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Research Article

Impact of Seasonality and Vertical Transmission on Mosquito Population in the Dynamics of Dengue Disease

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Abstract: Many epidemic diseases are season-related. Dengue is one of them. Since it is associated with a mosquito's life cycle, it is genuinely affected by weather changes. In this paper, we model the dynamics of dengue disease transmission in the human population using two systems of delay differential equations. First, we carry out the modeling with the vertical transmission in the mosquito population and demonstrate its basic properties. Then, we implement the seasonality effect in a second model by choosing some of the parameters that are affected by weather changes to be periodically time-dependent and re-deriving these parameters. We illustrate the conditions when the disease-free periodic solution is locally asymptotically stable and when it is unstable. Simulations in this case were compatible with the theoretical results.

***Keywords*:** dengue fever model, delay differential equations, seasonality, vertical transmission, basic reproduction number, periodic solutions

MSC: 34C11, 34C12, 34C25, 34D20, 34D23, 92D30

1. Introduction

Dengue disease has been considered endemic in most tropical and subtropical areas [1]. In subtropical countries, the disease has shown a frequent resurgence with rainy seasons, where the outbreaks happen three to four months after the rainy season [2]. (See Figure 1 for average monthly rainfall in Jeddah city, Makkah Province, Saudi Arabia, and Figure 2 for cases reported in Makkah province in the three years 2017-2019.) Notice the rise in cases three months after the rainfall season. In dry seasons, the cases drop rapidly. This raises the question of how the virus survives throughout the dry seasons. Some of the hypotheses relate the survival to the vertical transmission of the virus in the mosquito population, where the female mosquito transmits the virus to the eggs (it defers from horizontal transmission, where the virus is transmitted from one individual to another from the same generation). The infected egg carries the virus until it hatches at the beginning of the rainy season [2]. This observation has been confirmed by the data from Belo Horizonte, Brazil (Eiras & Blackmer, 2003) [3]. Therefore, it can be confirmed that seasonality and vertical transmission are two crucial aspects of the existence of vector-borne diseases.

Mathematically, vertical transmission has been implemented in several models of different epidemiological diseases, like malaria, bluetongue, Chagas, and dengue. The most common models are those of Anderson and May [4], Brauer [5], Busenberg and Cooke [6-9], and Li and Smith [10]. Recent studies of vertical transmission in mosquitoes

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have been precisely illustrated in [11] and [12-14].

On the other hand, the seasonal effect on vector-borne diseases forms a major factor in their dynamics in populations, where weather changes have a major role in vectors' life cycles as well as human habits. In the work of Abdelrazec and Gumel [15] and Okuneye et al. [16], the impact of temperature and rainfall on the dynamics of mosquito populations has been studied by driving non-autonomous non-linear differential equations. Our implementation of seasonality in this paper defers by allowing some of the parameters to periodically change with time t. As a result, some terms of the vector differential equations will need to be derived.

In this paper, we will consider both factors: seasonality and vertical transmission of dengue disease in mosquito populations. This approach has been considered in the work of Coutinhoa et al. [17]. The classification of the mosquito population in their mathematical model considers the egg stage, which classifies both susceptible and infected egg classes. However, our approach considers only the mature mosquitoes and implements the egg stage in a delay term of the mosquito's birth rate. Our justification for this is that the infected eggs do not infect humans directly, and our study concerns the dynamics of dengue disease in the human population and its interactions with infected mosquitoes via mature female bites.

The structure of this paper is to formulate an autonomous differential equation system implementing vertical transmission. We will then study its basic properties and the linear stability of its disease-free solutions. We move on in the next section to reformulate the same model with seasonality effects on some of the parameters that are mostly affected by weather changes. followed by analyzing the disease-free solutions of the non-autonomous model, where the disease-free equilibrium of susceptible mosquitoes is no longer expected to be a steady state but rather develops a periodic behavior. Then, a numerical analysis section will illustrate the numerical solution of the disease-free periodic solution. A discussion and conclusions are in the last section.

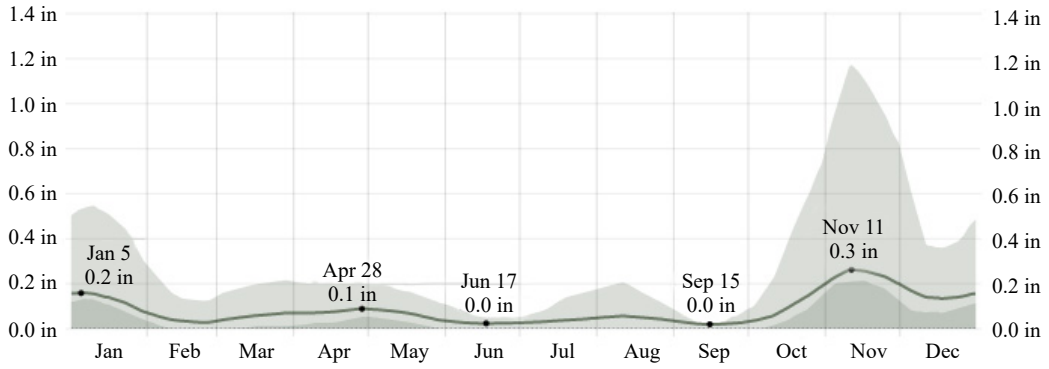

Figure 1. Average monthly rainfall in Jeddah [18]

