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Research Article

Convergence Analysis of Resolvent Equation Technique for Co-Variational Inequality Problem in Banach Spaces

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Abstract: In this work, a co-variational inequality problem and a co-resolvent equation problem are introduced and investigated. It is shown that the fixed point problem is equivalent to the problem of co-variational inequality. The co-variational inequality problem and the co-resolvent equation problem are equivalent due to this resemblance. The co-resolvent equation problem is solved using an iterative approach that we provide at the final stage. Our findings may be seen as an enhancement of several established findings.

Keywords: co-variational inequality, co-resolvent equation, equivalence, algorithm

MSC: 47J20, 49J40, 65K15, 90C33

1. Introduction

The variational inequality is an influential unifying tool for solving problems in pure and applied sciences. The extension and generalisation of variational inequalities in a variety of different directions, which has been done for the purpose of applications in the fields of science and technology by many researchers (see, [1]) after the variational inequality introdced and studied by Stampacchia [2]. This theory can be applied as a potent mathematical instrument in order to address issues that arise in several subfields of research, such as engineering, optimization, economics, equilibrium theory, non-linear programming, elasticity, and so on, see e.g., [3–11] and references therein.

In 2000, Alber et al. [12] introduced and studied co-variational inequality problem, they obtained existence and convergence results for the solution of co-variational inequality. In 2012, Ahmad et al. [13] extended co-variational inequality and studied by different approach. Recently, Ahmad et al. [14], studied co-variational inequality involving two Yosida approximation operator.

There are many different numerical approaches that may be used to solve variational inequalities, propose iterative algorithms for variational inequalities, and analyse those algorithms. Some of these numerical methods include the projection technique and its variant forms, the auxiliary principle technique, the Newton and descending framework, and others can be found in [15–22] and references therein.

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Because resolvent equations generalise and expand the Wiener-Hopf equations, the resolvent operator approaches are effective for solving variational inequalities and other related problems. It is possible to utilise this method to demonstrate that variational inequalities and resolvent equations are equivalent to one another. The resolvent equation methodology is used in order to construct appropriate numerical methods for the purpose of resolving variational inequalities and optimisation problems.

Motivated by the aformentioned works, this study presents a generalisation of co-variational inequality solved by the idea of resolvent equations in Banach spaces through the incorporation co-resolvent equations which is a refinement of the problems studied in [13, 14]. This investigation establishes a connection between the co-variational inequality problem and the co-resolvent equation problem. A proposed approach for addressing the co-resolvent equation problem involves the use of an iterative algorithm. Eventually, a proof of an existence and convergence result is presented.

2. Preliminaries

In this study, we make the assumption that E is a real Banach space equipped with its norm $\|\cdot\|$, E^* is the topological dual of E, d is the metric induced by the norm $\|\cdot\|$, $CB(E)$ (respectively, 2^E) denotes the family of all nonempty closed and bounded subsets (respectively, all nonempty subsets) of E, $\mathscr{D}(\cdot,\cdot)$ known as the Hausdorff metric on $CB(E)$ and is defined as

$$\mathscr{D}(P, Q) = \max\left\{\sup_{x\in P} d(x, Q), \sup_{y\in Q} d(P, y)\right\},$$

where $d(x, Q) = \inf_{y\in Q} d(x, y)$ and $d(P, y) = \inf_{x\in P} d(x, y)$. Additionally, it is assumed that $\langle\cdot,\cdot\rangle$ represents the duality pairing between E and E^*, and $\mathscr{F}\colon E \longrightarrow 2^{E^*}$ is the normalized duality mapping that has been defined by

$$\mathscr{F}(x) = \{f \in E^*\colon \langle x, f\rangle = \|x\|\|f\| \text{ and } \|f\| = \|x\|\}, \quad \forall x \in E.$$

The property of uniform smoothness of the space E may be defined as follows: for every given $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $x, y \in X$, $\|x\| = 1$ and $\|y\| \le \delta$, the following condition holds:

$$\|x+y\| + \|x-y\| \le 2 + \varepsilon\|y\|.$$

The function

$$\rho_E(t) = \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1\colon \|x\| = 1, \|y\| = t\right\}$$

is called the modulus of smoothness of the space E. The space E is uniformly smooth if and only if, $\lim_{t\to 0}\frac{\rho_E(t)}{t} = 0$.

Remark 1 All Hilbert spaces, $L_p(\ell_p)$ spaces $(p \ge 2)$ and Sobolev spaces $W^{p,m}(p \ge 2)$ exhibit uniform smoothness. However, for $1 \le p < 2$, both $L_p(\ell_p)$ and $W^{p,m}$ spaces demonstrate p-uniform smoothness.

