Research Article

**Numerical Solutions of Fuzzy Differential Equations by Harmonic Mean and Cubic Mean of Modified Euler’s Method**

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Received: 18 July 2023; Revised: 15 August 2023; Accepted: 4 September 2023

**Abstract:** We aimed to solve first-order differential equations using two novel techniques: the harmonic mean and the cubic mean of Euler’s modified approach for fuzzy primary value in this research proposal. We present a new formulation of Euler’s classic approach based on Zadeh’s extension concept to address this dependency issue in a fuzzy situation. In the literature, numerical approaches for solving differential equations with fuzzy main values often disregard this issue. With a few examples, we show how our approach outperforms more traditional fuzzy approaches based on Euler’s method.

**Keywords:** modified Euler’s method, harmonic mean method, cubic mean method, differential equation, fuzzy initial value, fuzzy solutions

**MSC:** 34A07

**1. Introduction**

Differential equations play an important role in modern life, just as they do in science and engineering. This proposed research solves the ordinary differential equation (ODE) problems \[1\]. The tangent line method is another name for the Euler’s method, and it is a simple way to solve first order differential equation (FODE) problems \[2-5\]. Leonhard Euler \[6\] discovered a method for rapid training, simple implementation, and low computational cost. However, the accuracy factor leads the investigator to use a new intricate technique to replace the Euler method. Ultimately, the investigators of this study hope to improve the accuracy factor \[7\] and determine the exact solution by implementing the various mean values. Thus, changes to the current Euler method result in the Modified Euler’s method \[8-9\]. The modification method is used to determine the average gradients. The proposed method is intended to be an improvement on Euler’s method, incorporating the harmonic mean and cubic mean.

In recent years, the uncertainty in fuzzy differential equations has become increasingly important in fuzzy analysis. The terms “fuzzy differential equation” \[10\], “fuzzy differential and integral equations” \[11\], “fuzzy differential inclusion” \[12\], and “fuzzy differential inclusion” \[13\] are used interchangeably because the differential equations with...
fuzzy initial values [14] or fuzzy boundary values are calculated alongside the functions for fuzzy numbers (Seikkala [15], Abbasbandy [16] and Osmo [17]).

The Zadeh’s extension principle is most commonly used in fuzzy set theory and fuzzy set operations. While fuzzy logic and fuzzy control systems are not directly used for solving differential equations, they can be applied to differential equations in some cases. Fuzzy control systems, which use fuzzy logic and reasoning to design controllers for systems described by differential equations, can be used to design controllers for systems described by differential equations. These systems may be characterized by uncertainty, imprecise inputs, or complex relationships that are difficult to accurately model using traditional methods.

In such cases, the fuzzy control system can use fuzzy sets and rules to map input variables (which may be related to the state of the system described by differential equations) to output actions, thereby creating a control strategy that accounts for the problem’s imprecision and uncertainty. While the Zadeh’s extension principle does not directly solve differential equations, it does serve as the foundation for fuzzy logic and control, which can be applied to a wide range of engineering and scientific problems, including differential equations. In cases where precise mathematical models may be difficult to establish, the key is to use fuzzy logic to manage uncertainty and approximate solutions.

In this paper, we discuss the following topics: Section 2 determines some basic terms for triangular fuzzy numbers and fuzzy derivatives [17], as well as the results. In Section 3, we propose two modified Euler’s methods for solving the first order fuzzy differential equation with a fuzzy initial value condition. Section 4 illustrates and solves a numerical example using the proposed method. Furthermore, the results of the proposed methods’ approximation solutions are compared to the other Euler methods with different step sizes of h.

2. Preliminaries

In this section we will discuss the completion of Harmonic mean and Cubic mean of Modified Euler’s [18-19] approach towards to resolve the first order differential equations accompany with the fuzzy initial value problem.

2.1 Triangular fuzzy number [19]

A triangular fuzzy number \( \tilde{u} \) can be defined by a triplet \((a, b, c)\), the membership function is defined as follows.

\[
\tilde{u}(x) = \begin{cases} 
0, & \text{if } x < a \\
\frac{x-a}{b-a}, & \text{if } a \leq x < b \\
\frac{c-x}{c-b}, & \text{if } b \leq x \leq c \\
0, & \text{if } x > c
\end{cases}
\]

The \( \alpha \)-level of the fuzzy number \( \tilde{u} \) is \( \tilde{u}_\alpha = [a + (b - c)\alpha, c - (c - b)\alpha] \) for any \( \alpha \in [0, 1] \).