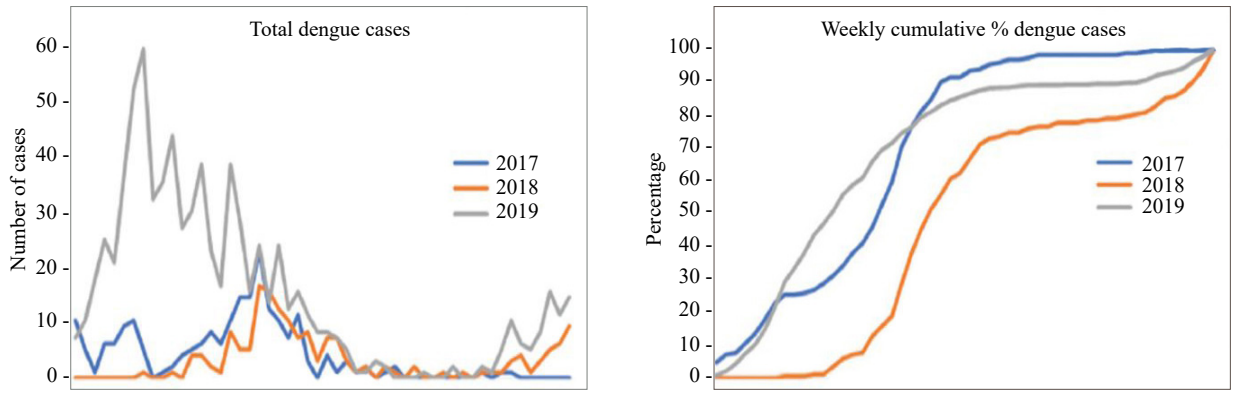

Figure 2. Dengue cases in Makkah, Kingdom of Saudi Arabia, by weeks. (a) The total number of confirmed dengue cases was plotted against the week numbers in 2017, 2018, and 2019. (b) Cumulative percentages of the total number of confirmed dengue cases were plotted against the week number in 2017, 2018, and 2019 [19]

2. Autonomous model formulation

As mentioned above, we encounter the two main factors that, in our opinion, play a key role in transmitting dengue disease from mosquitoes to humans: vertical transmission of the virus from an infected female mosquito to its egg and weather changes due to varying rain seasons. During the dry season, mosquito population growth is postponed until the beginning of the rainy season, normally in late autumn in subtropical regions. The vertical transmission of the virus guarantees to maintain disease persistence within the population during that time. Therefore, the two factors, seasonality and vertical transmission in mosquitoes, coincide. First, we will formulate the autonomous differential equations of the model, and then we will add the seasonality effect to some of the parameters. Replacing the constant parameters with periodic functions will not be that simple. This step will undoubtedly necessitate re-deriving some of these periodic parameters.

Our approach to implementing vertical transmission in the vector population will be to consider only the mature class of females since they are the direct principle for transmitting the virus to humans via bites when the mature female seeks to feed on host blood. However, the previous stages of a mosquito's life cycle are modeled as a delay in the birth rate of the female. The time of these stages from laying eggs until maturation (egg, larva, and pupa) is clumped in constant delay time τ since it is predictable. This approach has been implemented in Gourley et al.'s [20] work. Therefore, the probability that a single egg will survive these stages is given by $e^{-(\rho_e+\kappa_e)\tau}$ where ρ_e is the per-capita natural mortality rate caused by dry weather or any other natural causes, and κ_e is the per-capita death rate caused by using *larvicides* (larva insecticide). Considering the larvicide rate here will help us measure the effect of using it early enough to prevent the vertical transmission of the virus to the next generation of mosquitoes. The probability that an egg will inherit the disease from the mother will be taken as $p \in [0,1]$, and we denote the per-capita laying egg rate for female mosquitoes by $b_m N_m(t))$, where $N_m(t)$ is the total population of adult mosquitoes, $N_m(t) = S_m(t) + I_m(t)$, with susceptible mosquitoes $S_m(t)$ and infectious ones $I_m(t)$.

We likewise need to gauge the impact of insecticides on mature mosquitoes in terms of the per-capita death rate κ_m. Subsequently, we add an extra variable $K_m(t)$ to denote the biomass of the mosquitoes that die insecticidally. Moreover, the natural mortality rate of mosquitoes due to other causes is taken as ρ_m. We will introduce the classes of the host (human) population as $S_h(t)$ for the number of susceptible, $I_h(t)$ for the number of infectious, and the total number of human population will then be $N_h(t) = S_h(t) + I_h(t)$. Due to its short duration in comparison to the yearly scale we are considering, we will not take into consideration the exposed but not yet infectious class. Also, since the recovered patient will develop immunity against the dengue infection, we will not consider the recovery class. Instead, we will represent the δ_h rate that will represent the removal per-capita rate, and we mean by removal rate the rate at which an individual patient of dengue fever will either recover or die because of the disease. We need to mention that dengue fever can cause death if not treated promptly. However, in most endemic regions, patient mortality due to dengue fever is significantly low [1]. All the parameters of the model are illustrated in Table 1.

Table 1. Description of variables and parameters of the model (1)

Variables	Description
$S_m(I_m)$	Number of susceptible (infectious) female mosquitoes at time t
$S_h(I_m)$	Number of susceptible (infectious) human at time t
κ_m	The biomass of mosquitoes died insecticidally at time t
Parameters	
$b_m(.)$	*Per capita* laying eggs rate for female mosquitoes
$b_h(.)$	Overall (*not per capita*) birth rate of human
$\rho_m(\rho_e)$	Natural *per capita* death rate for mature (pre-mature) mosquitoes
$\kappa_m(\kappa_e)$	Mortality *per capita* rate of mature (pre-mature) mosquitoes due to insecticide (larvicide)
c	Clearance rate of dead mosquitoes due to insecticide (larvicide)
ρ_h	Mortality *per capita* rate of humans caused by causes other than dengue
δ_h	*Per capita* removal rate of dengue in human
$\beta_{mh}(\beta_{hm})$	Dengue virus transmission probability from mosquitoes to humans (human to mosquitoes)

Therefore, the autonomous system of differential equations that models dengue disease in a population without the seasonality effect will take the form

$$\begin{cases} \dfrac{dS_h}{dt} = b_h(S_h + I_m)(t) - \beta_{mh}\dfrac{S_h(t)I_m(t)}{N_h(t)} - \rho_h S_h(t), \\ \dfrac{dI_h}{dt} = \beta_{mh}\dfrac{S_h(t)I_m(t)}{N_h(t)} - (\rho_h + \delta_h)I_h(t), \\ \dfrac{dS_m}{dt} = e^{-(\rho_e+\kappa_e)\tau}b_m(S_m + (1-p)I_m)(t-\tau) - \beta_{hm}\dfrac{S_m(t)I_h(t)}{N_h(t)} - (\rho_m + \kappa_m)S_m(t), \\ \dfrac{dI_m}{dt} = e^{-(\rho_e+\kappa_e)\tau}b_m(pI_m(t-\tau)) + \beta_{hm}\dfrac{S_m(t)I_h(t)}{N_h(t)} - (\rho_m + \kappa_m)I_m(t), \\ \dfrac{dK_m}{dt} = \kappa_m(S_m(t) + I_m(t)) - cK_m. \end{cases} \tag{1}$$