Remark 2 ([23]) If a Banach space E has the property of uniform smoothness, it may be inferred that it also possesses the properties of smoothness and reflexivity.

The subsequent inequalities will be used in the presentation of our main result, and the demonstration of these inequalities may be located in the reference [12].

Proposition 1 ([12]) Let E be a uniformly smooth Banach space and J be the normalized duality mapping from E to E^*. Then for all $x, y \in E$, we have

(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle$;

(ii) $\langle x-y, J(x)-J(y)\rangle \leq 2k^2\rho_E\left(\frac{4\|x-y\|}{k}\right)$,

where $k = \sqrt{\frac{\|x\|^2+\|y\|^2}{2}}$.

The following definitions are needed in the sequel.

Definition 1 Let $T: E \longrightarrow CB(E)$ be a set-valued mapping, $J: E \longrightarrow E^*$ and $g: E \longrightarrow E$ be the single-valued mappings. Then,

(i) T is said to be $\mathscr{D}$-Lipschitz continuous with constant $\delta_T > 0$ if,

$$\mathscr{D}(T(x)-T(y)) \leq \delta_T\|x-y\|, \quad \forall x, y \in E;$$

(ii) J is said to be strongly monotone if, there exists a constant $\alpha > 0$ such that

$$\langle J(x)-J(y), x-y\rangle \geq \alpha\|x-y\|^2, \quad \forall x, y \in E;$$

(iii) g is said to be strongly accretive if for any $x, y \in E$, there exists $j(x-y) \in \mathscr{F}(x-y)$ and a constant $\gamma > 0$ such that

$$\langle j(x-y), g(x)-g(y)\rangle \geq \gamma\|x-y\|^2.$$

Definition 2 Let $\phi: E \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a proper functional such that

$$\phi(y)-\phi(x) \geq \langle f^*, y-x\rangle, \quad \forall x, y \in E, f^* \in E^*.$$

The point f^* is called subgradient of ϕ at x. The set of all subgradients of ϕ at x is denoted by $\partial\phi(x)$. The mapping $\partial\phi: E \longrightarrow 2^{E^*}$, defined by

$$\partial\phi(x) = \{f^* \in E^*: \phi(y)-\phi(x) \geq \langle f^*, y-x\rangle, \quad \forall y \in E\},$$

is said to be subdifferential of ϕ at x.

Definition 3 ([6]) Let E be a Banach space with its dual space E^*, $\phi: E \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a proper subdifferentiable (may not convex) functional, and $J: E \longrightarrow E^*$ be a mapping. If for any given point $x^* \in E^*$ and $\rho > 0$, there is a unique point $x \in E$ satisfying

$$\langle J(x)-x^*, y-x\rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \quad \forall y \in E.$$

The mapping $x^* \to x$, denoted by $J_\rho^{\partial\phi}(x^*)$, is said to be J-proximal mapping of ϕ. We have $x^* - J(x) \in \rho\partial\phi(x)$, it follows that

$$J_\rho^{\partial\phi}(x^*) = (J + \rho\partial\phi)^{-1}(x^*).$$

Remark 3 Given a Hilbert space E, let ϕ be a convex, lower semicontinuous, and proper functional defined on E, and consider J as the identity mapping. In this context, the J-proximal mapping of ϕ can be seen as equivalent to the resolvent operator of ϕ on Hilbert spaces.

Theorem 1 ([6]) Let E be a reflexive Banach space with its dual space E^*, and ϕ: $E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous subdifferentiable proper functional which may not be convex. Let J: $E \longrightarrow E^*$ be α-strongly monotone continuous mapping. Then for any $\rho > 0$, $x^* \in E^*$, there exists a unique $x \in E$ such that

$$\langle J(x) - x^*, y - x\rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \quad \forall y \in E.$$

That is $x = J_\rho^{\partial\phi}(x^*)$ and so the J-proximal mapping of ϕ is well-defined and $\frac{1}{\alpha}$-Lipschitz continuous.

Let T, A: $E \longrightarrow CB(E)$ be two set-valued mappings, J: $E \longrightarrow E^*$, N: $E \times E \longrightarrow E$ and f, h, g: $E \longrightarrow E$ be single-valued mappings. Let ϕ: $E \times E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be such that for each fixed $x \in E$, $\phi(\cdot, x)$ is lower semicontinuous, subdifferentiable functional (may not convex) on E satisfying $g(E) \cap dom(\partial\phi(\cdot, x)) \neq \emptyset$, where $\partial\phi(\cdot, x)$ is the subdifferential of $\phi(\cdot, x)$. We consider the following problem of finding $x \in E$, $u \in T(x)$ and $v \in A(x)$ such that $g(x) \in dom(\partial\phi(\cdot, x))$ and

$$\langle J(N(f(u), h(v)), y - g(x))\rangle \geq \phi(g(x), x) - \phi(y, x), \quad \forall y \in E. \tag{1}$$

Problem (1) is called co-variational inequality problem.

