(n-2)}, \tag{43}$$

$$\omega = \frac{(1+n)^2(2-n)b_3^3 + (1+n)(2+n)^2(n-2)Bb_3b_4 + 2(n+2)^3 b_2 b_4^2}{(1+n)(2+n)^2(n-2)b_3 b_4}. \tag{44}$$

Here λ is a constant parameter that is not restricted and can take any arbitrary value.

Upon the substitution of equations (41) and (42) into equation (38), the solution formulae of equation (9) takes the following form:

$$V(\xi) = \pm\frac{\lambda}{2n}\sqrt{\frac{A(n+1)}{b_4}} - \frac{(n+1)b_3}{2(n+2)b_4} \pm \frac{1}{n}\sqrt{\frac{A(n+1)}{b_4}}\left(\frac{G'(\xi)}{G(\xi)}\right). \tag{45}$$

Through the substitution of the general solutions of the second-order linear ODE into equation (45), we derive the traveling wave solutions. If $\Delta = \lambda^2 - 4\mu > 0$, the resulting solution can be described as a hyperbolic function traveling wave, expressed as:

$$q(x, y, z, t) = \left\{\pm\frac{\lambda}{2n}\sqrt{\frac{A(n+1)}{b_4}} - \frac{(n+1)b_3}{2(n+2)b_4}\right.$$

$$\left.\pm\frac{1}{2n}\sqrt{\frac{A(n+1)\left(\lambda^2-4\mu\right)}{b_4}}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right)\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\kappa_1 x-\kappa_2 y-\kappa_3 z+\left\{\frac{(1+n)^2(2-n)b_3^3+(1+n)(2+n)^2(n-2)Bb_3b_4+2(n+2)^3 b_2 b_4^2}{(1+n)(2+n)^2(n-2)b_3b_4}\right\}t+\theta\right)}, \tag{46}$$

where

$$\lambda^2 - 4\mu = \frac{4n^2(n+2)^3 b_2 b_4^2 - n^2(n+1)^2(2-n)b_3^3}{Ab_3b_4(n+1)(n+2)^2(n-2)}. \tag{47}$$

However, in the case where $C_1 \neq 0$ and $C_2 = 0$, the solution of equation (1) takes the form of a domain wall, which can be represented by

$$q(x, y, z, t) = \left\{\pm\frac{\lambda}{2n}\sqrt{\frac{A(n+1)}{b_4}} - \frac{(n+1)b_3}{2(n+2)b_4}\right.$$

$$\left.\pm\frac{1}{2n}\sqrt{\frac{A(n+1)\left(\lambda^2-4\mu\right)}{b_4}}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(B_1x+B_2y+B_3z-vt\right)\right)\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\kappa_1x-\kappa_2y-\kappa_3z+\left\{\frac{(1+n)^2(2-n)b_3^3+(1+n)(2+n)^2(n-2)Bb_3b_4+2(n+2)^3b_2b_4^2}{(1+n)(2+n)^2(n-2)b_3b_4}\right\}t+\theta\right)}. \tag{48}$$

Next, if we assume $C_1 = 0$ and $C_2 \neq 0$, the solution to equation (1) takes the form of a singular 1-soliton, which can be expressed as

$$q(x,y,z,t)=\left\{\pm\frac{\lambda}{2n}\sqrt{\frac{A(n+1)}{b_4}}-\frac{(n+1)b_3}{2(n+2)b_4}\right.$$

$$\left.\pm\frac{1}{2n}\sqrt{\frac{A(n+1)\left(\lambda^2-4\mu\right)}{b_4}}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(B_1x+B_2y+B_3z-vt\right)\right)\right\}^{\frac{1}{n}}$$

$$\times e^{i\left(-\kappa_1x-\kappa_2y-\kappa_3z+\left\{\frac{(1+n)^2(2-n)b_3^3+(1+n)(2+n)^2(n-2)Bb_3b_4+2(n+2)^3b_2b_4^2}{(1+n)(2+n)^2(n-2)b_3b_4}\right\}t+\theta\right)}. \tag{49}$$

5. Conclusions

The paper recovered optical bullets to the governing NLSE that is with CD and cross spatio-dispersive effects while the SPM is the one proposed by Kudryashov having quadrupled forms of power-law. The bullets are derived and classified as well. These novel and interesting results are of prime importance in quantum optics, which has led the way to the advancement of modern telecommunication engineering sciences.

This paper introduces several novel contributions in the field of nonlinear optics, particularly in the retrieval and control of optical bullets and domain walls. A comprehensive overview of the main contributions introduced in the paper can be given as: Retrieval of optical bullets and domain walls, consideration of cross spatio-dispersive effects, integration of Kudryashov's proposed form of self-phase modulation, overcoming limitations of conventional methods, insights into the interplay between self-phase modulation and spatio-dispersive effects. The current paper's novel contributions in integrating cross spatio-dispersive effects and Kudryashov's self-phase modulation, along with their applications in the retrieval and control of optical bullets and domain walls, significantly advance the field of nonlinear optics and offer exciting prospects for practical applications in all-optical signal processing and high-speed data transmission.

The results are next going to be used to address additional advanced topics such as the model can be extended with the presence of perturbation terms, both deterministic and stochastic. Additionally, the model needs to be addressed with fractional temporal evolution. Such projects are on the table and the results of those research activities will be soon made visible to the knowledge-hungry scientists. The novel results from additional integration approaches, as well as other forms of analysis, would be disseminated all across that would align with the pre-existing approaches [11-16].

Conflict of interest

The authors declare that there is no conflict of interest.

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