The terms involving delays, S_m and I_m, will need to be prescribed on the negative time from $[-\tau, 0]$ as follows:

$$S_m(\theta) = S_m^0(\theta) \geq 0, \quad I_m(\theta) = I_m^0(\theta) \geq 0$$

for $\theta \in [-\tau, 0]$. But the other variables do not need initial data for the previous time of $t = 0$ since they do not involve any delays, and therefore we only need their following initial data:

$$S_h(0) = S_h^0 \geq 0, \quad I_h(0) = I_h^0 \geq 0 \quad K_m(0) = K_m^0 \geq 0,$$

such that $S_h(0) + I_h(0) > 0$, since N_h appears in the denominator. This assumption is plausible in the sense that we need an initial human population to start the dynamic.

3. Basic properties of the autonomous model

In the absence of dengue disease in both vector and host populations, the birth rates of humans and mosquitoes will be $b_h(S_h)$ and $b_m(S_m)$, respectively. For non-negative arguments, we assume they are locally Lipschitz continuous. We add a negative argument extension to the definition of b_m by defining $b_m(N) = 0$ for $N < 0$. Also, at zero, we assume $b_h(0) = b_m(0) = 0$. This addition will preserve the continuity of Lipschitz. We also assume that $b_h(s) > 0$ and $b_m(t) > 0$ for all s, $t > 0$, which is biologically acceptable.

3.1 *Positivity*

Easily, one can establish the non-negativity of solutions using the standard approach of setting all variables in (1) to zero except the variable that appears on the left-hand side. Then, the non-negativity of that variable will follow. However, establishing that solutions to all variables, especially the disease variables, I_h and I_m, become and remain strictly positive, requires the necessary condition of an initial infected mosquito's presence. The following proposition illustrates this result.

Proposition 1. Suppose that $S_h(0), I_h(0), S_v(0), I_m(0), K_m(0) \geq 0$. Additionally, suppose that $I_m(0) > 0$. Then, for all $t > 0$, $I_h(t) > 0$.

Proof. Let $I_m(0) > 0$. Knowing that solutions are non-negative, from the fourth equation of (1), we get

$$\frac{dI_m(t)}{dt} \geq -(\rho_m + \kappa_m)I_m(t).$$

By solving this inequality, we get

$$I_m(t) \geq I_m(0)e^{-(\rho_m+\kappa_m)t} > 0,$$

for all $t > 0$.

From the equation of I_h(t) in (1),

$$\frac{dI_h(t)}{dt} = \beta_{mh}\frac{S_h(t)I_m(t)}{N_h(t)} - (\rho_h + \delta_h)I_h(t).$$

Solving this equation,

$$I_h(t) = I_h(0)e^{-(\rho_h+\delta_h)t} + \int_0^t e^{-(\rho_h+\delta_h)(t-s)}\beta_{mh}\frac{S_h(s)I_m(s)}{N_h(s)}ds. \tag{2}$$

$I_h(t)$ is strictly positive if $I_h(0) > 0$, but we need to prove the strict positivity in the case when $I_h(0) = 0$. So, we assume that there exists, such a $t^* > 0$ where $I_h(t^*) = 0$, then (2) will be

$$0 = \int_0^{t^*} e^{-(\rho_h+\delta_h)(t^*-s)}\beta_{mh}\frac{S_h(s)I_m(s)}{N_h(s)}ds,$$

but $\beta_{mh}\frac{S_h(s)I_m(s)}{N_h(s)}$ is non-negative and moreover, $I_m(t) > 0$ for all $t > 0$ as proven above.

Therefore, for all $s \in [0, t^*], S_h(s) \equiv 0$, more precisely, $S_h(0) = 0$, which implies that there is no human population since we assumed that also $I_h(0) = 0$, and this contradicts the initial condition $N_h(0) = S_h(0) + I_h(0) > 0$. Then, for all $t > 0$, $I_h(t) > 0$.

3.2 *Existence of disease-free equilibrium*

In the absence of dengue disease in the human population, a disease-free equilibrium exists if and only if

$$b_h(S_h^0) = \rho_h S_h^0, \tag{3}$$

and

$$b_m(S_m^0)e^{-(\rho_e+\kappa_e)\tau} = (\rho_m + \kappa_m)S_m^0. \tag{4}$$

The total human population in this case will be $N_h(t) = S_h^0$; similarly, $N_m(t) = S_m^0$, and all other variables remain identically zero. The assumption that $b_h(\cdot)$ and $b_m(\cdot)$ are Lipstchitz continuous guarantees that (3) and (4) have unique solutions $S_h^0 > 0$ and $S_m^0 > 0$, respectively.

3.3 *Extinction of the whole mosquito population*

Notice from (4) that as κ_m increases, S_m^0 decreases. If κ_m is sufficiently large, then for a realistic $b_m(\cdot)$, equation (4) has no root with $S_m^0 > 0$. This suggests that the whole population of mosquitoes will be extinct. We state this in the following theorem.

Theorem 1. Let $b_m(\cdot)$ be twice differentiable and concave. Moreover, let $b_m(0) = 0$, such that

$$b_m'(0)e^{-(\rho_e+\kappa_e)\tau} < (\rho_m + \kappa_m), \tag{5}$$

then, $N_m(t) \to 0$ as $t \to \infty$ for sufficiently large κ_m.