Special Cases: The following are particular examples of problem (1).

(I) If E is a real Banach space and J, f and h are identity mappings, then problem (1) can be simplified to the following problem: Find $x \in E$, $u \in T(x)$ and $v \in A(x)$ such that

$$\langle N(u, v), y - g(x)\rangle \geq \phi(g(x), x) - \phi(y, x), \quad \forall y \in E. \tag{2}$$

Problem (2) covers a range of problems that have been investigated by *Hassouni* and *Moudafi* [18], *Kazmi* [20], and *Ding* [15, 16] as special cases.

(II) If E is a real Hilbert space, $\phi(x, y) = \phi(x)$, for all x, $y \in E$, J is the identity mapping, and $N(f(u), h(v)) = f(u) - h(v)$, for all u, $v \in E$, then problem (1) transforms into the problem of determining $x \in E$, $u \in T(x)$ and $v \in A(x)$ such that

$$\langle f(u) - h(v), y - g(x)\rangle \geq \phi(g(x)) - \phi(y), \quad \forall y \in E. \tag{3}$$

The problem (3) was first proposed and subsequently investigated by *Huang* [24].

By carefully selecting the appropriate mappings in the formulation of problem (1), one may establish relationships to other existing problems that have been previously studied, see e.g., [8–11].

In relation to the co-variational inequality problem (1), we consider the subsequent co-resolvent equation problem: Find $z \in E^*$, $x \in E$, $u \in T(x)$ and $v \in A(x)$ such that

$$J(N(f(u), h(v))) + \rho^{-1} R_\rho^{\partial\phi(\cdot, x)}(z) = 0, \tag{4}$$

where $\rho > 0$ is a constant, $R_\rho^{\partial\phi} = I - J\left(J_\rho^{\partial\phi}(z)\right)$, where $J\left(J_\rho^{\partial\phi}(z)\right) = \left[J\left(J_\rho^{\partial\phi}\right)\right](z)$ and I is the identity mapping.

3. Iterative algorithm and convergence result

Initially, we establish a correspondence between the co-variational inequality problem (1) and a fixed point problem. This correspondence can be simply shown by using the notion of the resolvent operator and the concept of subdifferentiability of $\phi(\cdot, x)$.

Lemma 1 Let (x, u, v), where $x \in E$, $u \in T(x)$ and $v \in A(x)$, be a solution of co-variational inequality problem (1) if and only if it is the solution of the following equation

$$g(x) = J_\rho^{\partial\phi(\cdot, x)} \{J(g(x)) - \rho J(N(f(u), h(v)))\}. \tag{5}$$

Proof. Let (5) holds, then

$$g(x) = J_\rho^{\partial\phi(\cdot, x)} \{J(g(x)) - \rho J(N(f(u), h(v)))\}$$

$$g(x) = [J + \rho\partial\phi]^{-1} \{J(g(x)) - \rho J(N(f(u), h(v)))\},$$

which implies that,

$$J(g(x)) + \rho\partial\phi(g(x), x) = J(g(x) - \rho J(N(f(u), h(v)))$$

$$J(N(f(u), h(v))) \in \partial\phi(g(x), x).$$

Therefore (x, u, v) is the solution of problem (1). That is,

$$\langle J(N(f(u), h(v)), y - g(x))\rangle \geq \phi(g(x), x) - \phi(y, x), \quad \forall y \in E,$$

holds. □

Now, we prove that co-variational inequality problem (1) is equivalent to the co-resolvent equation problem (4).

Lemma 2 If J is one-one, then co-variational inequality problem (1) has a solution (x, u, v), where $x \in E$, $u \in T(x)$ and $v \in A(x)$, if and only if co-resolvent equation problem (4) has a solution (z, x, u, v), $z \in E^*$, $x \in E$, $u \in T(x)$ and $v \in A(x)$, where

$$g(x) = J_\rho^{\partial\phi(\cdot, x)}(z), \tag{6}$$

and

$$z = J(g(x)) - \rho J(N(f(u), h(v))).$$

Proof. Let (x, u, v) be a solution of co-variational inequality problem (1). Then by Lemma 1, it is a solution of the following equation

$$g(x) = J_{\rho}^{\partial\phi(\cdot, x)}\left\{J(g(x)) - \rho J(N(f(u), h(v)))\right\}.$$