2.2 Modified Euler’s method using harmonic mean

Consider \( a, b \) and \( c \) are the fundamentals of Harmonic sequence. The Harmonic mean is calculated as \( c = \frac{2ab}{a+b} \). Then, the anticipated Harmonic mean of the two center points are definite by

\[
\left[ \frac{2x_0x_1}{x_0 + x_1}, \frac{2y_0y_1}{y_0 + y_1} \right].
\]

The equation also can be expressed as
Using an equation in the neighborhood of a point, such \( P(x_0, y_0) \), and the gradient up the equation, we may derive a new equation written as,

\[
y = y_0 + hf \left( \frac{2y_0 (x_0 + hf(x_0, y_0))}{x_0 + (x_0 + hf(x_0, y_0))}, \frac{2y_0 (y_0 + hf(x_0, y_0))}{y_0 + (y_0 + hf(x_0, y_0))} \right) .
\]  

(2)

By using the modified and stable slope function at the predictable central points of \((x_0, y_0)\) and \((x_1, y_1)\) to estimated \(y_{n+1}\), the Euler method will be more stable. The above-mentioned equation is termed as Harmonic Mean of Euler’s modified method [19]. It can be defined as,

\[
y_{n+1} = y_n + hf \left( \frac{2y_n (x_n + hf(x_n, y_n))}{x_n + (x_n + hf(x_n, y_n))}, \frac{2y_n (y_n + hf(x_n, y_n))}{y_n + (y_n + hf(x_n, y_n))} \right) .
\]  

(3)

2.3 Cubic mean of modified Euler’s method

Consider \( a \), \( b \) and \( c \) are the fundamental elements of Geometric Progression, in that case the Geometric mean of cubic average point is defined as \( c = \sqrt[3]{a^3 + b^3} \). As a result, the reckoning middling of Geometric cubic mean of the two midpoints are expressed as,

\[
\left( \frac{x_0 + x_1}{2} \right)^{\frac{1}{3}}, \left( \frac{y_0 + y_1}{2} \right)^{\frac{1}{3}}.
\]  

(5)

The equation can be defined as,

\[
\left( \frac{x_0 + (x_0 + h)^3}{2} \right)^{\frac{1}{3}}, \left( \frac{y_0 + hf(x_0, y_0)}{2} \right)^{\frac{1}{3}}.
\]  

(6)

If an equation passes to the course of a point, like \( P(x_0, y_0) \), in the midst of the slope throughout the equation, a new equation can be generated and it is given as,

\[
y = y_0 + hf \left( \left( \frac{x_0 + (x_0 + h)^3}{2} \right)^{\frac{1}{3}}, \left( \frac{y_0 + hf(x_0, y_0)}{2} \right)^{\frac{1}{3}} \right) .
\]  

(7)

Modified Euler method provide new sure and precise outcomes. The above equation is called as Cubic Mean of modified Euler’s method. It can be written as
\[ y_{n+1} = y_n + hf \left( \frac{x_n^3 + (x_n + h)^3}{2} \right)^\frac{3}{2} \]

(8)

3. Fuzzy Initial Value Problem (FIVP)

3.1 Technique for precisely solving FIVP

Take a look at the FIVP [20]

\[
x'(t) = \begin{cases} f(t, x) \\ x(t_0) = \left( \xi_0, \lambda_0, \eta_0 \right) \end{cases}
\]

(9)

Here a fuzzy starting condition is in term of triangular fuzzy numbers. Normal differential equations may be recast as an Eigen value problem. By using Taylor’s method [9, 12, 16] to find their accurate solutions. The solution to FIVP is obtained using fuzzy beginning conditions.