Proof. Recall that the total number of the mosquitoes is $N_m(t) = S_m(t) + I_m(t)$. By adding up S_m' and I_m' equations

from (1), we have

$$\frac{dN_m(t)}{dt} = e^{-(\rho_e+\kappa_e)\tau} b_m(N_m(t-\tau)) - (\rho_m+\kappa_m)N_m(t). \tag{6}$$

By the concavity of $b_m(\cdot)$ and Taylor expansion, $b_m(N_m) \le b_m'(0)N_m$ and equation (6) becomes

$$\frac{dN_m(t)}{dt} \le e^{-(\rho_e+\kappa_e)\tau} b_m'(0)N_m(t-\tau) - (\rho_m+\kappa_m)N_m(t). \tag{7}$$

Since $b_m'(0) > 0$, then as $N_m(t-\tau)$ increases, the right-hand side of (7) increases. Using the comparison argument of Smith [21], let the solution of

$$\frac{d\hat{N}_m(t)}{dt} = e^{-(\rho_e+\kappa_e)\tau} b_m'(0)\hat{N}_m(t-\tau) - (\rho_m+\kappa_m)\hat{N}_m(t),$$

be $\hat{N}_m(t)$ subjected to initial data as of $N_m(t)$. Then, by applying the comparison argument, $N_m(t) \le \hat{N}_m(t)$. Recall that

$$b_m'(0)e^{-(\rho_e+\kappa_e)\tau} < (\rho_m+\kappa_m),$$

then $\hat{N}_m(t) \to 0$ as $t \to \infty$. Regarding that $0 \le N_m(t) \le \hat{N}_m(t)$, then $N_m(t) \to 0$ as $t \to \infty$.

3.4 *Linear stability and the basic reproduction number*

In the absence of dengue disease, we proved that the dengue-free equilibrium exists under certain conditions. We now linearize the system (1) in order to investigate its linear stability. We will only consider the disease equations (I_h and I_m) around this equilibrium and introduce a small perturbation of the disease. Then, the linearized, decoupled infection subsystem will take the form

$$\begin{aligned}
\frac{d\tilde{I}_h(t)}{dt} &= \beta_{mh}\tilde{I}_m(t) - (\rho_h+\delta_h)\tilde{I}_h(t),\\
\frac{d\tilde{I}_m(t)}{dt} &= (\rho_m+\kappa_m)p\tilde{I}_m(t-\tau) + \beta_{hm}\frac{S_m^0}{S_h^0}\tilde{I}_h(t) - (\rho_m+\kappa_m)\tilde{I}_m(t),
\end{aligned} \tag{8}$$

recalling that $b_m(S_m^0)e^{-(\rho_m+\kappa_m)\tau} = \rho_m+\kappa_m$. Seeking a non-trivial solution of the form

$$e^{\lambda t}(c_1, c_2) = (\tilde{I}_h, \tilde{I}_m),$$

we get the following characteristic equation:

$$\lambda^2 + \lambda[(\rho_m+\kappa_m)(1-p) + (\rho_h+\delta_h)] + (\rho_h+\delta_h)(\rho_m+\kappa_m)(1-p) - \beta_{mh}\beta_{hm}\frac{S_m^0}{S_h^0} = 0. \tag{9}$$

According to the Routh-Hurwitz criterion, this equation has negative real parts if and only if

$$\frac{\beta_{mh}\beta_{hm}\dfrac{S_m^0}{S_h^0}}{(\rho_h+\delta_h)(\rho_m+\kappa_m)(1-p)} < 1.$$

Therefore, we can state the following theorem.

Theorem 2. If the assumption of (3) and (4) hold and

$$R_0 = \frac{\beta_{mh}\beta_{hm}\dfrac{S_m^0}{S_h^0}}{(\rho_h+\delta_h)(\rho_m+\kappa_m)(1-p)} < 1, \tag{10}$$

then the dengue-free equilibrium $(S_h^0, 0, S_m^0, 0, 0)$ is locally asymptotically stable to perturbations involving small introductions of disease.

4. The model with seasonality (the non-autonomous system)

As we mentioned before, to implement seasonality into the autonomous system, we will need to re-derive some of the terms with parameters that we believe are the most affected by weather changes. The first term we will re-derive is $e^{-(\rho_e+\kappa_e)\tau}$ in the rate of maturity of mosquitoes (taken before as $e^{-(\rho_e+\kappa_e)\tau}b_m(S_m+I_m)(t-\tau)$ for $p = 0$, where the vertical transmission is zero), by considering ρ_e and κ_e as periodic functions of time $\rho_e(t)$ and $\kappa_e(t)$, respectively. Let the density of larvae at time t be $l(t, a)$, where a is age. Therefore, for mosquitoes that are still in the larval stage, the age-structure equation of McKendrick-von Foerster is

$$\frac{\partial l(t,a)}{\partial t} + \frac{\partial l(t,a)}{\partial a} = -(\rho_e(t)+\kappa_e(t))l(t,a). \tag{11}$$

By solving this partial differential equation, we get the formula $l(t, a)$. Firstly, we take the mosquito's egg-hatching rate as a decreasing function called $b_m(\cdot)$, and that is when $a = 0$, as follows: $l(t, 0) = b_m((S_m + (1\text{-}p)I_m)(t))$.

Define

$$l^{\alpha}(a) = l(a+\alpha, a).$$

Therefore,

$$\begin{aligned}
\frac{dl^{\alpha}(a)}{da} &= \left(\frac{\partial l}{\partial t}\frac{\partial t}{\partial a} + \frac{\partial l}{\partial a}\frac{\partial a}{\partial a}\right)_{t=a+\alpha}, \\
&= \left(\frac{\partial l}{\partial t} + \frac{\partial l}{\partial a}\right)_{t=a+\alpha}, \\
&= \left(-(\rho_e(t)+\kappa_e(t))l(t,a)\right)_{t=a+\alpha}, \\
&= -\left(\rho_e(a+\alpha)+\kappa_e(a+\alpha)\right)l^{\alpha}(a).
\end{aligned}$$

Now, we solve this ordinary differential equation (ODE) as follows

$$l^{\alpha}(a) = l^{\alpha}(0)\exp\left\{-\int_0^a (\rho_e(\eta+\alpha)+\kappa_e(\eta+\alpha))d\eta\right\}.$$

Rewriting this equation in form $l^{\alpha}(a) = l(a+\alpha, a)$, we have

$$l(a+\alpha,a) = l(\alpha,0)\exp\left\{-\int_0^a (\rho_e(\eta+\alpha)+\kappa_e(\eta+\alpha))d\eta\right\}.$$