Using the fact that $R_{\rho}^{\partial\phi(\cdot, x)} = \left[I - J\left(J_{\rho}^{\partial\phi(\cdot, x)}\right)\right]$ and (6), we have

$$\begin{aligned}
&R_{\rho}^{\partial\phi(\cdot, x)}\left\{J(g(x)) - \rho J(N(f(u), h(v)))\right\}\\
=&J(g(x)) - \rho J(N(f(u), h(v))) - J\left[J_{\rho}^{\partial\phi(\cdot, x)}\left\{J(g(x)) - \rho J(N(f(u), h(v)))\right\}\right]\\
=&J(g(x)) - \rho J(N(f(u), h(v))) - J(g(x))\\
=&-\rho J(N(f(u), h(v))),
\end{aligned}$$

which implies that

$$J(N(f(u), h(v))) + \rho^{-1}R_{\rho}^{\partial\phi(\cdot, x)}(z) = 0,$$

with $z = J(g(x)) - \rho J(N(f(u), h(v)))$, i.e., (z, x, u, v) is the solution of co-resolvent equation problem (4).

Conversely, let (z, x, u, v) be the solution of co-resolvent equation problem (4), then

$$\rho J(N(f(u), h(v))) = -R_{\rho}^{\partial\phi(\cdot, x)}(z) = J\left(J_{\rho}^{\partial\phi(\cdot, x)}(z)\right) - z. \tag{7}$$

From (6) and (7), we have

$$\begin{aligned}
&\rho J(N(f(u), h(v)))\\
=&J\left(J_{\rho}^{\partial\phi(\cdot, x)}\{J(g(x)) - \rho J(N(f(u), h(v)))\}\right) - J(g(x)) - \rho J(N(f(u), h(v))),
\end{aligned}$$

which implies that

$$J(g(x)) = J\left(J_{\rho}^{\partial\phi(\cdot,\,x)}\{J(g(x)) - \rho J(N(f(u),\,h(v)))\}\right).$$

Since J is one-one, we have

$$g(x) = J_{\rho}^{\partial\phi(\cdot,\,x)}\left\{J(g(x)) - \rho J(N(f(u),\,h(v)))\right\},$$

i.e., $(x,\,u,\,v)$ is the solution of co-variational inequality problem (1). □

Alternative Proof. Let

$$z = J(g(x)) - \rho J(N(f(u),\,h(v))).$$

Then from (6), we have

$$g(x) = J_{\rho}^{\partial\phi(\cdot,\,x)}(z),$$

and

$$z = J\left(J_{\rho}^{\partial\phi(\cdot,\,x)}(z)\right) - \rho J(N(f(u),\,h(v))).$$

By using the fact that $J\left(J_{\rho}^{\partial\phi(\cdot,\,x)}(z)\right) = \left[J\left(J_{\rho}^{\partial\phi(\cdot,\,x)}\right)\right](z)$, it follows that

$$J(N(f(u),\,h(v))) + \rho^{-1}R_{\rho}^{\partial\phi(\cdot,\,x)}(z) = 0,$$

which is the required co-resolvent equation problem (4). □

Following is the iterative algorithm to solve our problem.

Algorithm 1 For any $z_0 \in E^*$, $x_0 \in E$, $u_0 \in T(x_0)$ and $v_0 \in A(x_0)$, let

$$z_1 = J(g(x_0)) - \rho J(N(f(u_0),\,h(v_0))) \in E^*,$$

and take $x_1 \in E$ such that

$$g(x_1) = J_{\rho}^{\partial\phi(\cdot,\,x_1)}(z_1).$$

Since $u_0 \in T(x_0)$ and $v_0 \in A(x_0)$, by Nadler's Theorem [25], there exists $u_1 \in T(x_1)$ and $v_1 \in A(x_1)$ such that

$$\|u_0 - u_1\| \leq (1+1)\mathscr{D}(T(x_0), T(x_1));$$

$$\|v_0 - v_1\| \leq (1+1)\mathscr{D}(A(x_0), A(x_1)),$$

where $\mathscr{D}(\cdot, \cdot)$ is the Hausdorff metric on $CB(E)$. Let

$$z_2 = J(g(x_1)) - \rho J(N(f(u_1), h(v_1))),$$

and take any $x_2 \in E$ such that

$$g(x_2) = J_\rho^{\partial\phi(\cdot, x_2)}(z_2).$$

Continuing the above process inductively, we can obtain the following scheme:

For any $z_0 \in E^*$, $x_0 \in E$, $u_0 \in T(x_0)$ and $v_0 \in A(x_0)$, compute the sequences $\{z_n\}$, $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ by the iterative schemes such that