3.2 Proposed methods to FIVP

In this case, we examine the fuzzy initial value issue as

\[
x'(t) = \begin{cases} f(t, x) \\ x(t_0) = \left( \xi_0, \lambda_0, \eta_0 \right) \end{cases}
\]

(10)

The triangular fuzzy starting condition may be expressed using an \( r \)-cut technique as

\[ \left[ (\cdot - \xi_0) + \xi_0 \cdot (\cdot - \eta_0) \right], \quad 0 \leq r \leq 1. \]

The current modified Euler’s method has been designed to solve the FIVP. At this time, all the new upper and lower bound possible permutation are evaluated by implementing the Euler’s modified method [21]. The grid points are calculated as \( t_n \) and the results are shown below.

\[
\Sigma_{n+1}^{(1)} (t_{n+1} : r) = \Sigma (t_n : r) + F [x(t_n : r)], \\
\Sigma_{n+1}^{(1)} (t_{n+1} : r) = \Sigma (t_n : r) + G [x(t_n : r)], \\
\Sigma_{n+1}^{(2)} (t_{n+1} : r) = \Sigma (t_n : r) + G [x(t_n : r)], \\
\Sigma_{n+1}^{(2)} (t_{n+1} : r) = \Sigma (t_n : r) + F [x(t_n : r)].
\]

(11)
Then are intended for inferior and superior value of the independent variable \(x\), from that we adopt the values from minimum to maximum. So that it produces the better exact solutions.

\[
\bar{x}_{n+1} = \min \left\{ \bar{x}(t_n : r) + F \left[ x(t_n : r) \right], \bar{x}(t_n : r) + G \left[ x(t_n : r) \right] \right\},
\]

\[
\tilde{x}_{n+1} = \max \left\{ \bar{x}(t_n : r) + G \left[ x(t_n : r) \right], \bar{x}(t_n : r) + F \left[ x(t_n : r) \right] \right\}.
\]

(12)

### 3.2.1 Harmonic mean of modified Euler’s method

The anticipated Harmonic mean of Modified Euler’s method is definite described by

\[
\bar{x}_{n+1} = \bar{x}(t_n : r) + F \left[ x(t_n : r) \right],
\]

\[
\tilde{x}_{n+1} = \tilde{x}(t_n : r) + G \left[ x(t_n : r) \right].
\]

(13)

Where

\[
F = h f \left[ \frac{2(t_n : r)((t_n : r) + h)}{(t_n : r) + ((t_n : r) + h)} \right],
\]

\[
G = h f \left[ \frac{2(t_n : r)((t_n : r) + h)}{(t_n : r) + ((t_n : r) + h)} \right].
\]

### 3.2.2 Cubic mean of modified Euler’s method

The proposed Cubic mean of Modified Euler’s method is defined by

\[
\bar{x}_{n+1} = \bar{x}(t_n : r) + F \left[ x(t_n : r) \right],
\]

\[
\tilde{x}_{n+1} = \tilde{x}(t_n : r) + G \left[ x(t_n : r) \right]
\]

(14)

where

\[
F = h f \left[ \left( \frac{(t_n : r)^3 + ((t_n : r) + h)^3}{2} \right)^{\frac{1}{3}} \right],
\]

\[
G = h f \left[ \left( \frac{(t_n : r)^3 + ((t_n : r) + h)^3}{2} \right)^{\frac{1}{3}} \right].
\]
4. Result and discussion by numerical example

Consider the differential equation with fuzzy initial value problem,

\[ x' = -t^3 x, \quad x(0) = (0.8 + 0.2r, 1.25 - 0.25r) \text{ where } 0 \leq r \leq 1. \]  

(15)

Solution:
The exact solution is given by

\[ x(t; r) = x(t; r) e^{-\frac{t^4}{4}} \quad \text{and} \quad \bar{x}(t; r) = \bar{x}(t; r) e^{-\frac{t^4}{4}}, \]

(16)

then the solutions at \( t = 1 \),

\[ x(1; r) = [(0.8 + 0.2r)e^{-0.25}, (1.25 - 0.25r)e^{-0.25}] \quad 0 \leq r \leq 1. \]  

(17)

The precise and fairly accurate solution is acquired by Euler’s method. The harmonic mean of modified Euler method and also cubic mean of modified Euler method by means of \( h = 0.1 \) is given below:

<table>
<thead>
<tr>
<th>( r )</th>
<th>Exact Solution</th>
<th>Harmonic Mean Solution</th>
<th>Cubic Mean Solution</th>
<th>Euler Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.623046265</td>
<td>0.9735009788</td>
<td>0.62306666947</td>
<td>0.9735415542</td>
</tr>
<tr>
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<td>0.9540309593</td>
<td>0.6386432595</td>
<td>0.9540707231</td>
</tr>
<tr>
<td>0.2</td>
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<td>0.6542199244</td>
<td>0.9345998920</td>
</tr>
<tr>
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<td>0.9150909201</td>
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</tr>
<tr>
<td>0.5</td>
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<td>0.8761873988</td>
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<tr>
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<td>0.7165265839</td>
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</tr>
<tr>
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<tr>
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<td>0.8177408222</td>
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<td>0.8177749055</td>
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<tr>
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<td>0.7982708026</td>
<td>0.76325666785</td>
<td>0.7983040744</td>
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<td>0.7788007381</td>
<td>0.7787709147</td>
<td>0.7788324333</td>
</tr>
</tbody>
</table>

Table 1. Numerical Solutions of Fuzzy Differential Equation with \( h = 0.1 \)
The Fuzzy differential equation problem is solved by known method. That is Euler’s method, the approximate solution is displayed in the above Table 1. Also, we solved our proposed two methods like Harmonic Mean and Cubic Mean, the approximate solution is displayed in the above Table 1. Table 2 shows Error Analysis between Exact and Proposed Methods when $h = 0.1$. Here the overall analysis of the solution obtained by our proposed methods gives nearly accurate solution other than the Euler’s method. So, it can be concluded that our recommended procedure is better than the Euler’s method.

The Fuzzy differential equation problem is solved by a known method. That is Euler’s method, the approximate solution is displayed in the above Table 3. Also, we solved our proposed two methods like Harmonic Mean and Cubic Mean, the approximate solution is displayed in the above Table 3. Table 4 shows Error Analysis between the Exact and Proposed Methods when $h = 0.001$. Here the overall analysis of the solution obtained by our proposed methods gives nearly more accurate solution other than the Euler’s method. So, it concludes that our recommended procedure is more better than Euler’s method.
### Table 3. Numerical Solutions of Fuzzy Differential Equation with h = 0.001

<table>
<thead>
<tr>
<th>r</th>
<th>Exact Solution</th>
<th>Harmonic Mean Solution</th>
<th>Cubic Mean Solution</th>
<th>Euler Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Lower Fuzzy</td>
<td>Upper Fuzzy</td>
<td>Lower Fuzzy</td>
<td>Upper Fuzzy</td>
</tr>
<tr>
<td>0.0</td>
<td>0.6230406265</td>
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<td>0.7788011076</td>
</tr>
</tbody>
</table>

### Table 4. Error Analysis between Exact and Proposed Methods h = 0.001

<table>
<thead>
<tr>
<th>r</th>
<th>Harmonic Mean Solution</th>
<th>Cubic Mean Solution</th>
<th>Euler Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Lower Fuzzy</td>
<td>Upper Fuzzy</td>
<td>Lower Fuzzy</td>
</tr>
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</table>
Figure 1 shows that, the value of Exact solution are marked at Green Colour, Harmonic Solution are marked at Red Colour, Geometric Solution are marked at Blue Colour and Euler solution marked at Black Colour when $h = 0.1$. Every solution Fuzzy triangular number format clearly our proposed solution graph more accurate.
For the different values of step size $h = 0.01, 0.001$, this technique provides the very Exact solution for fuzzy differential equations as shown in the Figure 2 and Figure 3 and the numerical calculations for this paper are performed using C-programming language with matplotlib.

5. Conclusion

In comparison to the prior approach, the new strategy we introduce as part of our research will result in an accurate solution. Various upper and lower bound values ($h = 0.1, 0.01, 0.001, 0.0001$) will be used to assess our method. Other than the Euler’s method, our modified Euler’s method provides the most comprehensive and accurate solution. When compared to our proposed approach, $h = 0.1$ is the optimum value. Euler’s approach is more precise when $h = 0.01$.

Acknowledgements

We are thankful to the editor and referee for their fruitful comments. Also, the third author would like to thank REVA University for their encouragement and support in carrying out this research work.

Conflict of interest

The authors declare no competing financial interest.

References