Put $\alpha = t - a$, then the above equation will be

$$l(t,a) = l(t-a,0)\exp\left\{-\int_0^a (\rho_e(\eta+t-a)+\kappa_e(\eta+t-a))d\eta\right\}. \tag{12}$$

Let a be the age when a larva turns into a mosquito, so the rate at which mosquito larva enter the population is $l(t, \tau)$. Put $a = \tau$ in (12), and then the mosquito maturity rate can be expressed as the product of its hatching rate and the probability of surviving the life stages before maturity as

$$l(t,\tau) = b_m\left((S_m + (1-p)I_m)(t-\tau)\right)e^{-\int_0^\tau (\rho_e(\eta+t-\tau)+\kappa_e(\eta+t-\tau))d\eta}. \tag{13}$$

Therefore, the susceptible vectors equation is giving by

$$\begin{aligned} S_m'(t) &= b_m((S_m + (1-p)I_m)(t-\tau))e^{-\int_0^\tau (\rho_e(\eta+t-\tau)+\kappa_e(\eta+t-\tau))d\eta} \\ &\quad - \beta_{hm}\frac{S_m(t)I_m(t)}{N_h(t)} - (\rho_m(t)+\kappa_m(t))S_m(t), \end{aligned} \tag{14}$$

and the model for the dengue disease with seasonality is as follows:

$$\begin{aligned} \frac{dS_h(t)}{dt} &= b_h((S_h + I_h)(t)) - \beta_{mh}\frac{S_h(t)I_m(t)}{N_h(t)} - \rho_h S_h(t), \\ \frac{dI_h(t)}{dt} &= \beta_{mh}\frac{S_h(t)I_m(t)}{N_h(t)} - (\rho_h + \delta_h)I_h(t), \\ \frac{dS_m(t)}{dt} &= b_m((S_m + (1-p)I_m)(t-\tau))e^{-\int_0^\tau (\rho_e(\eta+t-\tau)+\kappa_e(\eta+t-\tau))d\eta} - \beta_{hm}\frac{S_m(t)I_m(t)}{N_h(t)} - (\rho_m(t)+\kappa_m(t))S_m(t), \\ \frac{dI_m(t)}{dt} &= b_m(pI_m(t-\tau))e^{-\int_0^\tau (\rho_e(\eta+t-\tau)+\kappa_e(\eta+t-\tau))d\eta} + \beta_{hm}\frac{S_m(t)I_m(t)}{N_h(t)} - (\rho_m(t)+\kappa_m(t))I_m(t), \\ \frac{dK_m(t)}{dt} &= \kappa_m(S_m(t)+I_m(t)) + cK_m(t). \end{aligned} \tag{15}$$

Note that in the absence of seasonality, that is, when $\tau \to 0$, the system will be reduced to the autonomous system (1).

4.1 *Disease-free periodic solution*

Regarding the representation of some of the parameters as periodic coefficients, we never again expect a disease-free equilibrium to be a steady-state. Alternatively, the disease-free solution will be periodic.

Obviously, the steady state values of the susceptible human population will remain the same since we assume that the human population is not affected by seasonality. However, it is not the case for mosquitoes. The disease-free periodic solution

$$(S_h(t), I_h(t), S_m(t), I_m(t), K_m(t)) = (S_h^0, 0, S_m^0, 0, 0)$$

exists if

$$b_h(S_h^0) = \rho_h S_h^0, \tag{16}$$

and the function $S_m^0(t)$ satisfies

$$\frac{dS_m^0(t)}{dt} = b_m(S_m^0(t-\tau))e^{-\int_0^\tau(\rho_e(\eta+t-\tau)+\kappa_e(\eta+t-\tau))d\eta} - (\rho_m(t)+\kappa_m(t))S_m^0(t). \tag{17}$$

Let $S_m^0(t)$ be the periodic solution of this equation. The plausibility of this solution will require that the non-periodic solution is linearly unstable. We examine that and linearize (17) about its non-periodic solution

$$\frac{dS_m^0(t)}{dt} = b_m'(0)S_m^0(t-\tau)e^{-\int_0^\tau(\rho_e(\eta+t-\tau)+\kappa_e(\eta+t-\tau))d\eta} - (\rho_m(t)+\kappa_m(t))S_m^0(t). \tag{18}$$

We suggest a form of

$$S_m^0(t) = e^{\lambda t}\omega(t)$$

as a solution where $\omega(t)$ is a periodic function of period T. Then, by substituting this solution in (18), we get

$$\lambda e^{\lambda t}\omega(t) + e^{\lambda t}\frac{d\omega(t)}{dt} = b_m'(0)e^{\lambda(t-\tau)}\omega(t-\tau)e^{-\int_0^\tau(\rho_e(\eta+t-\tau)+\kappa_e(\eta+t-\tau))d\eta} - (\rho_m(t)+\kappa_m(t))e^{\lambda t}\omega(t).$$

So,

$$\frac{d\omega(t)}{dt} = e^{-\lambda\tau}b_m'(0)\omega(t-\tau)e^{-\int_0^\tau(\rho_e(\eta+t-\tau)+\kappa_e(\eta+t-\tau))d\eta} - (\lambda+\rho_m(t)+\kappa_m(t))\omega(t).$$

Solving this equation by recasting it as

$$\frac{d\omega(t)}{dt} + \omega(t)(\rho_m(t)+\kappa_m(t)+\lambda) = \omega(t-\tau)e^{-\lambda\tau}b_m'(0)e^{-\int_0^\tau(\rho_e(\eta+t-\tau)+\kappa_e(\eta+t-\tau))d\eta},$$

which can be written as

$$\frac{d}{dt}\left(\omega(t)e^{\int_0^t(\rho_m(s)+\kappa_m(s)+\lambda)ds}\right) = \omega(t-\tau)e^{\int_0^t(\lambda+\rho_m(s)+\kappa_m(s))ds}e^{-\lambda\tau}b_m'(0)e^{-\int_0^\tau(\rho_e(\eta+t-\tau)+\kappa_e(\eta+t-\tau))d\eta}.$$