1.

$$g(x_n) = J_\rho^{\partial\phi(\cdot, x_n)}(z_n); \tag{8}$$

2.

$$u_n \in T(x_n), \ \|u_n - u_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right)\mathscr{D}(T(x_n), T(x_{n+1})); \tag{9}$$

3.

$$v_n \in A(x_n), \ \|v_n - v_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right)\mathscr{D}(A(x_n), A(x_{n+1})); \tag{10}$$

4.

$$z_{n+1} = J(g(x_n)) - \rho J(N(f(u_n), h(v_n))), \tag{11}$$

for $n = 0, 1, 2, \cdots$ and $\rho > 0$ is a constant.

Now, we have the main result.

Theorem 2 Let E be a uniformly smooth Banach space with the module of smoothness $\rho_E(t) \leq Ct^2$, for some $t > 0$. Let T, A: $E \longrightarrow CB(E)$ be the $\mathscr{D}$-Lipschitz continuous mappings with constants δ_T and δ_A, respectively. Let f, h: $E \longrightarrow E$ be the Lipschitz continuous mappings with constants λ_f and λ_h, respectively, and g: $E \longrightarrow E$ be a Lipschitz continuous mapping with constant λ_g and strongly accretive with constant $\gamma > 0$. Let J: $E \longrightarrow E^*$ be a Lipschitz continuous mapping with constant λ_j and strongly monotone with constant $\alpha > 0$, and N: $E \times E \longrightarrow E$ be Lipschitz continuous in both the arguments with constants λ_{N_1} and λ_{N_2}, respectively. Let ϕ: $E \times E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a mapping such that for each fixed $x \in E$, $\phi(\cdot, x)$ is lower semicontinuous, subdifferentiable, proper functional satisfying $g(x) \in dom\partial\phi(\cdot, x)$, for all $x \in E$. Suppose that there exists a constant $\rho > 0$ such that for each $x, y \in E$, $x^* \in E^*$

$$\left\| J_{\rho}^{\partial\phi(\cdot,x)}(x^*) - J_{\rho}^{\partial\phi(\cdot,y)}(x^*) \right\| \leq \mu \|x - y\|, \tag{12}$$

where $J^{\partial\phi(\cdot,x)}$ is J-proximal mapping of ϕ and the following condition is satisfied:

$$\left| \rho - \frac{\alpha\mu - \lambda_g\lambda_j}{\lambda_j\left(\lambda_{N_1}\lambda_f\delta_T + \lambda_{N_2}\lambda_h\delta_A\right)} \right| < \frac{\alpha\sqrt{1 - 2\gamma + 64C\lambda_g^2}}{\lambda_j\left(\lambda_{N_1}\lambda_f\delta_T + \lambda_{N_2}\lambda_h\delta_A\right)}, \tag{13}$$

$\alpha\mu > \lambda_g\lambda_j$, then there exists $z \in E^*$, $x \in E$, $u \in T(x)$ and $v \in A(x)$ satisfying co-resolvent equation problem (4), and the iterative sequences $\{z_n\}$, $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ generated by Algorithm 1 converge strongly to z, x, u and v, respectively.

Proof. From Algorithm 1, we have

$$\begin{aligned} &\|z_{n+1} - z_n\| \\ &= \|J(g(x_n)) - \rho J(N(f(u_n), h(v_n))) - \{J(g(x_{n-1})) - \rho J(N(f(u_{n-1}), h(v_{n-1})))\}\| \\ &\leq \|J(g(x_n)) - J(g(x_{n-1}))\| + \rho \|J(N(f(u_n), h(v_n))) - J(N(f(u_{n-1}), h(v_{n-1})))\|. \end{aligned} \tag{14}$$

By the Lipschitz continuity of J and g, we have

$$\|J(g(x_n)) - J(g(x_{n-1}))\| \leq \lambda_j \|g(x_n) - g(x_{n-1})\| \leq \lambda_j\lambda_g \|x_n - x_{n-1}\|. \tag{15}$$

By the Lipschitz continuity of J, f, h, $\mathscr{D}$-Lipschitz continuity of T, A and Lipschitz continuity of N in both the arguments, we have

$$\begin{aligned} &\|J(N(f(u_n), h(v_n))) - J(N(f(u_{n-1}), h(v_{n-1})))\| \\ &\leq \lambda_j \{\lambda_{N_1} \|f(u_n) - f(u_{n-1})\| + \lambda_{N_2} \|h(v_n) - h(v_{n-1})\|\} \\ &\leq \lambda_j\lambda_{N_1}\lambda_f \|u_n - u_{n-1}\| + \lambda_j\lambda_{N_2}\lambda_h \|v_n - v_{n-1}\| \\ &\leq \lambda_j\lambda_{N_1}\lambda_f [\mathscr{D}(T(x_n), T(x_{n-1}))] + \lambda_j\lambda_{N_2}\lambda_h [\mathscr{D}(A(x_n), A(x_{n-1}))] \\ &\leq \lambda_j\lambda_{N_1}\lambda_f\delta_T \|x_n - x_{n-1}\| + \lambda_j\lambda_{N_2}\lambda_h\delta_A \|x_n - x_{n-1}\| \end{aligned}$$

$$= \lambda_j \left(\lambda_{N_1} \lambda_f \delta_T + \lambda_{N_2} \lambda_h \delta_A\right) \|x_n - x_{n-1}\|. \tag{16}$$

Combining (15) and (16) with (14), we obtain

$$\|z_{n+1} - z_n\| \leq \left[\lambda_j \lambda_g + \rho \lambda_j \left(\lambda_{N_1} \lambda_f \delta_T + \lambda_{N_2} \lambda_h \delta_A\right)\right] \|x_n - x_{n-1}\|. \tag{17}$$

Using condition (12) and Lipschitz continuity of $J_\rho^{\partial\phi(\cdot,x)}$, we get

$$\begin{aligned}
&\|x_n - x_{n-1}\| \\
=&\left\|J_\rho^{\partial\phi(\cdot,x_n)}(z_n) - J_\rho^{\partial\phi(\cdot,x_{n-1})}(z_{n-1}) - [g(x_n) - x_n - \{g(x_{n-1}) - x_{n-1}\}]\right\| \\
\leq&\left\|J_\rho^{\partial\phi(\cdot,x_n)}(z_n) - J_\rho^{\partial\phi(\cdot,x_{n-1})}(z_{n-1}) + J_\rho^{\partial\phi(\cdot,x_{n-1})}(z_n) - J_\rho^{\partial\phi(\cdot,x_{n-1})}(z_n)\right\| \\
&+ \|x_n - x_{n-1} - \{g(x_n) - g(x_{n-1})\}\| \\
\leq&\left\|J_\rho^{\partial\phi(\cdot,x_n)}(z_n) - J_\rho^{\partial\phi(\cdot,x_{n-1})}(z_n)\right\| + \left\|J_\rho^{\partial\phi(\cdot,x_{n-1})}(z_n) - J_\rho^{\partial\phi(\cdot,x_{n-1})}(z_{n-1})\right\| \\
&+ \|x_n - x_{n-1} - \{g(x_n) - g(x_{n-1})\}\| \\
\leq&\mu\|x_n - x_{n-1}\| + \frac{1}{\alpha}\|z_n - z_{n-1}\| + \|x_n - x_{n-1} - \{g(x_n) - g(x_{n-1})\}\|.
\end{aligned} \tag{18}$$

By Proposition 1, we have

$$\begin{aligned}
&\|x_n - x_{n-1} - \{g(x_n) - g(x_{n-1})\}\|^2 \\
\leq&\|x_n - x_{n-1}\|^2 - 2\left\langle g(x_n) - g(x_{n-1}), J(x_n - x_{n-1} - \{g(x_n) - g(x_{n-1})\})\right\rangle \\
=&\|x_n - x_{n-1}\|^2 - 2\left\langle g(x_n) - g(x_{n-1}), J(x_n - x_{n-1})\right\rangle - \\
&2\left\langle g(x_n) - g(x_{n-1}), J(x_n - x_{n-1} - \{g(x_n) - g(x_{n-1})\}) - J(x_n - x_{n-1})\right\rangle
\end{aligned}$$

$$\leq \|x_n - x_{n-1}\|^2 - 2\gamma \|x_n - x_{n-1}\|^2 + 4k^2 \rho_E \left(\frac{4\|g(x_n) - g(x_{n-1})\|}{k} \right)$$

$$\leq \|x_n - x_{n-1}\|^2 - 2\gamma \|x_n - x_{n-1}\|^2 + 64C\|g(x_n) - g(x_{n-1})\|^2$$

$$\leq \left(1 - 2\gamma + 64C\lambda_g^2\right) \|x_n - x_{n-1}\|^2,$$

which implies that

$$\|x_n - x_{n-1} - \{g(x_n) - g(x_{n-1})\}\| \leq \sqrt{\left(1 - 2\gamma + 64C\lambda_g^2\right)} \|x_n - x_{n-1}\|. \tag{19}$$