By integrating from $-\infty$ to t, we get

$$\omega(t)e^{\int_0^t(\lambda+\rho_m(s)+\kappa_m(s))ds} = \int_{-\infty}^t e^{\int_0^\zeta(\lambda+\rho_m(s)+\kappa_m(s))ds}e^{-\lambda\tau}b_m'(0)\omega(\zeta-\tau)e^{-\int_0^\tau(\rho_e(\eta+\zeta-\tau)+\kappa_e(\eta+\zeta-\tau))d\eta}d\zeta.$$

Therefore,

$$\omega(t) = \int_{-\infty}^t e^{-\int_\zeta^t(\lambda+\rho_m(s)+\kappa_m(s))ds}e^{-\lambda\tau}b_m'(0)\omega(\zeta-\tau)e^{-\int_0^\tau(\rho_e(\eta+\zeta-\tau)+\kappa_e(\eta+\zeta-\tau))d\eta}d\zeta. \tag{19}$$

Let $\phi(t)$ be a function that is defined as

$$\phi(t) = |b_m'(0)|\int_{-\infty}^t e^{-\int_\zeta^t(\rho_m(s)+\kappa_m(s))ds}e^{-\int_0^\tau(\rho_e(\eta+\zeta-\tau)+\kappa_e(\eta+\zeta-\tau))d\eta}d\zeta. \tag{20}$$

Recalling that all of $\rho_e(\cdot), \rho_m(\cdot), \kappa_e(\cdot),$ and $\kappa_m(\cdot)$ are periodic of period T, thus, $\phi(t)$ is periodic as well with period T. Now, we establish the condition when the non-periodic solution is locally asymptotically stable in the next theorem.

Theorem 3. If $\phi(t) < 1$ for all $t \in \mathbb{R}$, then the non-periodic solution $S_m^0(t) \equiv 0$ of (17) is locally asymptotically stable.

Proof. For a contradiction, we suppose that the non-periodic solution $S_m^0(t) \equiv 0$ is unstable. Therefore, there exists λ such that its real part is positive, as we get from equation (19).

$$
\begin{aligned}
|\omega(t)| &\le \int_{-\infty}^{t} e^{-\int_{\zeta}^{t}(Re\lambda+\rho_m(s)+\kappa_m(s))ds} e^{-Re\lambda\tau} e^{-\int_0^{\tau}(\rho_e(\eta+\zeta-\tau)+\kappa_e(\eta+\zeta-\tau))d\eta} |b_m'(0)||\omega(\zeta-\tau)|\,d\zeta \\
&\le \int_{-\infty}^{t} e^{-\int_{\zeta}^{t}(\rho_m(s)+\kappa_m(s))ds} e^{-\int_0^{\tau}(\rho_e(\eta+\zeta-\tau)+\kappa_e(\eta+\zeta-\tau))d\eta} |b_m'(0)||\omega(\zeta-\tau)|\,d\zeta, \\
&\le \max_{\zeta\in\mathbb{R}} |\omega(\zeta)| \int_{-\infty}^{t} e^{-\int_{\zeta}^{t}(\rho_m(s)+\kappa_m(s))ds} e^{-\int_0^{\tau}(\rho_e(\eta+\zeta-\tau)+\kappa_e(\eta+\zeta-\tau))d\eta} |b_m'(0)|\,d\zeta.
\end{aligned}
$$

Therefore,

$$|\omega(t)| \le \phi(t) \max_{\zeta\in\mathbb{R}} |\omega(\zeta)|.$$

Since $|\omega(t)|$ is periodic, then there exists t^*, such that

$$|\omega(t^*)| = \max_{\zeta\in\mathbb{R}} |\omega(\zeta)|.$$

Then,

$$\max_{\zeta\in\mathbb{R}} |\omega(\zeta)| \le \phi(t^*) \max_{\zeta\in\mathbb{R}} |\omega(\zeta)|.$$

Thus, $\phi(t^*) \ge 1$, which contradicts the assumption.

The numerical solution of $S_m(t)$ in Figure 3 shows the eradication of the whole population of mosquitoes when $\phi(t) < 1$ for all t, and the disease-free periodic solution no longer exists as a result of the population extinction.

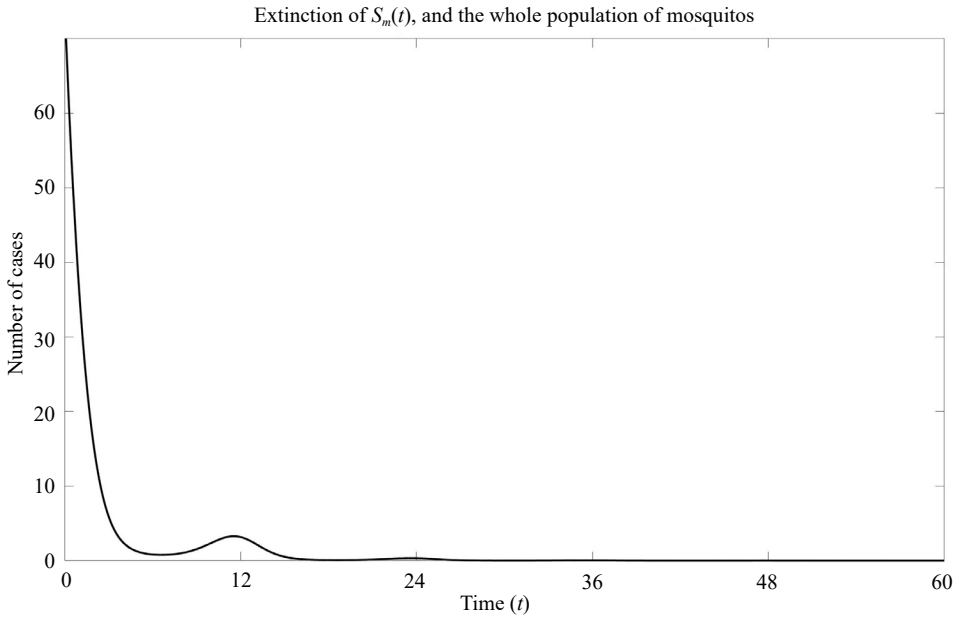

Figure 3. When $\phi(t) < 1$, the non-periodic solution of (17) is locally asymptotically stable for all t.