Using (19), (18) becomes

$$\|x_n - x_{n-1}\| \leq \left[\sqrt{\left(1 - 2\gamma + 64C\lambda_g^2\right)} + \mu \right] \|x_n - x_{n-1}\| + \frac{1}{\alpha} \|z_n - z_{n-1}\|,$$

which implies that

$$\|x_n - x_{n-1}\| \leq \frac{1}{\alpha \left[\sqrt{\left(1 - 2\gamma + 64C\lambda_g^2\right)} + \mu \right]} \|z_n - z_{n-1}\|. \tag{20}$$

By the application of (20), (17) reduces to

$$\|z_{n+1} - z_n\| \leq \frac{\left[\lambda_j \lambda_g + \rho \lambda_j \left(\lambda_{N_1} \lambda_f \delta_T + \lambda_{N_2} \lambda_h \delta_A\right)\right]}{\alpha \left[\sqrt{\left(1 - 2\gamma + 64C\lambda_g^2\right)} + \mu \right]} \|z_n - z_{n-1}\|,$$

i.e.,

$$\|z_{n+1} - z_n\| \leq \Theta \|z_n - z_{n-1}\|, \tag{21}$$

where $\Theta = \dfrac{\lambda_j \lambda_g + \rho \lambda_j \left(\lambda_{N_1} \lambda_f \delta_T + \lambda_{N_2} \lambda_h \delta_A\right)}{\alpha \left[\sqrt{\left(1 - 2\gamma + 64C\lambda_g^2\right)} + \mu \right]}$.

From (13), we have $\Theta < 1$, and consequently $\{z_n\}$ is a cauchy sequence in E^*. Since E^* is a Banach space, there exists $z \in E^*$ such that $z_n \to z$ as $n \to \infty$. From (18), we can see that $\{x_n\}$ is also a cauchy sequence in E. Therefore, there exists $x \in E$ such that $x_n \to x$ as $n \to \infty$. Since the mappings T and A are $\mathscr{D}$-Lipschitz continuous, it follows from (9) and (10) of Algorithm 1 that $\{u_n\}$ and $\{v_n\}$ are also cauchy sequences, we may assume that $u_n \to u$ and $v_n \to v$.

Since J, N, g, f and h are continuous mappings, and by (11) of Algorithm 1, it follows that

$$J(g(x_n)) - \rho J(N(f(u_n), h(v_n))) = z_{n+1} \to z = J(g(x)) - \rho J(N(f(u), h(v))), n \to \infty, \tag{22}$$

and

$$J_\rho^{\partial\phi(\cdot, x_n)}(z_n) = g(x_n) \to g(x) = J_\rho^{\partial\phi(\cdot, x)}(z), n \to \infty. \tag{23}$$

By (22), (23) and Lemma 2, we have

$$J(N(f(u), h(v))) + \rho^{-1} R_\rho^{\partial\phi(\cdot, x)}(z) = 0.$$

Finally, we prove that $u \in T(x)$ and $v \in A(x)$. In fact, since $u_n \in T(x_n)$ and

$$d(u_n, T(x)) \leq \max\left\{d(u_n, T(x)), \sup_{q \in T(x)} d(T(x), q)\right\}$$

$$= \mathscr{D}(T(x_n), T(x)).$$

Therefore, we have

$$d(u, T(x)) \leq \|u - u_n\| + d(u_n, T(x))$$

$$\leq \|u - u_n\| + \mathscr{D}(T(x_n), T(x))$$

$$\leq \|u - u_n\| + \delta_T \|x_n - x\| \to 0, \text{ as } n \to \infty,$$

which implies that $d(u, T(x)) = 0$. Since $T(x) \in CB(E)$, it follows that $u \in T(x)$. Similarly, we can prove that $v \in A(x)$. By Lemma 2, the required result follows. This completes the proof. □

To substantiate the assertion made in problem (1), we provide the following example.

Example 1 Let us suppose that $E = \mathbb{R}$. Define $T, A: E \longrightarrow CB(E)$ by $T(x) = \{\frac{3x}{5} + 1\}$, $A(x) = \{\frac{x}{11}\}$, g and ϕ by $g(x) = x - 1$ and $\phi(y, x) = y + x$, for all $x, y \in E$.

We define for $x \in E$, $u \in T(x)$ and $v \in A(x)$

(i) $J(x) = 2x$,

(ii) $N(x, y) = 2x + 3y$,

(iii) $f(u) = \frac{u}{2}$,

(iv) $h(v) = \frac{v}{3}$,

(v) $\partial\phi(\cdot, x) = \{\alpha \in \mathbb{R}: \phi(\cdot, y) - \phi(\cdot, x) \geq \alpha(y - x)\}$.