The time unit is a month, and therefore, $T = 12$ is a one-year period. For the periodic functions we used: $\rho_e(t) = \rho_e^{(0)} + \rho_e^{(1)} \sin(\frac{\pi}{6}t)$ with $\rho_e^{(0)} = 1$ and $\rho_e^{(1)} = 0.2, \kappa_e(t) = \kappa_e^{(0)} + \kappa_e^{(1)} \sin(\frac{\pi}{6}t)$ with $\kappa_e^{(0)} = 1$ and $\kappa_e^{(1)} = 0.1, \rho_m(t) = \rho_m^{(0)} + \rho_m^{(1)} \sin(\frac{\pi}{6}t)$ with $\rho_m^{(0)} = 1$ and $\rho_m^{(1)} = 0.4, \kappa_m(t) = \kappa_m^{(0)} + \kappa_m^{(1)} \sin(\frac{\pi}{6}t)$ with $\kappa_m^{(0)} = 0.8$ and $\kappa_m^{(1)} = 0.3$. The function of mosquito birth rate we used is $b_m(S_m(t)) = 2S_m(t)e^{\frac{S_m(t)}{200}}$, where the time delay is $\tau = 0.1$

We should highlight the fact that the system (15) will convert into the autonomous system (1) in the case of constant coefficients instead of periodic ones, and equation (17) becomes

$$\frac{dS_m^0(t)}{dt} = b_m(S_m^0(t-\tau))e^{-(\rho_e+\kappa_e)\tau} - (\rho_m+\kappa_m)S_m^0(t).$$

and for all t, the condition $\phi < 1$ will be

$$\begin{aligned} 1 > \phi(t) = & \ |b_m'(0)| \int_{-\infty}^{t} e^{-(\rho_m+\kappa_m)(t-\zeta)} e^{-(\rho_e+\kappa_e)\tau} d\zeta, \\ = & \ |b_m'(0)| e^{-(\rho_e+\kappa_e)\tau} \int_{-\infty}^{t} e^{-(\rho_m+\kappa_m)(t-\zeta)} d\zeta, \\ = & \ \frac{|b_m'(0)| e^{-(\rho_e+\kappa_e)\tau}}{\rho_m+\kappa_m}, \end{aligned}$$

that is,

$$|b_m'(0)| e^{-(\rho_e+\kappa_e)\tau} < \rho_m+\kappa_m.$$

Recall that it is the same condition (5) of the whole mosquito population extinction.

The next theorem demonstrates the condition that guarantees the instability of the non-periodic solution (17). For all $t > 0$, let $\omega(t) > 0$ in the trail solution $S_m^0(t) \equiv e^{\lambda t}\omega(t)$ and that λ is real. This is plausible since the solutions to (17) and (18) are positive.

Theorem 4. Suppose that for all $t \in \mathbb{R}, \phi(t) > 1$. Then, the non-periodic solution of (17) is linearly unstable.

Proof. Considering equation (19) and the trail solution $S_m^0(t) \equiv e^{\lambda t}\omega(t)$ of the linearized equation (18), we use a contradiction argument and assume that λ is real. However, we need to investigate whether it is positive or negative. Then, the non-periodic solution will be linearly unstable if the solution (18) grows with time, which means λ is positive. So we assume for a contradiction that it is non-positive $\lambda \le 0$.

Then, equation (19) will be

$$\begin{aligned} \omega(t) &\ge \int_{-\infty}^{t} e^{-\int_{\zeta}^{t}(\rho_m(s)+\kappa_m(s))ds} e^{-\int_0^{\tau}(\rho_e(\eta+\zeta-\tau)+\kappa_e(\eta+\zeta-\tau))d\eta} b_m'(0)\omega(\zeta-\tau)d\zeta, \\ &\ge \min_{\zeta\in\mathbb{R}} \omega(\zeta) \int_{-\infty}^{t} e^{-\int_{\zeta}^{t}(\rho_m(s)+\kappa_m(s))ds} e^{-\int_0^{\tau}(\rho_e(\eta+\zeta-\tau)+\kappa_e(\eta+\zeta-\tau))d\eta} b_m'(0)d\zeta, \\ &= \phi(t)\min_{\zeta\in\mathbb{R}} \omega(\zeta). \end{aligned}$$

Let $\hat{t}$ be the value of t when $\omega(\hat{t}) = \min_{\zeta\in\mathbb{R}} \omega(\zeta)$. Therefore,

$$\begin{aligned} &\min_{\zeta\in\mathbb{R}} \omega(\zeta) \ge \phi(\hat{t})\min_{\zeta\in\mathbb{R}} \omega(\zeta). \\ &\phi(\hat{t}) \le 1, \end{aligned}$$

which is a contradiction.

This periodic solution of the disease-free solution is consistent with the numerical solution of equation (17) in Figure 4 when $\phi(t) > 1$.