Now, for $\rho = 1$, we define the J-proximal mapping

$$J_\rho^{\partial\phi}(x) = \frac{x-2}{2}.$$

We examine the sequence $\{x_n\}$ by using Algorithm 1 as

$$\begin{aligned} g(x_n) &= J_\rho^{\partial\phi(\cdot, x_n)}(z_n) \\ x_n &= \frac{z_n}{2}, \end{aligned} \tag{24}$$

and

$$\begin{aligned} z_{n+1} &= J(g(x_n)) - \rho J(N(f(u_n), h(v_n))) \\ z_{n+1} &= \frac{17x_n}{55} - 2. \end{aligned} \tag{25}$$

Using (24) in (25), we have

$$z_{n+1} = \frac{17z_n}{110} - 2.$$

The convergence of the sequence $\{z_n\}$ is shown in Figure 1 and computational table is also provided. As $\{z_n\}$ converges to -2.3656. Hence, $\{x_n\}$ converges to -1.1828.

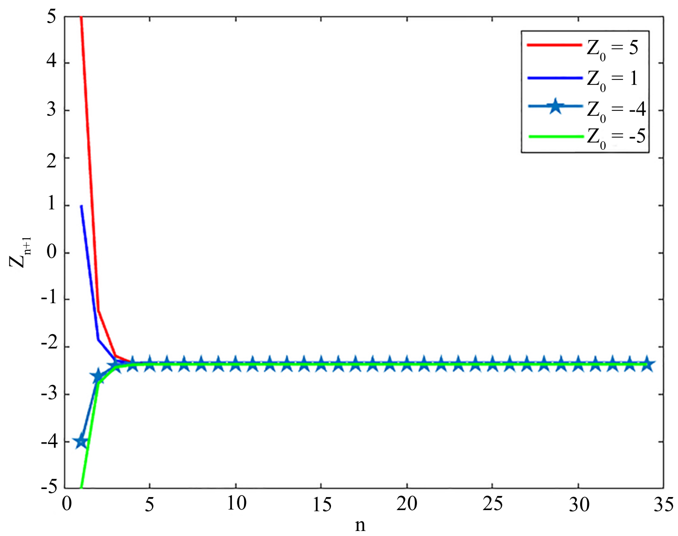

Figure 1. The sequence $\{z_n\}'s$ convergence for four distinct initial values

Table 1. The values of z_n with initial values $z_0 = -5$, $z_0 = -4$, $z_0 = 1$ and $z_0 = 5$

No. of Iteration	For $z_0 = -5$ z_n	For $z_0 = -4$ z_n	For $z_0 = 1$ z_n	For $z_0 = 5$ z_n
n = 1	−5	2.5	5	5
n = 2	−2.7727	−2.6182	−1.8455	−1.2273
n = 3	−2.4285	−2.2852	−2.1897	−2.1897
n = 4	−2.3753	−2.3532	−2.3384	−2.3384
n = 5	−2.3671	−2.3637	−2.3614	−2.3614
n = 10	−2.3656	−2.3656	−2.3656	−2.3656
n = 15	−2.3656	−2.3656	−2.3656	−2.3656
n = 20	−2.3656	−2.3656	−2.3656	−2.3656
n = 25	−2.3656	−2.3656	−2.3656	−2.3656
n = 27	−2.3656	−2.3656	−2.3656	−2.3656
n = 29	−2.3656	−2.3656	−2.3656	−2.3656
n = 30	−2.3656	−2.3656	−2.3656	−2.3656

4. Conclusion and future recommendations

This work presents a comprehensive analysis of a co-variational inequality problem and a co-resolvent equation problem. The scope of our study encompasses a broader range of problems compared to those previously explored in the literature, as shown by the works of [15, 16, 18, 20]. The proof of the equivalence lemma between the co-variational inequality problem and the co-resolvent equation problem is established by using the concepts of sub-differentiability and the resolvent operator. Moreover, a suggested iterative algorithm is presented for the purpose of addressing the problems at hand. Additionally, a strong convergence theorem is established to demonstrate that the sequences created by the algorithm exhibit strong convergence towards the solution of the primary problem. The findings of our study may be regarded as an advancement upon several previously established findings. To prove our main result the Lipschitz continuity of J-proximal mapping is needed where the Lipschitz constant $K = 1/\alpha$, which is often difficult and expensive to get in real applications. So, we tried to remove this condition but we couldn't get it. This was the challenge which we faced during the proof of our main results. So further investigation on convergence analysis of our proposed problems is needed to obtain more milder and checkable conditions.

Confilict of interest

The authors declare no competing financial interest.

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