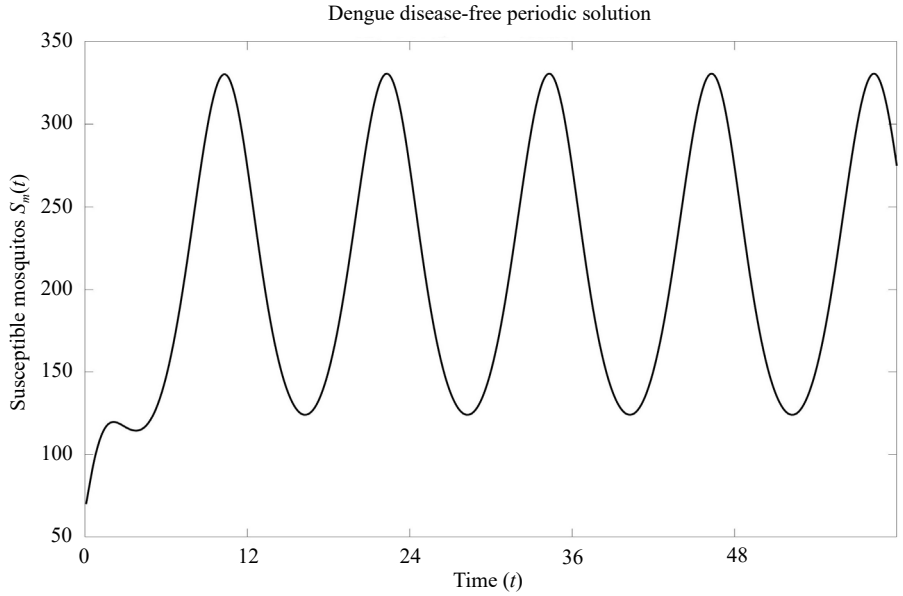

Figure 4. When $\phi(t) > 1$, the disease has a periodic solution of (17) for all t, where the time unit is a month with a period of one year. The periodic functions are taken as the ones in Figure 3, with the following values of the parameters: $\rho_e^{(0)} = 0.2, \rho_e^{(1)} = 0.1, \kappa_e^{(0)} = 0.1, \kappa_e^{(1)} = 0.1, \rho_m^{(0)} = 0.2, \rho_m^{(1)} = 0.1, \kappa_m^{(0)} = 0.5$, and $\kappa^{(1)}m = 0.3$. We used the mosquito birth rate function as shown in Figure 3

5. Numerical simulation

As referred to in the analytical study in the previous section, the solution of the susceptible mosquito disease-free equation (17) shows a periodic behavior when $\phi(t) > 1$, Figure 4, and an extinct behavior of the whole population of mosquitoes when $\phi(t) > 1$, Figure 3. The periodicity of the susceptible mosquito populations suggests a survival period during the dry seasons and rising again in the rainfall seasons. Since the mosquitoes that carry the dengue virus do not die as a result, we cannot anticipate that the dynamics of mosquito population growth will change when dengue disease is present. In this manner, when a mosquito becomes infected with the dengue virus, it is bound to bite a human during ideal season conditions for its endurance. The reports from international public health organizations support this observation, where they show a rise in dengue disease cases after rainy seasons, as shown in Figure 5 [22]. The simulations for ordinary differential equations with delays were carried out using the MATLAB routine DDE23. Default values were used for numerical parameters such as step sizes.

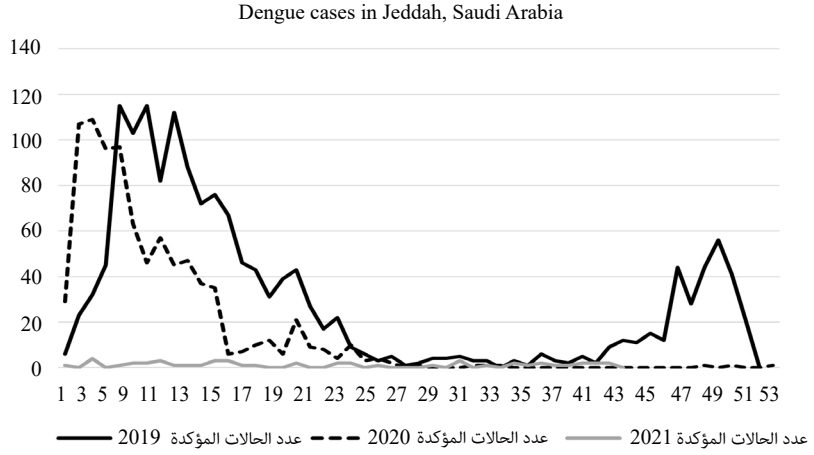

Figure 5. Confirmed cases were reported in Jeddah, Makkah, and Saudi Arabia from 2019-2021. The total number of confirmed dengue cases was plotted against the week numbers in 2019, 2020, and 2021 [3]

6. Discussion and conclusion

$\phi(t)$, which is defined in (20), is the function from which we derive the most insightful information. We have shown that when $\phi(t) < 1$, the whole population of mosquitoes will be extinct, whereas when $\phi(t) > 1$, the population of mosquitoes will adopt a periodic behavior. Regarding that, the disease's existence in the mosquito population does not affect its growth; in other words, it does not cause mosquito death. This refers to the period where the disease cases rise simultaneously with the population growth period, more precisely, three months after the growth peak. Having that said, one would suggest a scenario of eliminating the whole population of mosquitoes by decreasing the value of the equation. This can be done by increasing the parameters that appear in the exponential terms in the $\phi(t)$ function. Those are mainly the death rate parameters $\rho_m, \kappa_m, \rho_e,$ and κ_e.This means that insecticide mosquitoes are crucial in controlling dengue. However, reducing the mosquito population to a very low level to avoid extinction is not realistic in real life. Alternatively, the insecticide procedure can be effective if applied during the time period of nesting to lower the peak point of the population wave. Limited data on female mosque's egg-laying rate in the most common regions where dengue is considered an endemic disease is a basic obstacle. The accuracy of the numerical simulations is therefore quite vague. Establishing a collaboration with the public health authorities in these regions to set up a scientific mosquito trap to collect accurate data on their population growth behavior will indeed improve future studies' impact and, therefore, better control of the disease dynamics.

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Conflict of interest

There is no conflict of interest in this study.

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Appendix

Comparison principle

In Theorem 1, we used the comparison argument to prove the theorem. The rigorous background to this principle can be found in Smith's [21] book, page 78, Theorem 5.1.1, which we restate below with related notation to our models. First, consider the system of ordinary differential inequalities:

$$\frac{dy_i}{dt} \leq f_i(y_1(t), y_2(t), ..., y_n(t)), \qquad i = 1, 2, ..., n. \tag{21}$$

Let $(x_1(t), x_2(t), ..., x_n(t))$ be a solution of the system of corresponding differential equations

$$\frac{dx_i}{dt} = f_i(x_1(t), x_2(t), ..., x_n(t)), \qquad i = 1, 2, ..., n. \tag{22}$$

We state the following theorem without proof, where the proof can be found in [4] as mentioned above.

Theorem 5. For (21) and (22), suppose that, for each $i = 1, 2, ..., n$ the function $f_i(y_1, y_2, ..., y_n)$ is non-decreasing with respect to y_j for all $j \neq i$ (quasi-monotone condition). Suppose that $y_i(0) \leq x_i(0)$ for all $i = 1, 2, ..., n$. Then, $y_i(t) \leq x_i(t), i = 1, 2, ..., n$ for all $t > 0$, such that the solution of (22) is defined.